Further Pure Mathematics 1

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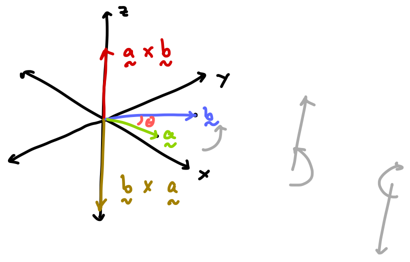
Chapter 1: Vectors

THE VECTOR PRODUCT

- notation: $\underset{\sim}{a} \times \underset{\sim}{b}$



- defn:
  - a vector of magn $|\underset{\sim}{a}||\underset{\sim}{b}| \sin\theta$, $\perp$ to the plane containing a & b.
  - $\underset{\sim}{a} \times \underset{\sim}{b} = |a||b| \sin\theta \, \hat{n}$

  ($\hat{n}$ = unit vector in dir. of a×b).

  ☆ $a \times b = -b \times a$ (cross product NOT commutative)

Cases

① Two // vectors

$\rightarrow$ $a \times b = 0$

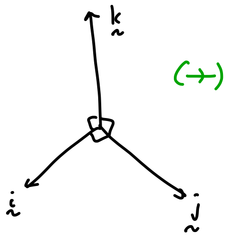
since $\theta = 0$.

② Two $\perp$ vectors

$\rightarrow$ $|a \times b| = |a||b| \sin 90$

$|a \times b| = |a||b|$.

③ Unit vectors



$\underset{\sim}{i} \quad \underset{\sim}{j} \quad \underset{\sim}{k} \quad \underset{\sim}{i} \quad \underset{\sim}{j}$

(+)

$\underset{\sim}{i} \times \underset{\sim}{j} = \underset{\sim}{k}$

$\underset{\sim}{j} \times \underset{\sim}{k} = \underset{\sim}{i}$

$\underset{\sim}{k} \times \underset{\sim}{i} = \underset{\sim}{j}$

(−)

$\underset{\sim}{j} \times \underset{\sim}{i} = -\underset{\sim}{k}$

$\underset{\sim}{k} \times \underset{\sim}{j} = -\underset{\sim}{i}$

$\underset{\sim}{i} \times \underset{\sim}{k} = -\underset{\sim}{j}$

$\underset{\sim}{i} \times \underset{\sim}{i} = \underset{\sim}{j} \times \underset{\sim}{j} = \underset{\sim}{k} \times \underset{\sim}{k} = 0$

☆ but $\underset{\sim}{i} \cdot \underset{\sim}{i} = \underset{\sim}{j} \cdot \underset{\sim}{j} = \underset{\sim}{k} \cdot \underset{\sim}{k} = 1$!!!

④ Two vectors in Cartesian form.

$\underset{\sim}{a} = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} \qquad \underset{\sim}{b} = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}$

$$\boxed{\text{result: } \underset{\sim}{a} \times \underset{\sim}{b} = \begin{pmatrix} a_2 b_3 - a_3 b_2 \\ a_3 b_1 - a_1 b_3 \\ a_1 b_2 - a_2 b_1 \end{pmatrix}.}$$

proof:

$$\underset{\sim}{a} \times \underset{\sim}{b} = (a_1 \underset{\sim}{i} + a_2 \underset{\sim}{j} + a_3 \underset{\sim}{k})(b_1 \underset{\sim}{i} + b_2 \underset{\sim}{j} + b_3 \underset{\sim}{k})$$

$$= \underset{\sim}{0} + a_1 b_2 \underset{\sim}{k} - a_1 b_3 \underset{\sim}{j}$$

$$+ (-a_2 b_1) \underset{\sim}{k} + \underset{\sim}{0} + a_2 b_3 \underset{\sim}{i}$$

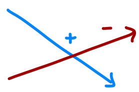
$$+ a_3 b_1 \underset{\sim}{j} - a_3 b_2 \underset{\sim}{i} + \underset{\sim}{0}$$

$$= (a_2 b_3 - a_3 b_2) \underset{\sim}{i} + (a_3 b_1 - a_1 b_3) \underset{\sim}{j} + (a_1 b_2 - a_2 b_1) \underset{\sim}{k}.$$

Shortcut

$\underset{\sim}{a} \times \underset{\sim}{b} = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} \times \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}$ ① ② ③

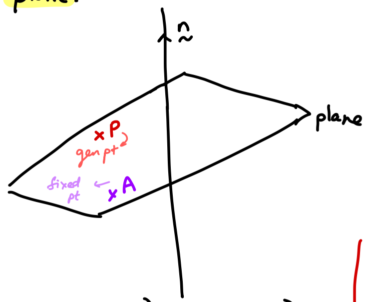
tip: cover "rows" in descending order, and:



EQ^{N} OF A PLANE

A plane can be located in space by :

① The direction of a vector $\perp$ to the plane and one point in the plane.



let $\underset{\sim}{r} = \overrightarrow{OP}$, $\underset{\sim}{a} = \overrightarrow{OA}$.

$\therefore \overrightarrow{AP} = \underset{\sim}{r} - \underset{\sim}{a}$.

$\overrightarrow{AP} \perp \underset{\sim}{n} = 0$

$\Rightarrow (\underset{\sim}{r} - \underset{\sim}{a}) \cdot \underset{\sim}{n} = 0$

$\therefore \underset{\sim}{r} \cdot \underset{\sim}{n} = \underset{\sim}{a} \cdot \underset{\sim}{n} \quad (= d)$.

$\boxed{\underset{\sim}{r} \cdot \underset{\sim}{n} = d} \rightarrow$ eq^{n} of a plane.

$\hookrightarrow$ $\underset{\sim}{n}$ is a vector $\perp$ to the plane, ie a normal vector.

Cartesian eq^{n} of a plane.

$\underset{\sim}{r} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$. Let $\underset{\sim}{n} = \begin{pmatrix} a \\ b \\ c \end{pmatrix}$.

$\boxed{\text{Then } d = ax + by + cz.}$ ← known as the Cartesian eq^{n} of the plane.

$x = 0 \longrightarrow$ the yz plane

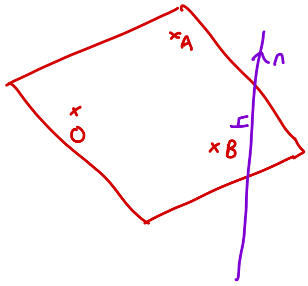
$y = 0 \longrightarrow$ the xz plane

$z = 0 \longrightarrow$ the xy plane.

② Three non-collinear points.

eg : $\overrightarrow{OA} = \begin{pmatrix} 5 \\ -1 \\ -3 \end{pmatrix}$, $\overrightarrow{OB} = \begin{pmatrix} -4 \\ 4 \\ -1 \end{pmatrix}$.

plane of OAB eq^{n} ?



Let $\underset{\sim}{n}$ be a normal vector to the plane.

$\boxed{\underset{\sim}{n} \perp \overrightarrow{OA} \ \& \ \underset{\sim}{n} \perp \overrightarrow{OB}.}$

$\therefore \underset{\sim}{n} = \overrightarrow{OA} \times \overrightarrow{OB}$.

$= \begin{pmatrix} 5 \\ -1 \\ -3 \end{pmatrix} \times \begin{pmatrix} -4 \\ 4 \\ -1 \end{pmatrix}$

$= \begin{pmatrix} 13 \\ 17 \\ 16 \end{pmatrix}$

Hence the eq^{n} of the plane

is $\underset{\sim}{r} \cdot \begin{pmatrix} 13 \\ 17 \\ 16 \end{pmatrix} = \underset{\sim}{a} \cdot \underset{\sim}{n} = 0$ (why? contains origin).

$\Rightarrow 13x + 17y + 16z = 0$.

To find the eq^{n} of a plane:

steps

1) must know pv of a pt in the plane

☆ 2) find a vector $\perp$ to the plane

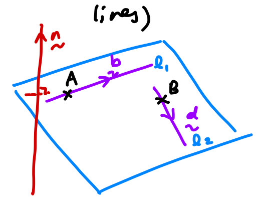
3) the eq^{n} of the plane is

$\underset{\sim}{r} \cdot \underset{\sim}{n} = \underset{\sim}{a} \cdot \underset{\sim}{n}$

$\underset{\sim}{n}$ can be found by taking the vector product of two vectors // to the plane.

③ Two concurrent lines in the plane

(contains 2 non // lines or // to 2 nonparallel lines)

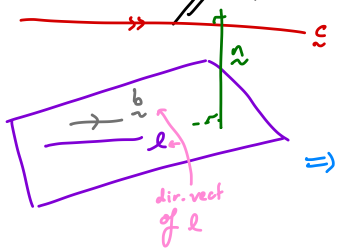


$\underset{\sim}{n} \perp \underset{\sim}{b} \ \& \ \underset{\sim}{n} \perp \underset{\sim}{d}$

$\therefore \underset{\sim}{n} = \underset{\sim}{b} \times \underset{\sim}{d}$.

⋮

④ A line in the plane & a vector // to the plane.



$\underset{\sim}{n} \perp \underset{\sim}{b} \ \& \ \underset{\sim}{c}$, $\underset{\sim}{c}$ // the plane

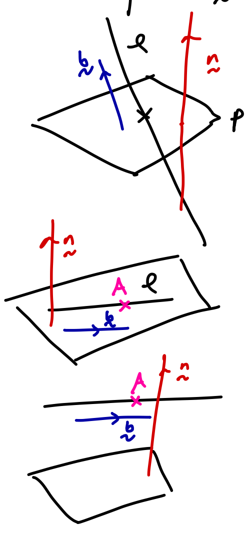
$\Rightarrow \underset{\sim}{n} = \underset{\sim}{b} \times \underset{\sim}{c}$.

GIVEN A LINE & A PLANE...

There are 3 cases.

let $l: \underset{\sim}{r} = \underset{\sim}{a} + \lambda \underset{\sim}{b}$

$p: \underset{\sim}{r} \cdot \underset{\sim}{n} = d$



(1) The line intersects the plane.

$\underset{\sim}{n} \cdot \underset{\sim}{b} \neq 0$

(2) The line lies in the plane.

$\underset{\sim}{n} \cdot \underset{\sim}{b} = 0$

$\underset{\sim}{a} \cdot \underset{\sim}{n} = d.$

(3) The line is // to the plane.

$\underset{\sim}{b} \cdot \underset{\sim}{n} = 0$

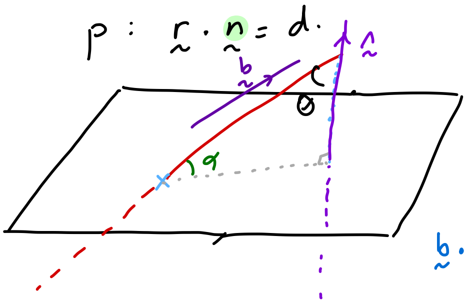
$\underset{\sim}{a} \cdot \underset{\sim}{n} \neq d.$

dot product of 2 vectors!

TO FIND THE ACUTE ∡ B/W A LINE & A PLANE.

$l: \underset{\sim}{r} = \underset{\sim}{a} + \lambda \underset{\sim}{b}$

$p: \underset{\sim}{r} \cdot \underset{\sim}{n} = d.$



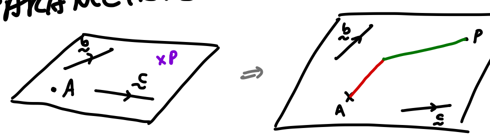
$\underset{\sim}{b} \cdot \underset{\sim}{n} \rightarrow$ find the ∡.

The acute ∡ b/w a line & a plane is the acute ∡ b/w the line & the projection (shadow) of the line onto the plane.

$\underset{\sim}{b} \cdot \underset{\sim}{n} = |\underset{\sim}{b}||\underset{\sim}{n}|\boxed{\cos\theta}.$ $\cos\theta = \sin\alpha.$

$\alpha = 90 - \theta.$

A VECTOR EQN OF A PLANE IN PARAMETRIC FORM.



green line // to $\underset{\sim}{c}$

red line // to $\underset{\sim}{b}$

$\underset{\sim}{a} \rightarrow$ pv of A

let $P \rightarrow$ general pt in the plane w/ pv $\underset{\sim}{r}$. ie $\vec{OP} = \underset{\sim}{r}$.

$\vec{OA} = \underset{\sim}{a}$

$\Rightarrow \vec{AP} = \lambda \underset{\sim}{b} + \mu \underset{\sim}{c}.$

$\vec{OP} = \underset{\sim}{a} + \lambda \underset{\sim}{b} + \mu \underset{\sim}{c}$

ie $\boxed{\underset{\sim}{r} = \underset{\sim}{a} + \lambda \underset{\sim}{b} + \mu \underset{\sim}{c}.}$

pv of any pt in the plane.

pv of a particular pt in the plane.

a vector // to the plane.

a vector // to the plane.

*but $\underset{\sim}{b}$ is not // to $\underset{\sim}{c}$.

$\underset{\sim}{n} \cdot \underset{\sim}{r} = d$

Conversion b/w Cartesian & parametric

Recall that since $\underset{\sim}{n}$ is the vect product of 2 vects // to the plane, $\underset{\sim}{n} = \underset{\sim}{b} \times \underset{\sim}{c}.$

and $d = \underset{\sim}{a} \cdot \underset{\sim}{n}.$

eg[1] $\underset{\sim}{r} = \begin{pmatrix}2\\3\\-4\end{pmatrix} + \theta\begin{pmatrix}1\\3\\2\end{pmatrix} + \phi\begin{pmatrix}5\\8\\-1\end{pmatrix}.$

$\Rightarrow \underset{\sim}{n} = \begin{pmatrix}1\\3\\2\end{pmatrix} \times \begin{pmatrix}5\\8\\-1\end{pmatrix}$

$= \begin{pmatrix}-19\\11\\-7\end{pmatrix}.$

$d = \underset{\sim}{a} \cdot \underset{\sim}{n}$

$= \begin{pmatrix}2\\3\\-4\end{pmatrix} \cdot \begin{pmatrix}-19\\11\\-7\end{pmatrix}$

$= -38 + 33 + 28$

$= 23.$

∴ cartesian eqn is $-19x + 11y - 7z = 23.$

eg[2] $4x + 4y + 5z = 40$

$\Rightarrow \underset{\sim}{r} = \underset{\sim}{a} + \lambda \underset{\sim}{b} + \mu \underset{\sim}{c}$

$\underset{\sim}{a} = \begin{pmatrix}0\\p\\0\end{pmatrix}, \underset{\sim}{b} = \begin{pmatrix}0\\q\\r\end{pmatrix}, \underset{\sim}{c} = \begin{pmatrix}s\\t\\0\end{pmatrix}$

$p, q, r, s, t \in \mathbb{Z}$

$\underset{\sim}{n} = \underset{\sim}{b} \times \underset{\sim}{c}$

$= \begin{pmatrix}0\\q\\r\end{pmatrix} \times \begin{pmatrix}s\\t\\0\end{pmatrix}$

$= \begin{pmatrix}-rt\\rs\\-qs\end{pmatrix} = \begin{pmatrix}4\\4\\5\end{pmatrix}.$

$\underset{\sim}{n} \cdot \underset{\sim}{a} = 40$

$\begin{pmatrix}0\\p\\0\end{pmatrix} \cdot \begin{pmatrix}4\\4\\5\end{pmatrix} = 40$

$4p = 40$ ∴ $p = 10.$

$-5 - 4t = 4$

$-rt = 4$

$rs = 4$

$-qs = 5.$

$s = 1 \rightarrow r = 4$

$q = -5$

$t = -1.$

∴ $\underset{\sim}{r} = \begin{pmatrix}0\\10\\0\end{pmatrix} + \lambda\begin{pmatrix}0\\-5\\4\end{pmatrix} + \mu\begin{pmatrix}1\\-1\\0\end{pmatrix}.$

eg[3] $\underset{\sim}{r} = \theta_1\begin{pmatrix}1\\-2\\-1\end{pmatrix} + \phi_1\begin{pmatrix}5\\-4\\-1\end{pmatrix}$

i) $\underset{\sim}{n} = \begin{pmatrix}1\\-2\\-1\end{pmatrix} \times \begin{pmatrix}5\\-4\\-1\end{pmatrix}$

$= \begin{pmatrix}-2\\-4\\6\end{pmatrix}$

∴ $l: \underset{\sim}{r} = \begin{pmatrix}4\\5\\7\end{pmatrix} + t\begin{pmatrix}-2\\-4\\6\end{pmatrix}.$

ii) $\begin{pmatrix}4-2t\\5-4t\\7+6t\end{pmatrix} = \begin{pmatrix}-1+7\theta_2+8\phi_2\\6+3\theta_2-9\phi_2\\3-2\theta_2+11\phi_2\end{pmatrix} \rightarrow$ very tedious!

cartesian:

$\underset{\sim}{n}_1 = \begin{pmatrix}7\\3\\-2\end{pmatrix} \times \begin{pmatrix}8\\-9\\11\end{pmatrix}$

$= \begin{pmatrix}15\\-93\\-87\end{pmatrix}.$

$d = \begin{pmatrix}15\\-93\\-87\end{pmatrix} \cdot \begin{pmatrix}-1\\6\\3\end{pmatrix}$

$= -15 - 558 - 261$

$= -834.$

eqn of this plane is $\underset{\sim}{r} \cdot \begin{pmatrix}15\\-93\\-87\end{pmatrix} = -834.$

$\underset{\sim}{r} = \begin{pmatrix}4\\5\\7\end{pmatrix} + \lambda\begin{pmatrix}-2\\-4\\6\end{pmatrix}.$

$\begin{pmatrix}4-2\lambda\\5-4\lambda\\7+6\lambda\end{pmatrix} \cdot \begin{pmatrix}15\\-93\\-87\end{pmatrix} = -834$

$60 - 30\lambda - 465 + 372\lambda$

$-609 - 522\lambda = -834$

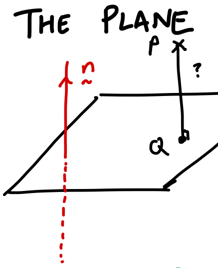
$-180\lambda = 180$

$\lambda = -1.$

∴ point $= \begin{pmatrix}4\\5\\7\end{pmatrix} - 1\begin{pmatrix}-2\\-4\\6\end{pmatrix}$

$= \begin{pmatrix}6\\9\\1\end{pmatrix}$

TO FIND THE ⊥ DIST FROM A PT TO THE PLANE



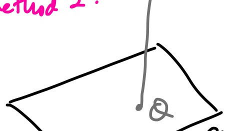
$\underset{\sim}{r} \cdot \underset{\sim}{n} = d$

method 1: i) find PV of Q

ii) ⊥ dist = $|PQ|$

method 2:

$P(\alpha, \beta, \gamma)$



$ax + by + cz - d = 0$

$$\boxed{s = \frac{|a\alpha + b\beta + c\gamma - d|}{\sqrt{a^2 + b^2 + c^2}}}$$

2 PLANES ... FIND

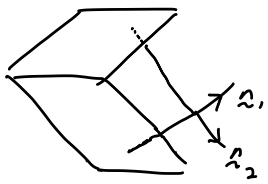
$P_1: \underset{\sim}{r} \cdot \underset{\sim}{n}_1 = d_1$

$P_2: \underset{\sim}{r} \cdot \underset{\sim}{n}_2 = d_2$

1) To find the acute ∡ b/w 2 planes.

do $|\underset{\sim}{n}_1 \cdot \underset{\sim}{n}_2|$ ($= |\underset{\sim}{n}_1||\underset{\sim}{n}_2|\cos\theta$)

ii) To find a vect eqn of the line of intersection of 2 planes.



$l \perp \underset{\sim}{n}_1$ & $l \perp \underset{\sim}{n}_2$

$\Rightarrow$ direction vector of l

$= \underset{\sim}{n}_1 \times \underset{\sim}{n}_2.$

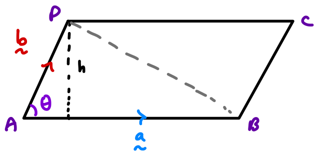
To find a pt on l, we need to solve the eqns of the planes.

∴ the vect eqn of l is

$\underset{\sim}{r} =$ pv of a pt on $l + \lambda(\underset{\sim}{n}_1 \times \underset{\sim}{n}_2).$

APPLICATIONS OF THE VECTOR PRODUCT.

① Area of a parallelogram.



$|\underset{\sim}{a} \times \underset{\sim}{b}| = |\underset{\sim}{a}||\underset{\sim}{b}| \sin\theta.$

$\Rightarrow$ area of //gram ABCD $= 2 \times$ area of ΔABD

$= 2\left(\frac{1}{2} ab \sin\theta\right)$

$= ab\sin\theta$

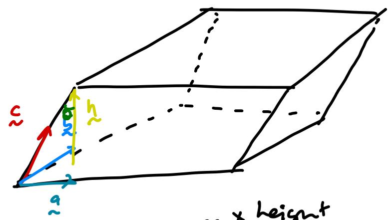
$= |\underset{\sim}{a} \times \underset{\sim}{b}|.$

$$\boxed{\therefore\ A = |\underset{\sim}{a} \times \underset{\sim}{b}|.}$$

② Area of triangle

$$\boxed{A = \frac{1}{2}|\underset{\sim}{a} \times \underset{\sim}{b}|.}$$

③ Volume of a parallelepiped



$V =$ base area $\times$ height

$= |\underset{\sim}{a} \times \underset{\sim}{b}| h$

$= |\underset{\sim}{a} \times \underset{\sim}{b}||\underset{\sim}{c}| \cos\theta$

$$\boxed{V = (\underset{\sim}{a} \times \underset{\sim}{b}) \cdot \underset{\sim}{c}.}$$

④ Tetrahedron

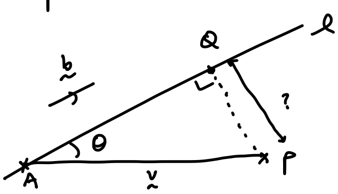
$V = \frac{1}{3}$ base area $\times$ height

$= \frac{1}{3} \times$ area of $\Delta \times$ height

$= \frac{1}{3} \times \frac{1}{2} |\underset{\sim}{a} \times \underset{\sim}{b}| h$

$$\boxed{V = \frac{1}{6}(\underset{\sim}{a} \times \underset{\sim}{b}) \cdot \underset{\sim}{c}.}$$

⑤ Perpendicular dist of a pt from a line.



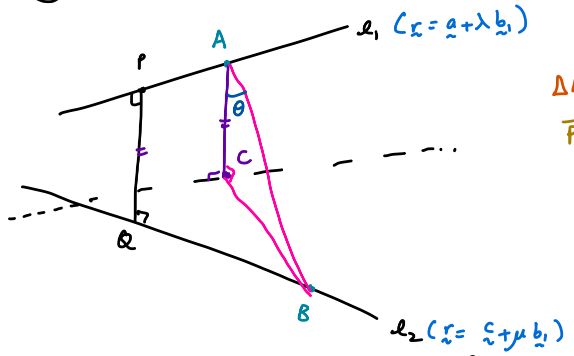
$PQ = v \sin\theta$

$= |\underset{\sim}{v}| \sin\theta$ → $\sin\theta$ (in vectors) $\Rightarrow$ cross product.

$|\underset{\sim}{b} \times \underset{\sim}{v}| = |\underset{\sim}{b}||\underset{\sim}{v}| \sin\theta$

$$\boxed{\therefore\ PQ = \frac{|\underset{\sim}{b} \times \underset{\sim}{v}|}{|\underset{\sim}{b}|}.}$$

⑥ Shortest distance b/w 2 skew lines



Let P & Q be the pts on l_1 & l_2 resp.

& PQ is the perpendicular to both l_1 & l_2.

$\Rightarrow$ shortest dist b/w l_1 & l_2 is PQ.

ΔACB is a right-angled triangle.

$\overrightarrow{PQ} \perp l_1$ & l_2

$\therefore \overrightarrow{PQ} = \underset{\sim}{b}_1 \times \underset{\sim}{b}_2$

$\therefore \overrightarrow{AC} \,//\, \overrightarrow{PQ}$

$\overrightarrow{AC} \,//\, \underset{\sim}{b}_1 \times \underset{\sim}{b}_2$

$\cos\theta = \frac{AC}{AB}$

$\therefore AC = AB\cos\theta$

$= |\overrightarrow{AB}| \cos\theta$

$|\overrightarrow{AC} \cdot \overrightarrow{AB}| = |\overrightarrow{AC}||\overrightarrow{AB}| \cos\theta$

$\frac{\lambda |(\underset{\sim}{b}_1 \times \underset{\sim}{b}_2) \cdot \overrightarrow{AB}|}{\lambda |\underset{\sim}{b}_1 \times \underset{\sim}{b}_2|} = |\overrightarrow{AB}| \cos\theta$

☆ $$\boxed{\therefore\ |\overrightarrow{PQ}| = \frac{|(\underset{\sim}{b}_1 \times \underset{\sim}{b}_2) \cdot \overrightarrow{AB}|}{|\underset{\sim}{b}_1 \times \underset{\sim}{b}_2|}.}$$

To find the pv of P & Q:

→ find λ & find μ.

P lies on l_1. $\therefore \overrightarrow{OP} = \underset{\sim}{a} + \lambda \underset{\sim}{b}_1$

Q lies on l_2. $\therefore \overrightarrow{OQ} = \underset{\sim}{c} + \mu \underset{\sim}{b}_2$

So:

1) Find $\overrightarrow{PQ}$

2) $\left.\begin{array}{l} \overrightarrow{PQ} \cdot \underset{\sim}{b}_1 = 0 \\ \overrightarrow{PQ} \cdot \underset{\sim}{b}_2 = 0. \end{array}\right\}$ 2 eqns involving λ & μ.

3) solve 2 eqns above simultaneously.

Chapter 2: Polynomial Equations

Relations between the roots & coefficients of a polynomial eqn.

Quadratic eqn:

$ax^2+bx+c=0.$

$\Rightarrow x^2+\frac{b}{a}x+\frac{c}{a}=0.$

let α & β be the roots

$\Rightarrow x^2+\frac{b}{a}x+\frac{c}{a} = (x-\alpha)(x-\beta).$

$x^2+\frac{b}{a}x+\frac{c}{a} = x^2-(\alpha+\beta)x+\alpha\beta$

$\Rightarrow \alpha+\beta = -\frac{b}{a}$ & $\alpha\beta = \frac{c}{a}.$

Cubic eqns

$ax^3+bx^2+cx+d=0$

$x^3+\frac{b}{a}x^2+\frac{c}{a}x+\frac{d}{a}=0=(x-\alpha)(x-\beta)(x-\gamma).$

$\rightarrow \alpha+\beta+\gamma = -\frac{b}{a}$

$\alpha\beta+\alpha\gamma+\beta\gamma = \frac{c}{a}$

$\alpha\beta\gamma = -\frac{d}{a}.$

Quartic eqns.

$ax^4+bx^3+cx^2+dx+e=0$

$=(x-\alpha)(x-\beta)(x-\gamma)(x-\delta)$

$\Rightarrow \alpha+\beta+\gamma+\delta = -\frac{b}{a}$

$\alpha\beta+\alpha\gamma+\alpha\delta+\beta\gamma+\beta\delta+\gamma\delta = \frac{c}{a}\ (=\Sigma\alpha\beta)$

$\alpha\beta\gamma+\alpha\beta\delta+\alpha\gamma\delta+\beta\gamma\delta = -\frac{d}{a}\ (=\Sigma\alpha\beta\gamma)$

$\alpha\beta\gamma\delta = \frac{e}{a}.$

Sum of the powers of the roots.

Let $S_n = \alpha^n+\beta^n+\gamma^n.$

$S_1 = \alpha+\beta+\gamma = -\frac{b}{a}.$

$S_2: \alpha^2+\beta^2+\gamma^2 = (\alpha+\beta+\gamma)^2-2(\alpha\beta+\alpha\gamma+\beta\gamma)$

$= \frac{b^2}{a^2}-2\left(\frac{+c}{a}\right)$

$S_2 = S_1^{\,2}-2\left(\frac{c}{a}\right)$

$S_{-1} = \alpha^{-1}+\beta^{-1}+\gamma^{-1}$

$= \frac{1}{\alpha}+\frac{1}{\beta}+\frac{1}{\gamma}$

$= \frac{\alpha\beta+\alpha\gamma+\beta\gamma}{\alpha\beta\gamma}$

$\boxed{S_{-1} = \frac{-c}{d}.}$

$S_n=\alpha^n+\beta^n+\gamma^n+\delta^n.$

$S_1=\alpha+\beta+\gamma+\delta = \frac{c}{a}.$

$S_2 = (\alpha+\beta+\gamma+\delta)^2-2\Sigma\alpha\beta.$

$S_{-1} = \frac{-d}{e}$

S_n?

Suppose $ax^3+bx^2+cx+d=0.$

$x=\alpha \quad a\alpha^3+b\alpha^2+c\alpha+d=0$ — ①

$x=\beta \quad a\beta^3+b\beta^2+c\beta+d=0$ — ②

$x=\gamma \quad a\gamma^3+b\gamma^2+c\gamma+d=0$ — ③

$+$

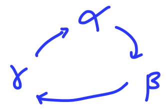
$\Rightarrow aS_3+bS_2+cS_1+3d=0.$

①$\times\alpha$+②$\times\beta$+③$\times\gamma$

$\Rightarrow aS_4+bS_3+cS_2+dS_1=0.$

$\boxed{\therefore aS_n+bS_{n-1}+cS_{n-2}+dS_{n-3}=0.}$

Application to the solutions of symmetrical simultaneous eqns in 3 unknowns



A "cyclic interchange" of α,β & γ means α is replaced by β, β by γ & γ by α.

$\Rightarrow$ hence, if an eqn in α,β & γ is unchanged by a cyclic interchg. we say that eqn is symmetrical.

Transformations of eqns

It is often useful to transform a given eqn into another, whose roots are related in some simple way to those of the original eqn.

eg: $x^2+3x+5=0$ has roots α & β.

Find a quadratic eqn whose roots are $\frac{1}{\alpha}$ & $\frac{1}{\beta}$.

This eqn is hence

$x^2+\left(\frac{1}{\alpha}+\frac{1}{\beta}\right)x+\frac{1}{\alpha\beta}=0$

$\alpha+\beta=-3$

$\alpha\beta=5$

$\Rightarrow x^2+\frac{\alpha+\beta}{\alpha\beta}x+\frac{1}{\alpha\beta}=0.$

$\Rightarrow x^2-\left(\frac{-3}{5}x\right)+\frac{1}{5}=0$

or $5x^2+3x+1=0.$

$x=\alpha,\beta$

$y=\frac{1}{\alpha},\frac{1}{\beta}$

$y=\frac{1}{x}.$ ↳ substn/transformation.

let $x=\frac{1}{y}$

$\Rightarrow \left(\frac{1}{y}\right)^2+3\left(\frac{1}{y}\right)+5=0.$

$\frac{1}{y^2}+\frac{3}{y}+5=0$

or $5y^2+3y+1=0.$

eg: The eqn $x^3+3x^2-2x+1=0$ has roots α,β & γ.

Find the cubic eqn whose roots are α^2, β^2 & γ^2.

For cubics & esp quartics, this method is tedious.

The nature of the roots of a polynomial eq$^{\underline{n}}$ with real coefficients.

Quadratic

$b^2 - 4ac$
- $< 0 \Rightarrow$ two complex roots
- $= 0 \Rightarrow$ real repeated root
- $> 0 \Rightarrow$ real and distinct

Cubic

ax^3+bx^2+cx+d

Since the eq$^{\underline{n}}$ is of degree 3 and has real coefficients, complex roots occur in conjugate pairs.

Hence, a cubic eq$^{\underline{n}}$ with real coefficients has either
- three real roots
- one real & a pair of conjugate complex roots.

* if the latter is true, then $\alpha^2+\beta^2+\gamma^2 < 0$.
- so not all roots $\in \mathbb{R}$

$\therefore$ eq$^{\underline{n}}$ has 1 real root and 2 complex conjugate roots.

Quartic

$ax^4 + bx^3 + cx^2 + dx + e = 0$

Since the quartic eq$^{\underline{n}}$ has real coefficients, the eq$^{\underline{n}}$ has:

(i) - two pairs of conjugate complex roots, or

(ii) - one pair of conjugate complex roots and two real roots, or

(iii) - four real roots

Let the roots be α, β, γ & δ.

Does $\alpha^2+\beta^2+\gamma^2+\delta^2 < 0$?

if yes $\Rightarrow$ either (ii) or (iii).

↳ Does $\alpha\beta\gamma\delta < 0$?

if no $\Rightarrow$ has to be option (ii).

Chapter 3:
Rational Functions

A rational function is the quotient of 2 polynomials.

① $\frac{A}{ax+b}$
② $\frac{Ax+B}{ax+b}$
③ $\frac{Ax+B}{ax^2+bx+c}$
④ $\frac{Ax^2+Bx+C}{ax^2+bx+c}$
⑤ $\frac{Ax^2+Bx+C}{ax+b}$

SKETCHING RATIONAL FUNCTIONS

Prominent features

① Asymptotes

⇒ as $x, y \to \infty$, the curve approaches a line, which is called an asymptote.

Key idea: if $x \to \pm\infty$, $y \to a$
⇒ $y=a$ is an asymptote

if $y \to \pm\infty$, $x \to a$
⇒ $x=a$ is an asymptote

if $x \to \pm\infty$, $y \to ax+b$
⇒ $y=ax+b$ is an oblique asymptote

• To find the eq^ns of asymptotes.

case ①: $y=\frac{A}{ax+b}$

$y \to \pm\infty$, $ax+b=0$
⇒ $x=-\frac{b}{a}$ is an asymptote.

$x \to \pm\infty$, $y \to 0$
⇒ $y=0$ is an asymptote

case ②: $y=\frac{Ax+B}{ax+b}$

$y \to \pm\infty$, $ax+b \to 0$
⇒ $x=-\frac{b}{a}$ is an asymptote.

$x \to \pm\infty$, $y \to \frac{A}{a}$
⇒ $y=\frac{A}{a}$ is an asymptote.

case ③: $y=\frac{Ax+B}{ax^2+bx+c}$

$y \to \pm\infty$ as $ax^2+bx+c \to 0$

$ax^2+bx+c=0$
→ $b^2-4ac=0$ ⇒ 1 asymptote
→ $b^2-4ac>0$ ⇒ 2 asymptotes
→ $b^2-4ac<0$ ⇒ no asymptotes

$x \to \pm\infty$, $y \to 0$
⇒ $y=0$ is an asymptote.

case ④: $y=\frac{Ax^2+Bx+C}{ax^2+bx+c}$

$y \to \pm\infty$, $ax^2+bx+c \to 0$
(same as ③)

$x \to \pm\infty$, $y \to \frac{A}{a}$
∴ $y=\frac{A}{a}$ is an asymptote.

Rules (shortcut)
↳ to find ↔ & I asymptotes.

1) To find the eq^ns of the asymptotes ∥ to the y-axis
⇒ equate the denominator to zero.

2) To find the eq^ns of the asymptotes ∥ to the x-axis
⇒ $y=\frac{\text{coeff of highest power of } x}{\text{coeff of highest power of } x}$

② Turning points

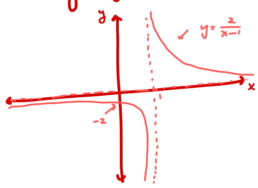
↳ to obtain:
method ①: calculus ($\frac{dy}{dx}$ & $\frac{d^2y}{dx^2}$)

Sketching

① $y=\frac{A}{ax+b}$ ② $\frac{Ax+B}{ax+b}$ } these two eq^ns have no turning pts.

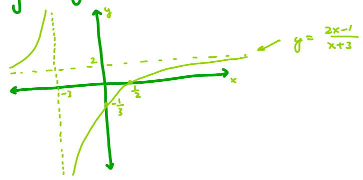
⇒ These curves have 2 asymptotes (one I, one ↔)

eg¹ $y=\frac{2}{x-1}$



eg² $y=\frac{2x-1}{x+3}$

$0=2x-1$
$x=\frac{1}{2}$



Case ⑤: $y=\frac{Ax^2+Bx+C}{ax+b}$

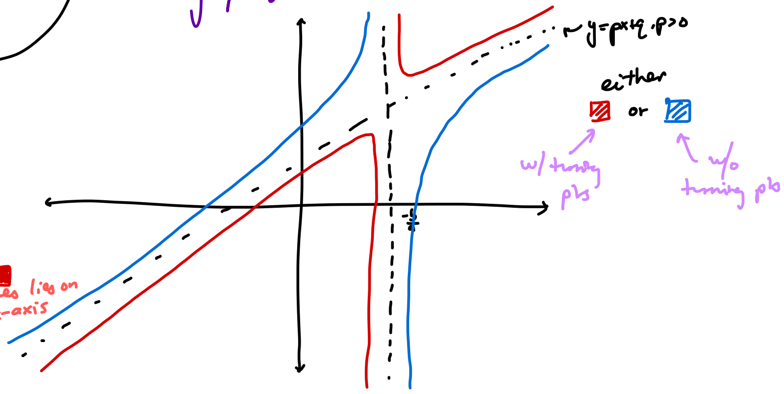
The eq^n of one of the 2 asymptotes is $ax+b=0$
⇒ $x=-\frac{b}{a}$.

This curve has an oblique asymptote.
↳ $Ax^2+Bx+C=(ax+b)(px+q)+R$

⇒ $y=px+q+\frac{R}{ax+b}$

As $x \to 0$ ⇒ $y \to px+q$.

⇒ The eq^n of the oblique asymptote is $y=px+q$.



③ x & y-intercepts

⇒ $x=0 \to y=?$
$y=0 \to x=?$

$y=0 ⇒ \frac{Ax^2+Bx+C}{ax+b}=0$
$Ax^2+Bx+C=0$

case 1) $B^2-4AC<0$ ⇒ curve cannot cross x-axis ⇒ [red]
2) $B^2-4AC=0$ ⇒ curve touches x-axis once → one of turning pts lies on x-axis ⇒ [red]
3) $B^2-4AC>0$ ⇒ curve touches x-axis twice
⇒ [red] or [blue]

pts of intersection inside "region"
pts of intersection outside of "region"

SISTER GRAPHS

① $y=f(x) \Rightarrow y=|f(x)|$

$\Rightarrow$ flip / reflect in x-axis

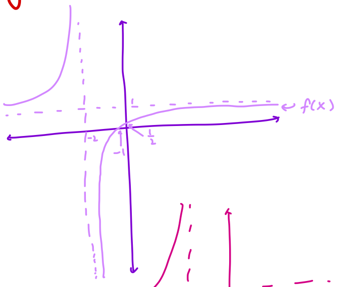
② $y=f(x) \Rightarrow y=f|x|$

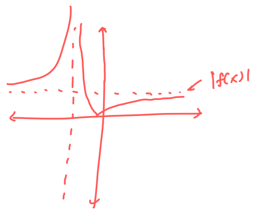
$\Rightarrow$ flip / reflect in y-axis

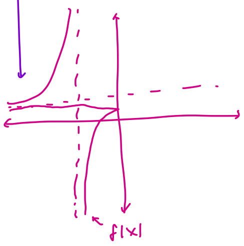
eg¹ $f(x) = \frac{x+1}{x+2}$

$y = 1 - \frac{1}{x+2}$

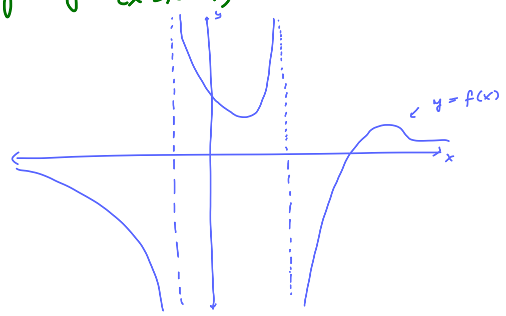
asymptotes: $x=-2$
$y=1$.

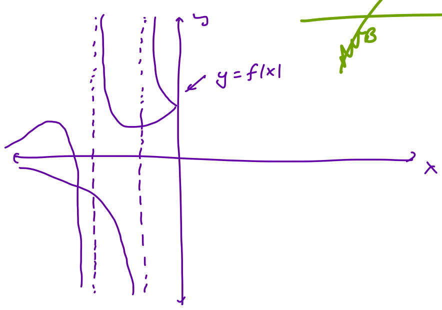


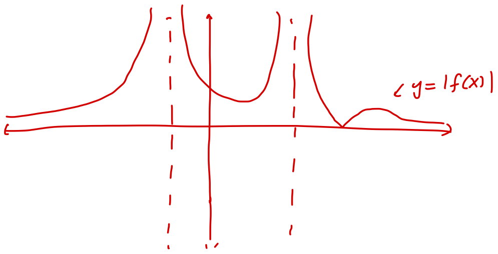




eg² $y = \frac{3x-9}{(x-2)(x+1)}$







③ $y^2 = f(x)$ $\Rightarrow$ curve symmetrical about the x-axis

$\Rightarrow y = \pm\sqrt{f(x)}$

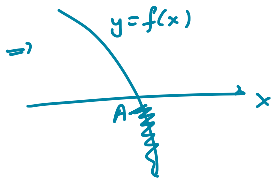
$\Rightarrow y^2 \geq 0 \quad \therefore f(x) \geq 0.$

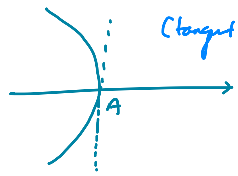
$\Rightarrow$ those pts below the x-axis must be discarded.

$\Rightarrow$ if (a,b) is a ~~turning pt~~ on the graph $y=f(x)$, where $b>0$

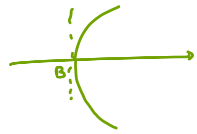
$\hookrightarrow$ $(a, \pm\sqrt{b})$ are the ~~turning pts~~ on the graph $y^2=f(x)$.

$\Rightarrow$





(tangent at A is ↕)



④ $y = \frac{1}{f(x)}$

$\rightarrow$ if (a,b) is a max pt of $y=f(x)$

$\Rightarrow (a, \frac{1}{b})$ is a min pt of $y = \frac{1}{f(a)}$.

$\rightarrow$ if (a,b) is a min pt of $y=f(x)$

$\Rightarrow (a, \frac{1}{b})$ is a max pt of $y = \frac{1}{f(x)}$.

$b \neq 0$

$\rightarrow$ if $x=a$ is an asymptote of $y=f(x)$

$\Rightarrow$ $y=f(x)$ touches the x-axis at $x=a$.

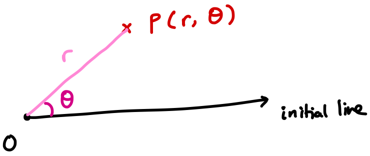
Chapter 4: Polar Coordinates

Polar coordinates are an alternative system to visualise curves.

$\Rightarrow$ some graphs have complex equations in Cartesian space.

Polar Frame of Reference

This system of reference consists of the pole, a fixed point, O, and a line in a fixed direction from O, called the initial line.



*$\theta > 0 \Rightarrow$ anticlockwise
$\theta < 0 \Rightarrow$ clockwise.

*$r > 0 \Rightarrow$ dist measured in dir OP
($r<0$ not in syllabus).

★ The polar coords of a pt are not unique.

eg $(-2, \frac{\pi}{6}) = (2, \frac{7\pi}{6}) = (2, -\frac{5\pi}{6})$.

$\Rightarrow$ however, by convention, we give the polar coordinates where
$r \geq 0$ & $-\pi < \theta \leq \pi$.

Relation b/w Cartesian & Polar.

If a point has coordinates (r, θ) in a polar frame of reference, its coordinates in the Cartesian plane is $(r\cos\theta, r\sin\theta)$.

ie: $x^2 + y^2 = r^2$

$\frac{y}{x} = \tan\theta$.

$\Rightarrow$ initial line is the x-axis

$\Rightarrow$ pole is origin.

Conversion b/w Cartesian & Polar and vv.

A polar eq^n of a curve is of the form $r = f(\theta)$.

eg^1 Find polar eq^n corresponding to the curve $(x^2+y^2)^2 = a^2(x^2-y^2)$.

$x^2+y^2 = r^2$ $\quad x = r\cos\theta$ $\quad y = r\sin\theta$

$\Rightarrow (r^2)^2 = a^2(r^2\cos^2\theta - r^2\sin^2\theta)$

$r^4 = a^2r^2(\cos^2\theta - \sin^2\theta)$

$\Rightarrow r^2 = a^2\cos 2\theta$.

eg^2 Find polar eq^n of $y^2 = (x+1)^2(3-x)$.

$y^2 = (x+1)^2(3-x)$

$(r\sin\theta)^2 = (r\cos\theta+1)^2(3 - r\cos\theta)$.

$r^2\sin^2\theta = (r^2\cos^2\theta + 2r\cos\theta + 1)(3 - r\cos\theta)$

$r^2\sin^2\theta = (3r^2\cos^2\theta + 6r\cos\theta + 3 - r^3\cos^3\theta - 2r^2\cos^2\theta - r\cos\theta)$

$r^2\sin^2\theta = -r^3\cos^3\theta + r^2\cos^2\theta + 5r\cos\theta + 3$

eg^3 Obtain Cartesian eq^n of $r^2(1+15\cos^2\theta) = 16$.

$r^2 + 15r^2\cos^2\theta = 16$

$(x^2+y^2) + 15(x)^2 = 16$

$16x^2 + y^2 = 16$.

eg^4 Obtain Cartesian eq^n of $r = \frac{1}{\sin\theta\cos\alpha + \cos\theta\sin\alpha}$

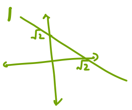
$r\sin\theta\cos\alpha + r\cos\theta\sin\alpha = 1$

$\Rightarrow y\cos\alpha + x\sin\alpha = 1$.

$\alpha = \frac{\pi}{4} \Rightarrow \frac{\sqrt{2}}{2}y + \frac{\sqrt{2}}{2}x = 1$

$y + x = \sqrt{2}$

$y = -x + \sqrt{2}$.



Sketching Polar Curves

The shape of a curve can be determined from its polar eq^n by listing corresponding values of θ & r and plotting these coords.

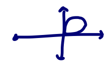
★ $\theta = \alpha$
$\Rightarrow$ a line segment from O.

Important observations

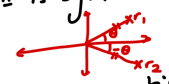
① $r = 0$, $\theta = \alpha \Rightarrow \theta = \alpha$ is a tangent to the curve at O.

② Ensure you know r_{min} & r_{max}.

③ If $r = 0$ when $\theta = \alpha$ & $\theta = \beta \Rightarrow$ curve has a loop b/w α & β. $\Rightarrow$



④ If $r(\theta) = r(-\theta)$, the eq^n is symmetric along the initial line. $\Rightarrow$



(most likely $r = f(\cos\theta)$)

⑤ If $r(\theta) = r(\pi - \theta)$, the eq^n is symmetric along the line $\theta = \frac{\pi}{2}$.

(most likely $r = f(\sin\theta)$)

⑥ If $r(\theta) = [-r]\theta$, the eq^n is symmetric along the pole.

(most likely $r^2 = f(\theta)$)

★ to sketch polar curves for $0 \leq \theta \leq 2\pi$, the amount of tabulated work can be reduced if we know the lines of symmetry.

Results

① If r is a funct of $\cos m\theta$ only, the curve is symmetrical about the lines $m\theta = 0, \pi, 2\pi$ ..

② If r is a funct of $\sin m\theta$ only, the curve is symmetrical about the lines $m\theta = \frac{\pi}{2}, \frac{3\pi}{2}$ ..

eg^5 Show Cartesian eq^n of $r = a\cos 3\theta$ is $(x^2+y^2)^2 = a(x^3 - 3xy^2)$.

$r = a\cos 3\theta$

$r = a(4\cos^3\theta - 3\cos\theta)$

$r = 4a\left(\frac{x}{r}\right)^3 - 3a\left(\frac{x}{r}\right)$

$x = r\cos\theta$
$\therefore \cos\theta = \frac{x}{r}$

$(\times r^3)$ $r^4 = 4ax^3 - 3ar^2x$

$(x^2+y^2)^2 = 4ax^3 - 3a(x^2+y^2)x$
$= 4ax^3 - 3ax^3 - 3ay^2x$

$\Rightarrow (x^2+y^2)^2 = a(x^3 - 3xy^2)$.

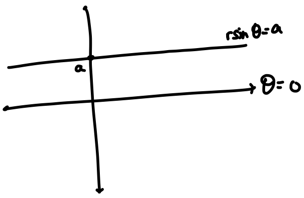
SPECIAL CURVES IN POLAR COORDINATES

① $\theta = \alpha$

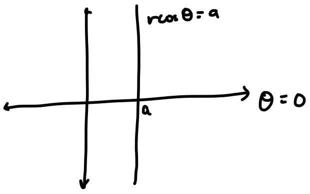
⇒ line ray

⇒ in Cartesian: $y = (\tan\alpha)x$.

② $r\sin\theta = a$

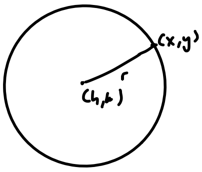


③ $r\cos\theta = a$



lines.

④ Circles.



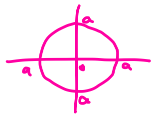
$(x-h)^2 + (y-k)^2 = r^2$

⇒ eq^n of circle w/ centre (h,k) & radius r.

⇒ $\boxed{r = a}$ → polar eq^n

$r^2 = a^2$

$x^2 + y^2 = a^2$. → circle centred on the pole and w/ radius a.



$\boxed{r = 2a\cos\theta}$

$r^2 = 2ar\cos\theta$

$x^2 + y^2 = 2ax$

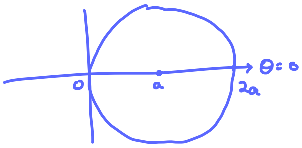
$x^2 + y^2 = 2ax$

$x^2 - 2ax + y^2 = 0$

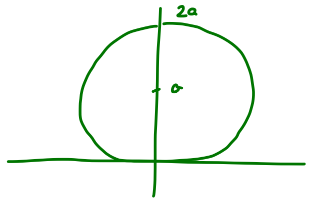
$(x-a)^2 - a^2 + y^2 = 0$

$(x-a)^2 + y^2 = a^2$.

→ circle centred at $(a, 0)$ w/ radius a.



$\boxed{r = 2a\sin\theta}$ → circle centred at $(0, a)$ w/ radius a.



from $\cos\theta \Rightarrow \sin\theta$

⇒ rotate abt pole 90° AC.

eg¹ $r = a + b\sin\theta$ $0 \le \theta \le 2\pi$.

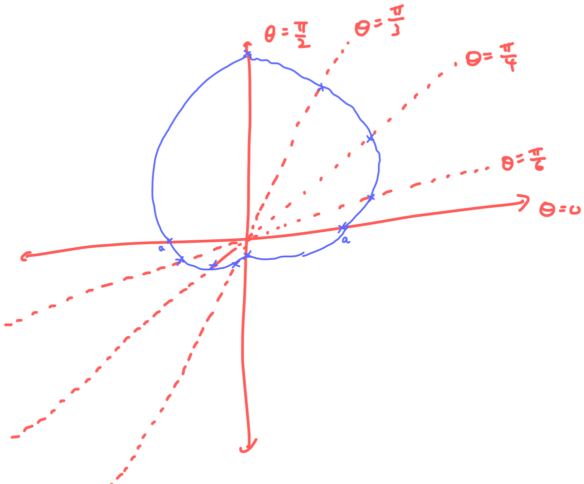
$a > b > 0$

Since r is a function of $\sin\theta$, it is symmetrical along the line $\theta = \frac{\pi}{2}$.

θ	0	30	60	90	..	180	210	240	270
r	a	$a+\frac{b}{2}$	$a+\frac{\sqrt{3}}{2}b$	$a+b$		a	$a-\frac{1}{2}b$	$a-\frac{\sqrt{3}}{2}b$	$a-b$

The greatest value of r is $a+b$.

The least value of r is $a-b$.



eg² $r = a(1+\cos\theta)$ $0 \le \theta \le 2\pi$

↓ whole curve.

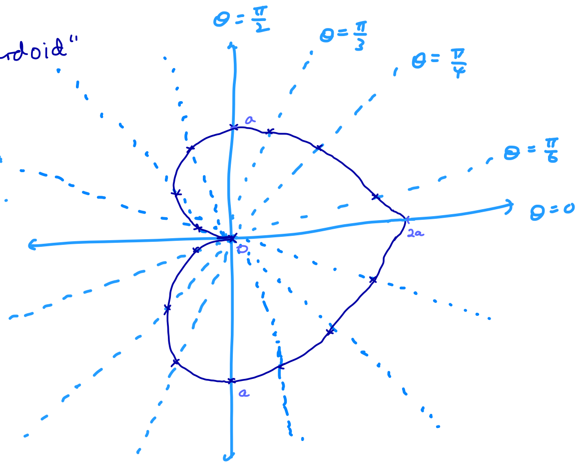
since func of $\cos\theta$ only ⇒ symmetrical along $\theta = 0$.

θ	0	30	60	90	120	150	180
r	$2a$	$1.87a$	$1.5a$	a	$0.5a$	$0.13a$	0

"cardioid"

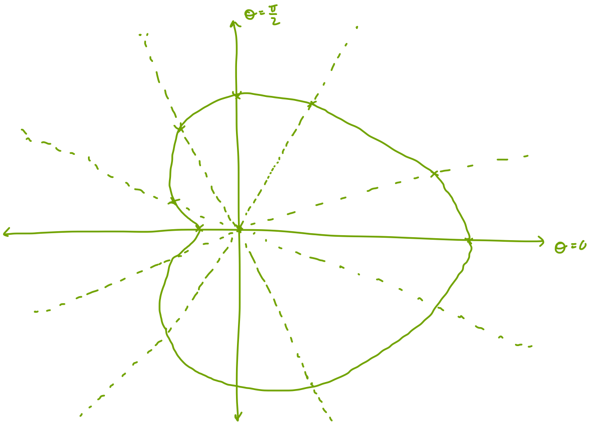
$r_{max} = 2a$

$r_{min} = 0$.

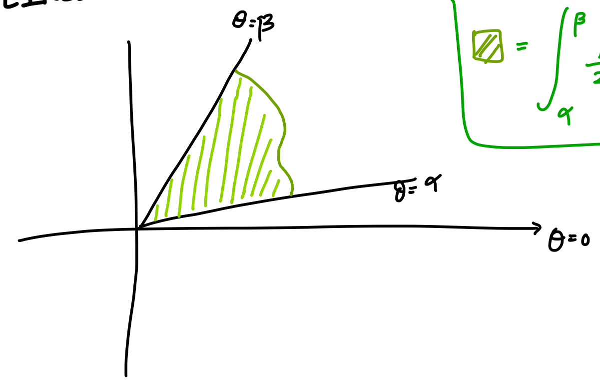


eg3 $r = a(2+\cos\theta)$

θ	0	30	60	90	120	150	180
r	3a	2.83a	2.5a	2a	1.5a	1.17a	a



AREA OF THE SECTOR BOUNDED BY A POLAR CURVE AND THE LINES $\theta = \alpha$, $\theta = \beta$.



$$\boxed{\text{Area} = \int_{\alpha}^{\beta} \frac{1}{2} r^2 \, d\theta}$$

TANGENT // TO THE INITIAL LINE

This satisfies $\frac{dy}{dx} = 0$

ie $\frac{dy/d\theta}{dx/d\theta} = 0$

$\frac{dy}{d\theta} = 0$, $y = r\sin\theta$

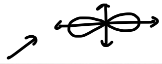
$\Rightarrow \frac{d(r\sin\theta)}{d\theta} = 0$

$\frac{d}{d\theta}(r\sin\theta) = 0$

$r\cos\theta + \frac{dr}{d\theta}\sin\theta = 0$

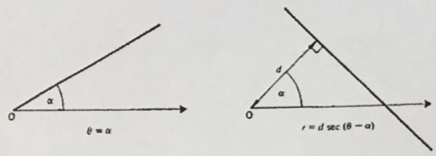
- ☆ For tangent // to initial line, $\boxed{\frac{dy}{d\theta} = 0}$
- ☆ For tangent ⊥ to initial line, $\boxed{\frac{dx}{d\theta} = 0}$.
- ☆ For ↓ or ↑ val. of r, $\boxed{\frac{dr}{d\theta} = 0}$.

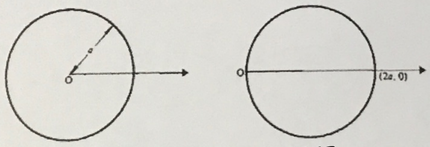
MORE COMMON CURVES



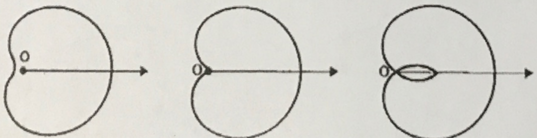
But if only $r>0$

Common Polar Lines and Curves



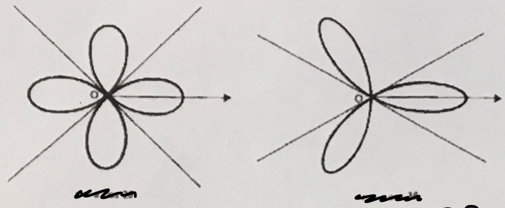


$r = a$ $\quad$ $r = 2a\cos\theta$

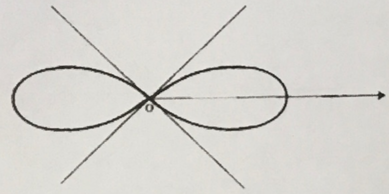


$a > b$ $\quad$ $a = b$ $\quad$ $a < b$

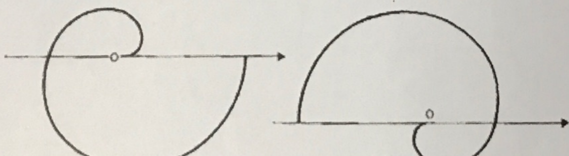
$r = a + b\cos\theta$



$r = a\cos 2\theta$ $\quad$ $r = a\cos 3\theta$



$r^2 = a^2\cos 2\theta$



$k>0$ $\quad$ $k<0$

$r = k\theta$

$$\frac{\cos 2\theta}{\cos^2\theta} = \frac{\cos^2\theta - \sin^2\theta}{\cos^2\theta}$$

$$= 1 - \tan^2\theta$$

Chapter 5:
Summation of Series

Let the series U be $u_1, u_2 \dots u_r$.

$\Rightarrow$ Let $S_n = \sum_{k=1}^{r} u_k$.

Examples

1) $1+2+3 \dots + n = \sum_{k=1}^{n} k$.

2) $\sum_{k=1}^{n} k^2$

3) $\sum_{k=1}^{n} k^3$

2) $\frac{1}{1\times 2} + \frac{1}{2\times 3} + \frac{1}{3\times 4} \dots + \frac{1}{n(n+1)} = \sum_{r=1}^{n} \frac{1}{r(r+1)}$

If we have $u_1, u_2 \dots \infty$

$\Rightarrow S_\infty = \sum_{n=1}^{\infty} u_n$.

If S_∞ is a constant $\Rightarrow$ series converges.

If S_∞ is infinite $\Rightarrow$ series diverges.

FINDING SUM OF SERIES.

Two methods:

1. Method of differences.

If $u_r = f(r+1) - f(r)$,

then $S_n = \sum_{r=1}^{n} [f(r+1) - f(r)]$

$= f(2) - f(1) + f(3) - f(2) \dots + f(n+1) - f(n)$

$$S_n = f(n+1) - f(1).$$

2. Quote results from MF19.

i) $\sum_{r=1}^{n} r = \frac{n}{2}(n+1)$

ii) $\sum_{r=1}^{n} r^2 = \frac{n(n+1)(2n+1)}{6}$

iii) $\sum_{r=1}^{n} r^3 = \frac{n^2(n+1)^2}{4}$.

Proof

1) $\sum_{r=1}^{n} r = 1+2+3 \dots + n$.

$= \frac{n}{2}(a+l)$

$= \frac{n}{2}(n+1)$.

Chapter 6: Mathematical Induction

A method of proving.

Steps:

1) Prove $n=1$

2) Prove $n=k+1$ assuming $n=k$ holds.

APPLICATIONS

SUMMATION OF SERIES

$\sum_{k=1}^{n+1} u_k = \sum_{k=1}^{n} u_k + u_{n+1}$.

THEOREMS ON DIVISIBILITY ($d \in \mathbb{Z}^+$)

Idea: let $f(n)$ be the expression, and write down $f(n+1)$.

① Show $f(1) \equiv 0 \pmod d$.

② Assume that test holds true for $f(k)$.

⇒ from here, prove that $f(k+1)$ is also divisible by the integer.

Method

$f(k) = \cdots\cdots$ ①

$f(k+1) = \cdots\cdots$ ②

} eliminate one of the terms. (eg constant term).

eg' Prove by induction $3^{4n-2} + 17^n + 22 \equiv 0 \pmod{16}$ $\forall n \in \mathbb{N}$

Let $f(n) = 3^{4n-2} + 17^n + 22$

$\Rightarrow f(1) = 3^2 + 17 + 22$
$= 48 = 3(16)$.

∴ Claim is true for $n=1$.

Assume it is true for $n=k$.

objective: prove it is true for $n=k+1$.

Method $f(k) = 3^{4k-2} + 17^k + 22$ — ①

$f(k+1) = 3^{4k+2} + 17^{k+1} + 22$ — ②

② − ① ⇒ $f(k+1) - f(k) = 3^{4k+2} + 17^{k+1} - 3^{4k-2} - 17^k$
$= 3^{4k-2}(3^4 - 1) + 17^k(17-1)$
$= 80(3^{4k-2}) + 16(17^k)$
$= 16\left(5(3^{4k-2}) + 17^k\right)$.

∴ $f(k+1) = 16\left(5(3^{4k-2}) + 17^k\right) + f(k)$.

∵ We assumed $f(k) \equiv 0 \pmod{16}$, it implies that $f(k+1) \equiv 0 \pmod{16}$.
Thus the claim is true for $n=k+1$.

By induction, the claim is true for $n = 1, 2 \cdots$ ie true for $n \in \mathbb{N}$. QED.

<20 <30 <40 ·

$0 < x < 20$ $20 \le x < 30$

NTH DERIVATIVE OF A FUNCTION

i.e. $\frac{d^n}{dx^n} f(x) = ?$

eg' if $y = e^{-x}\sin(\sqrt{3}\,x)$, prove by ind. that $\frac{d^n y}{dx^n} = (-2)^n e^{-x} \sin(x\sqrt{3} - \frac{1}{3}n\pi)$.

EVALUATION OF TERMS DEFINED BY A RECURRENCE RELATION

$u_{n+1} = f(u_n)$, given u_1.

Find u_n.

eg' $u_1 = 1$, $u_{n+1} = 3u_n + 2$.

Pbi $u_n = 2(3^{n-1}) - 1$.

Claim: $u_n = 2(3^{n-1}) - 1$ for the formula $u_{n+1} = 3u_n + 2$

$n=1$ → LHS = 1 RHS = $2(3^0) - 1$
$= 1$ (= LHS)

⇒ claim holds for $n=1$.

Assume it is true for $n=k$:

⇒ $u_k = 2(3^{k-1}) - 1$.

⇒ $u_{k+1} = 3u_k + 2$
$= 3\left[2(3^{k-1}) - 1\right] + 2$
$= 2(3^k) - 3 + 2$
$= 2(3^k) - 1$.

∴ Claim is also true for $n = k+1$.

Hence, by induction, claim is true for $n \in \mathbb{N}$. QED.

INEQUALITIES

To prove $a < b$, we might need to prove $a - b < 0$.

WRITE DOWN A CONJECTURE BASED ON A LTD TRIAL & FOLLOWED BY INDUCTIVE PROOF

Chapter 7: Matrices

A matrix is a rectangular array of numbers.

Size

$$\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \\ a_{m1} & a_{m2} & & a_{mn} \end{pmatrix}$$

m rows, n columns

- order $= m \times n$.

- a_{ij} refers to the element in the matrix at row i & column j.

eg. $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ 2×2 $\quad A = \begin{pmatrix} 1 \\ -2 \end{pmatrix}$ 2×1 (column vector)

$B = \begin{pmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{pmatrix}$ 3×2 $\quad B = (-3, -2, -1)$ 1×3 (row vector)

$C = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix}$ 3×3

SOME SPECIAL MATRICES

① Diagonal matrices

All entries are 0 except on the main diagonal.

$\begin{pmatrix} \lambda & 0 \\ 0 & \mu \end{pmatrix}$ $\quad \begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{pmatrix}$

② Transpose of a matrix

If a matrix A has a transpose $B = A^T$, B is defined as $b_{ij} = a_{ji}$.

eg $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \Rightarrow B = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$.

③ Square Matrices

Order is $n\times n$.

④ Null (Zero) Matrix

The zero matrix, denoted by O:

- has the property such that $A + O = A \quad \forall$ possible A,
- is defined by any matrix w/ all entries equal to zero.

⑤ Identity matrix

The identity matrix, denoted by I:

- has the property such that $AI = IA = A \quad \forall$ possible A
- and is defined as a diagonal matrix w/ the entries on the main diagonal all equal to 0.

PROPERTIES OF MATRICES

① Equality.

If $A = B$, then $a_{ij} = b_{ij} \quad \forall i, j$

② Addition / Subtraction of 2 matrices.

If $C = A + B$, then $c_{ij} = a_{ij} + b_{ij} \quad \forall i, j$

" $C = A - B$, then $c_{ij} = a_{ij} - b_{ij}$ "

☆ only defined iff order of A = order of B.

③ Scalar multiplication.

If $B = \lambda A$, where $\lambda \in \mathbb{R}$, then $b_{ij} = \lambda a_{ij} \quad \forall i, j$.

④ Associative.

$\alpha(\beta A) = (\alpha\beta)A$.

⑤ Commutative.

$\alpha(A+B) = \alpha A + \alpha B$

$(\alpha+\beta)A = \alpha A + \beta A$.

⑥ Matrix Multiplication.

If $C = AB$, where A is a $m\times n$ matrix & B is a $n\times k$ matrix,

then $c_{ij} = \sum_{r=1}^{n} a_{ir} b_{rj}$, and C is a $m\times k$ matrix.

$(= a_{i1}b_{1j} + a_{i2}b_{2j} \cdots + a_{in}b_{nj}.)$

eg. $\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}\begin{pmatrix} 1 & 3 \\ 2 & 4 \end{pmatrix} = \begin{pmatrix} 1\times1+2\times2 & 1\times3+2\times4 \\ 3\times1+4\times2 & 3\times3+4\times4 \end{pmatrix} = \begin{pmatrix} 5 & 11 \\ 11 & 25 \end{pmatrix}$

Note 1 $\quad AB \neq BA$, in general.

exceptions: $AO = OA = O$

$AI = IA = I$

$A^{-1}A = A^{-1}A = I$.

Terminology

$AB \Rightarrow A$ "pre multiplied by" B

$BA \Rightarrow A$ "post multiplied by" B

⑦ Matrix Powers

$A^n = \underbrace{AAA\cdots A}_{n}$.

$\rightarrow A^n$ can only exist if A is a square matrix!

Result 1. $\quad I^n = I$.

Result 2. $\quad$ If $D = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}$, then $D^n = \begin{pmatrix} \lambda_2^n & 0 \\ 0 & \lambda_1^n \end{pmatrix}$.

⑧ Inverse of Matrices.

The inverse of a matrix A, A^{-1}, has the property that $AA^{-1} = A^{-1}A = I$.

☆ Only square matrices

- 2×2: Let $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$

i.e: $A = \frac{1}{ad-bc}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$.

Determinant of A, $\det(A) = ad - bc$.

If $\det(A) = 0$, matrix is "singular".

$\det(A) \neq 0$, matrix is "non-singular".

$$\boxed{\begin{array}{l} * \text{ for any 2 matrices,} \\ \det AB = \det A \times \det B \end{array}}$$

The inverse:
1) swap a & d
2) chg signs of b & c.
3) divided by $\det A$.

- 3×3: Let $A = \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix}$.

$$\det A = a\begin{vmatrix} e & f \\ h & i \end{vmatrix} - b\begin{vmatrix} d & f \\ g & i \end{vmatrix} + c\begin{vmatrix} d & e \\ g & h \end{vmatrix}$$
$$= a(ei-hf) - b(di-gf) + c(dh-eg).$$
$$= -d(bi-ch) + e(ai-gc) - f(ah-bg)$$
$$= g(bf-ec) - h(af-cd) + i(ae-bd)$$

eg find det A, $A = \begin{pmatrix} 1 & -1 & 2 \\ 1 & 1 & 3 \\ 4 & 0 & 5 \end{pmatrix}$

$$\det A = 1(1\times 5 - 3\times 0) - (-1)(1\times 5 - 4\times 3) + 2(1\times 0 - 1\times 4)$$
$$= 1(5) + 1(-7) + 2(-4)$$
$$= -10.$$

Steps

① Take each element of the matrix in turn & replace by its minor.

For an element a_{ij}, if we cross out the row & column in which it lies, we call the determinant of what is left as the "minor".

② Find the cofactor of the matrix. (Cof A)

$C_{ij} = (-1)^{i+j} M_{ij}$, M_{ij} = minor of a_{ij}.

③ Find the adjoint of A. (Adj A)

⇒ the transpose of the cofactor of A.

④ Find the determinant of A.

⑤ $A^{-1} = \frac{1}{\det A}\,\mathrm{adj}(A)$.

eg' $A = \begin{pmatrix} 1 & -1 & 0 \\ 2 & 1 & 3 \\ 3 & 0 & 4 \end{pmatrix}$

$$(\mathrm{Cof}\ A) = \begin{pmatrix} +(1\times 4-0) & -(2\times 4 - 3\times 3) & +(0-3) \\ -(-1\times 4-0) & +(1\times 4-0) & -(1\times 0+3) \\ +(-3-0) & -(1\times 3-0) & +(1\times 1+2) \end{pmatrix}$$
$$= \begin{pmatrix} 4 & +1 & -3 \\ +4 & 4 & -3 \\ -3 & -3 & 3 \end{pmatrix}.$$

$$\mathrm{Adj}\ A = \begin{pmatrix} 4 & 1 & -3 \\ 4 & 4 & -3 \\ -3 & -3 & 3 \end{pmatrix}^T$$
$$= \begin{pmatrix} 4 & 4 & -3 \\ 1 & 4 & -3 \\ -3 & -3 & 3 \end{pmatrix}.$$

$$\det A = 1(4) - (-1)(8-9) + 0$$
$$= 3.$$

$$\therefore A^{-1} = \frac{1}{3}\begin{pmatrix} 4 & 4 & -3 \\ 1 & 4 & -3 \\ -3 & -3 & 3 \end{pmatrix}.$$

TRANSFORMATIONS (only 2x2)

A "transformation" of the plane is an one-to-one mapping from the set of points in the plane onto itself.

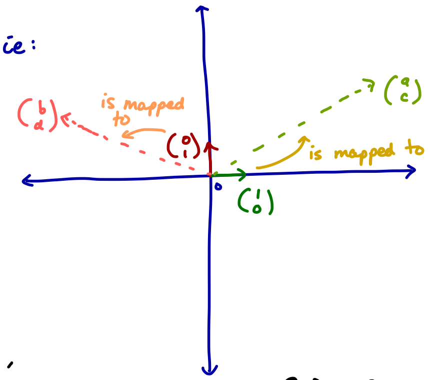
Visualising transformations

Let $\underset{\sim}{u}$ represent the column vector $\begin{pmatrix} x \\ y \end{pmatrix} = x\begin{pmatrix} 1 \\ 0 \end{pmatrix} + y\begin{pmatrix} 0 \\ 1 \end{pmatrix}$,

and A represent the matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$.

Let $\underset{\sim}{v}$ represent the product $A\underset{\sim}{u} = \begin{pmatrix} x' \\ y' \end{pmatrix}$.

ie:



$$\underset{\sim}{v} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$$
$$= \begin{pmatrix} a & b \\ c & d \end{pmatrix} x \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \begin{pmatrix} a & b \\ c & d \end{pmatrix} y \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$
$$= x\begin{pmatrix} a \\ c \end{pmatrix} + y\begin{pmatrix} b \\ d \end{pmatrix}.$$

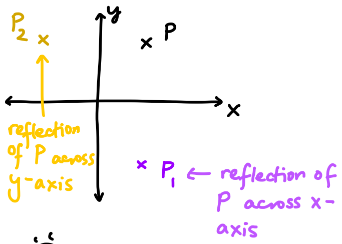
$\Rightarrow$ So, $A\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} a \\ c \end{pmatrix}$ & $A\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} b \\ d \end{pmatrix}$.

Hence, $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ is mapped to $\begin{pmatrix} a \\ c \end{pmatrix}$ under A,

and $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$ is mapped to $\begin{pmatrix} b \\ d \end{pmatrix}$ under A.

REFLECTIONS

x or y axises



For P_1:

$x_1 = x$, $y_1 = -y$.

$\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ $\begin{pmatrix} 0 \\ -1 \end{pmatrix}$

$$\boxed{\Rightarrow \begin{pmatrix} x_1 \\ y_1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}.}$$

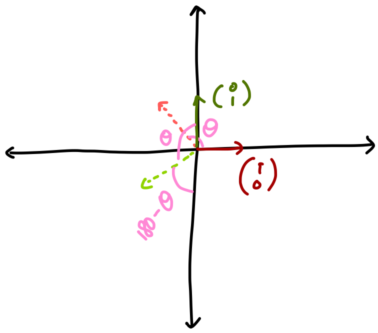
For P_2:

$x_1 = -x$, $y_1 = y$

$\begin{pmatrix} -1 \\ 0 \end{pmatrix}$ $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$

$$\boxed{\Rightarrow \begin{pmatrix} x_1 \\ y_1 \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}.}$$

ROTATIONS (AROUND THE ORIGIN)



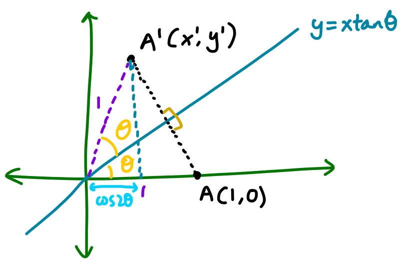
$\begin{pmatrix} 1 \\ 0 \end{pmatrix} \longrightarrow \begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix}$

$\begin{pmatrix} 0 \\ 1 \end{pmatrix} \longrightarrow \begin{pmatrix} -\sin\theta \\ \cos\theta \end{pmatrix}$

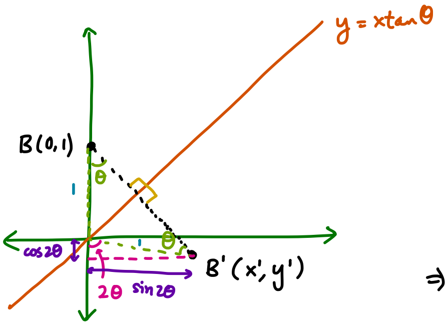
$\Rightarrow$ matrix is $\boxed{\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}}$.

(rotation anticlockwise by angle of θ)

$y = mx$ or $y = x\tan\theta$ (ie $m = \tan\theta$)



The column vector $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ is mapped to $\begin{pmatrix} \cos 2\theta \\ \sin 2\theta \end{pmatrix}$.

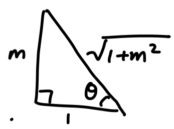


The column vector $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$ is mapped to $\begin{pmatrix} \sin 2\theta \\ -\cos 2\theta \end{pmatrix}$.

Hence, reflection in the line $y = x\tan\theta$ can be modelled using the matrix

$$\boxed{X = \begin{pmatrix} \cos 2\theta & \sin 2\theta \\ \sin 2\theta & -\cos 2\theta \end{pmatrix}}.$$

Recall that $m = \tan\theta$.

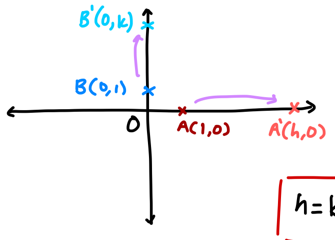


$\Rightarrow \cos\theta = \frac{1}{\sqrt{1+m^2}}$, $\sin\theta = \frac{m}{\sqrt{1+m^2}}$.

$$\therefore \cos 2\theta = \cos^2\theta - \sin^2\theta$$
$$= \frac{1}{1+m^2} - \frac{m^2}{1+m^2} = \frac{1-m^2}{1+m^2}.$$
$$\sin 2\theta = 2\cos\theta\sin\theta$$
$$= 2\frac{1}{\sqrt{1+m^2}}\frac{m}{\sqrt{1+m^2}} = \frac{2m}{1+m^2}.$$

$$\boxed{\Rightarrow X = \begin{pmatrix} \frac{1-m^2}{1+m^2} & \frac{2m}{1+m^2} \\ \frac{2m}{1+m^2} & \frac{m^2-1}{1+m^2} \end{pmatrix}.}$$

STRETCH



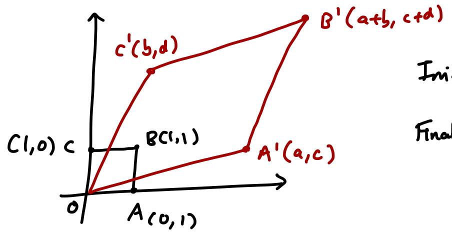
$\begin{pmatrix}1\\0\end{pmatrix} \to \begin{pmatrix}h\\0\end{pmatrix}$

$\begin{pmatrix}0\\1\end{pmatrix} \to \begin{pmatrix}0\\k\end{pmatrix}$

$\Rightarrow$ $\boxed{\text{matrix is } \begin{pmatrix}h & 0\\0 & k\end{pmatrix}.}$ (centre O) ($\uparrow$ by h, $\to$ by k)

$\boxed{h = k \Rightarrow \text{enlargement}}$ (centre O) (scale factor = h).

AREA OF A FIGURE UNDER A TRANSFORMATION.



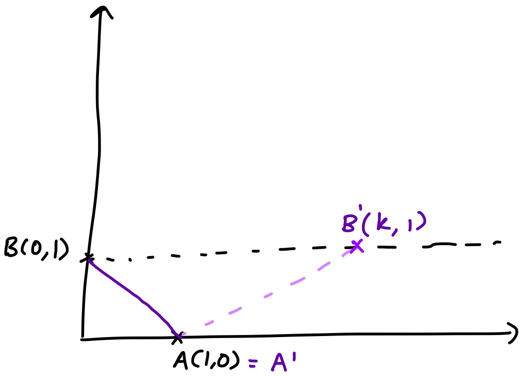
Initial area = 1

Final area = ad − bc

💡: M maps an area of 1 sq. unit to an area of (det M) units.

SHEAR TRANSFORMATIONS

① Invariant line = x-axis



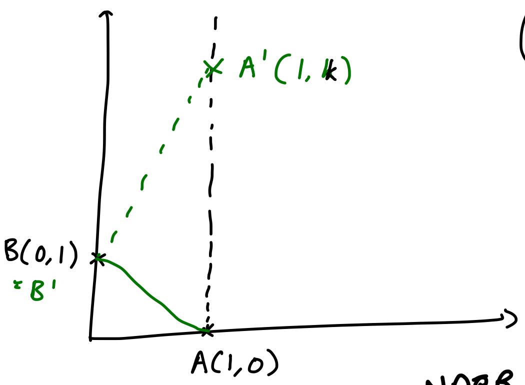
$\begin{pmatrix}1\\0\end{pmatrix} \to \begin{pmatrix}1\\0\end{pmatrix}$

$\begin{pmatrix}0\\1\end{pmatrix} \to \begin{pmatrix}k\\1\end{pmatrix}$

$\Rightarrow \boxed{M = \begin{pmatrix}1 & k\\0 & 1\end{pmatrix}.}$

* k =

2. Invariant line = y-axis



$\begin{pmatrix}1\\0\end{pmatrix} \to \begin{pmatrix}1\\k\end{pmatrix}$ $\begin{pmatrix}0\\1\end{pmatrix} \to \begin{pmatrix}0\\1\end{pmatrix}$

$\Rightarrow \boxed{M = \begin{pmatrix}1 & 0\\k & 1\end{pmatrix}}$

INVARIANT LINES UNDER A TRANSFORMATION.

💡: If a line $y = mx$ does not change after a transformation T, we say that that line is an invariant to T.

Determining the invariant line to a transformation.

* We consider the transformation T modelled by the matrix $M = \boxed{\begin{pmatrix}2 & 1\\2 & 3\end{pmatrix}}$.

Method #1. Let $y = mx$ be the invariant line.

$\Rightarrow M\begin{pmatrix}x\\y\end{pmatrix} = k\begin{pmatrix}x\\y\end{pmatrix}$.

$\begin{pmatrix}2 & 1\\2 & 3\end{pmatrix}\begin{pmatrix}x\\y\end{pmatrix} = k\begin{pmatrix}x\\y\end{pmatrix}$

$= \begin{pmatrix}kx\\ky\end{pmatrix}$.

$\Rightarrow \begin{pmatrix}2x+y\\2x+3y\end{pmatrix} = \begin{pmatrix}kx\\ky\end{pmatrix}$.

The system of eq^{n}s $\begin{cases}(2-k)x + y = 0\\ 2x + (3-k)y = 0\end{cases}$

Can be modelled by the matrix eq^{n} $\begin{pmatrix}2-k & 1\\2 & 3-k\end{pmatrix}\begin{pmatrix}x\\y\end{pmatrix} = 0$

The system of eq^{n}s has non-zero solutions if $\begin{pmatrix}2-k & 1\\2 & 3-k\end{pmatrix}$ is singular.

ie det M

$= (2-k)(3-k) - 2 = 0.$

$6 - 5k + k^2 - 2 = 0$

$k^2 - 5k + 4 = 0$

$(k-4)(k-1) = 0$

$\Rightarrow k = 4$ or $k = 1$.

$\therefore k = 4$

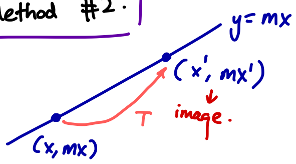
① or ② $\Rightarrow y = 2x$ $(m = 2)$

$k = 1$

① or ② $\Rightarrow y = -x$ $(m = -1)$

$\therefore y = 2x$ & $y = -x$ are the eq^{n}s of the invariant line.

Method #2.



$M\begin{pmatrix}x\\mx\end{pmatrix} = \begin{pmatrix}x'\\mx'\end{pmatrix}$

$\begin{pmatrix}2 & 1\\2 & 3\end{pmatrix}\begin{pmatrix}x\\mx\end{pmatrix} = \begin{pmatrix}x'\\mx'\end{pmatrix}$

$\begin{pmatrix}2x+mx\\2x+3mx\end{pmatrix} = \begin{pmatrix}x'\\mx'\end{pmatrix}$

$x' = x(2+m)$ — ①

$mx' = x(2+3m)$ — ②

$\frac{②}{①} \Rightarrow \frac{2+3m}{2+m} = m$

$2 + 3m = m^2 + 2m$

$0 = m^2 - m - 2$

$= (m-2)(m+1)$

$\Rightarrow m = 2,\ m = -1$

$\Rightarrow y = 2x$ and $y = -x$ are the invariant lines.