Further Pure Mathematics 2

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Chapter 1: Matrices

SYSTEMS OF LINEAR EQNS

We can use matrices to devise a general algorithm to solve a system of linear eq^n s in n unknowns.

eg. consider the system of eq^n s

$$\begin{cases} 1x + 1y + 2z = 9 \\ 2x + 4y - 3z = 1 \\ 3x + 6y - 5z = 0. \end{cases}$$

We can transform this into an "augmented matrix";

ie
$$\begin{array}{c} (r_1) \\ (r_2) \\ (r_3) \end{array} \left[\begin{array}{ccc|c} 1 & 1 & 2 & 9 \\ 2 & 4 & -3 & 1 \\ 3 & 6 & -5 & 0 \end{array}\right].$$

We can then use <u>elimination</u> to obtain values for x, y and z.

$$r_2 = r_2 - 2r_1 \Rightarrow \left[\begin{array}{ccc|c} 1 & 1 & 2 & 9 \\ 0 & 2 & -7 & -17 \\ 3 & 6 & -5 & 0 \end{array}\right]$$

$$r_3 = r_3 - 3r_1 \Rightarrow \left[\begin{array}{ccc|c} 1 & 1 & 2 & 9 \\ 0 & 2 & -7 & -17 \\ 0 & 3 & -11 & -27 \end{array}\right].$$

Then,

$$r_3 = r_3 - \tfrac{3}{2}r_2 \Rightarrow \left[\begin{array}{ccc|c} 1 & 1 & 2 & 9 \\ 0 & 2 & -7 & -17 \\ 0 & 0 & -\frac{1}{2} & -\frac{3}{2} \end{array}\right]$$

hence, $\underline{z = 3}$.

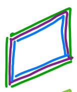
$\therefore 2y - 7(z) = -17$
$2y - 7(3) = -17 \quad \therefore \underline{y = 2}$

$\therefore x + y + 2z = 9$
$x + (2) + 2(3) = 9 \quad \therefore \underline{x = 1}.$

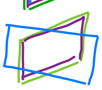
SYSTEM OF LINEAR EQ^N S HAVING ∞ SOLUTIONS.

The system of eq^n s has infinitely many solutions if the augmented matrix can be reduced to the indeterminate $0x + 0y + 0z = 0$.

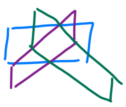
Possibilities:



① same plane listed 3 times;



② one plane listed twice and one other non-// plane;



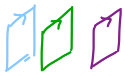
③ three planes meeting in a common line.

SYSTEM OF LINEAR EQ^N S W/ NO SOLUTIONS.

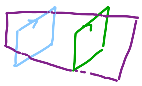
In 2 dimensions (ie. 2-variable LE), ∃ no solution if the equations represent <u>2 parallel lines</u>; eg $y = 2x + 5$ and $y = 2x - 3$.

In 3 <u>dimensions</u>, there are various possibilities for the relative positions of the <u>planes</u> if ∃ no solution to the system of LE:

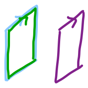
① 2 or more planes are parallel to each other



① 3 // planes

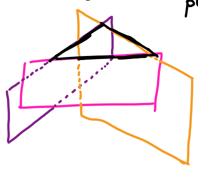


② 2 // planes and one other plane.



③ 1 plane listed <u>twice</u> and one other // plane.

② The planes form a "triangle". (intersecting pairs of planes form // lines.)



Row Echelon Form

A matrix is said to be in "row echelon form" if:

1. All zero rows are at the bottom.
2. The 1st non-zero entry from the left in each non-zero row is a constant.
   ↳ this is called the "leading coefficient" for that row.
3. Each leading coeff is to the right of all the leading coefficients in the rows above it.
   ↳ ie # of zeros increases as we go "down" the matrix.

e.g.

$$\begin{bmatrix} 1 & 3 & 2 & 0 & 3 \\ 0 & 0 & 1 & 2 & 4 \\ 0 & 0 & 0 & 4 & 3 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 3 & 0 & 5 & 3 \\ 0 & 0 & 1 & 0 & 4 \\ 0 & 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 0 & 1 & 2 & 6 & 3 \\ 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Solving Linear Eqns using "Gaussian Elimination" (Reduce to Echelon Form).

$$\begin{cases} ax+by+cz=d \\ ex+fy+gz=h \\ ix+jy+kx=l. \end{cases} \xrightarrow{\text{interpretation}} \left(\begin{array}{ccc|c} a & b & c & d \\ e & f & g & h \\ i & j & k & l \end{array}\right) \xrightarrow{\text{reduction}} \left(\begin{array}{ccc|c} a & b & c & d \\ 0 & p & q & t \\ 0 & 0 & r & u \end{array}\right) \Rightarrow \begin{array}{l} rz=u \\ py+qz=t \\ ax+by+cz=d. \end{array}$$

easily solvable.

Possibilities:

1. $r \neq 0$ ⟶ unique solution.
2. $r=0,\ u=0$ ⟶ many solutions.
3. $r=0,\ u\neq 0$ ⟶ no solutions.

Chapter 2: Eigenvalues and Eigenvectors

If A is an $n \times n$ matrix, then a non-zero vector $\underset{\sim}{x}$ in $\mathbb{R}^n$, ie $\underset{\sim}{x} = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}$ is called an eigenvector of A if $\exists$ a scalar λ such that

$$\boxed{A\underset{\sim}{x} = \lambda \underset{\sim}{x}.}$$

$\Rightarrow$ the scalar λ is called an eigenvalue of A, and $\underset{\sim}{x}$ is said to be an eigenvector of A corresponding to λ.

Finding the Eigenvalues of A

Any eigenvector of A, corresponding to the eigenvalue λ, satisfies

$A\underset{\sim}{x} = \lambda \underset{\sim}{x}$,

ie $A\underset{\sim}{x} = \lambda I \underset{\sim}{x}$,

ie $(A - \lambda I)\underset{\sim}{x} = \underset{\sim}{0}$.

The Characteristic Polynomial of A

For λ to be an eigenvalue, there must exist a non-zero solution, i.e.

$$\boxed{\det(A - \lambda I) = 0.}$$

Why? if $\det(A - \lambda I) \neq 0$, then $\underset{\sim}{x} = \underset{\sim}{0}\,(A - \lambda I)^{-1}$

$= \underset{\sim}{0}$ (since $(A - \lambda I)^{-1}$ exists).

☆ this is known as the characteristic equation of A, which can be shown to have the form

$$\boxed{\lambda^n + c_1 \lambda^{n-1} + c_2 \lambda^{n-2} \cdots + c_n = 0.}$$

↳ where λ = an eigenvalue of A.

Theorems.

$\Rightarrow$ for the following theorems, let λ & μ be eigenvalues of $\underset{\approx}{A}$ and $\underset{\approx}{B}$ respectively, and $\underset{\approx}{x}$ be the corresponding eigenvector.

Eigenvalues of A^n.

If $A\underset{\sim}{x} = \lambda \underset{\sim}{x}$, then $A^n \underset{\sim}{x} = \lambda^n \underset{\sim}{x}$, $\forall n \in \mathbb{Z}^+$.

(λ^n is an eigenvalue of A^n.)

Proof by induction. Trivial base case.
If claim is true for $n = k$, ie

$A^k \underset{\sim}{x} = \lambda^k \underset{\sim}{x}$.

then $A^{k+1}\underset{\sim}{x} = A A^k \underset{\sim}{x}$
$= A \lambda^k \underset{\sim}{x}$
$= \lambda^k (A\underset{\sim}{x})$
$= \lambda^k \lambda \underset{\sim}{x}$
$= \lambda^{k+1} \underset{\sim}{x}$. ∎

Eigenvalues of AB

If $A\underset{\sim}{x} = \lambda \underset{\sim}{x}$ & $B\underset{\sim}{x} = \mu \underset{\sim}{x}$, then $(AB)\underset{\sim}{x} = (\lambda\mu)\underset{\sim}{x}$.

($\lambda\mu$ is an eigenvalue of AB).

Proof. If $B\underset{\sim}{x} = \mu \underset{\sim}{x}$
$\Rightarrow AB\underset{\sim}{x} = A\mu\underset{\sim}{x}$
$= \mu A \underset{\sim}{x}$
$= \mu \lambda \underset{\sim}{x}$.

Eigenvalues of kA

If $A\underset{\sim}{x} = \lambda \underset{\sim}{x}$, then $kA\underset{\sim}{x} = k\lambda \underset{\sim}{x}$.

($k\lambda$ is an eigenvalue to kA).

Proof: $k(A\underset{\sim}{x}) = k(\lambda \underset{\sim}{x})$
$\Rightarrow kA\underset{\sim}{x} = (k\lambda)\underset{\sim}{x}$.

Eigenvalues of A-sI.

If $A\underset{\sim}{x} = \lambda \underset{\sim}{x}$, then $(A - sI)\underset{\sim}{x} = (\lambda - s)\underset{\sim}{x}$ $\forall s \in \mathbb{R}$.

($\lambda - s$ is an eigenvalue of $A - sI$).

Proof. $(A - sI)\underset{\sim}{x} = A\underset{\sim}{x} - s(I\underset{\sim}{x})$
$= A\underset{\sim}{x} - s\underset{\sim}{x}$
$= \lambda \underset{\sim}{x} - s\underset{\sim}{x}$
$= (\lambda - s)\underset{\sim}{x}$. ∎

Eigenvalues of A^{-1}.

If $A\underset{\sim}{x} = \lambda \underset{\sim}{x}$, then $A^{-1}\underset{\sim}{x} = \frac{1}{\lambda}\underset{\sim}{x}$.

($\frac{1}{\lambda}$ is an eigenvalue of A^{-1}.)

Proof. If $A\underset{\sim}{x} = \lambda \underset{\sim}{x}$, then
$\underset{\sim}{x} = A^{-1}\lambda \underset{\sim}{x}$.
$\therefore A^{-1}\underset{\sim}{x} = \frac{1}{\lambda}\underset{\sim}{x}$, as required.

Eigenvalue of A+B

If $A\underset{\sim}{x} = \lambda \underset{\sim}{x}$ and $B\underset{\sim}{x} = \mu \underset{\sim}{x}$, then $(A + B)\underset{\sim}{x} = (\lambda + \mu)\underset{\sim}{x}$.

($\lambda + \mu$ is an eigenvalue of $A+B$).

Proof. $(A\underset{\sim}{x} + B\underset{\sim}{x})$
$= (\lambda \underset{\sim}{x} + \mu \underset{\sim}{x})$
$= (\lambda + \mu)\underset{\sim}{x}$ $(= (A+B)\underset{\sim}{x})$.

FINDING λ & $\underset{\sim}{x}$ OF A SQUARE MATRIX.

2x2 MATRICES

- If A is 2x2, then $p(\lambda)$ is a quadratic polynomial.
  *Note: we only consider $\lambda \in \mathbb{R}$.

$\Rightarrow$ the eigenvalues of A satisfy
$\det(A-\lambda I) = 0.$

$\Rightarrow$ the eigenvectors of A satisfy
$A\underset{\sim}{x} = \lambda \underset{\sim}{x}.$

eg Find the eigenvalues of the matrix $\begin{pmatrix} 4 & -2 \\ 3 & -1 \end{pmatrix}$, and the corresponding eigenvectors.

Let $A = \begin{pmatrix} 4 & -2 \\ 3 & -1 \end{pmatrix}$. Then its characteristic eqn is

$\det(A-\lambda I) = 0$

$\Rightarrow \det\begin{pmatrix} 4-\lambda & -2 \\ 3 & -1-\lambda \end{pmatrix} = 0$

$\Rightarrow (4-\lambda)(-1-\lambda) + 6 = 0$

$+\lambda^2 - 3\lambda - 4 + 6 = 0$

$\lambda^2 - 3\lambda + 2 = 0$

$(\lambda-2)(\lambda-1) = 0$

$\Rightarrow \lambda = 2, \ \lambda = 1. \rightarrow$ eigenvalues.

$\lambda = 2 \rightarrow \begin{pmatrix} 2 & -2 \\ 3 & -3 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = 0.$

$2x - 2y = 0. \Rightarrow e_2 = \begin{pmatrix} t \\ t \end{pmatrix}, \ t \in \mathbb{R}$

$\lambda = 1 \rightarrow \begin{pmatrix} 3 & -2 \\ 3 & -2 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = 0$

$3x - 2y = 0 \Rightarrow e_1 = \begin{pmatrix} 2t \\ 3t \end{pmatrix}, \ t \in \mathbb{R}.$

3x3 MATRICES

- The characteristic eqn $\det(A-\lambda I) = 0$ if A is 3x3 is a cubic eqn. (let $A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$)

$\Rightarrow$ to obtain the eigenvectors, take the cross product of 2 unequal rows of the matrix $A - \lambda I$. $\longrightarrow$ Proof.

★ If A is in echelon form, the eigen values are the diagonal entries.

Proof. For any matrix A,

$A\underset{\sim}{x} = \lambda \underset{\sim}{x},$

or $(A-\lambda I)\underset{\sim}{x} = \underset{\sim}{0}.$

expanding.

$$\begin{pmatrix} a_{11}-\lambda & a_{12} & a_{13} \\ a_{21} & a_{22}-\lambda & a_{23} \\ a_{31} & a_{32} & a_{33}-\lambda \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

$\Rightarrow \begin{pmatrix} a_{11}-\lambda & a_{12} & a_{13} \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = 0$

$\Rightarrow (a_{11}-\lambda)x + (a_{12})y + (a_{13})z = 0.$

But this can be written as $\begin{pmatrix} a_{11}-\lambda \\ a_{12} \\ a_{13} \end{pmatrix} \cdot \begin{pmatrix} x \\ y \\ z \end{pmatrix} = 0.$

and hence

$\begin{pmatrix} x \\ y \\ z \end{pmatrix}$ is perpendicular to $\begin{pmatrix} a_{11}-\lambda \\ a_{12} \\ a_{13} \end{pmatrix}.$

Similarly, $\begin{pmatrix} x \\ y \\ z \end{pmatrix} \perp \begin{pmatrix} a_{21} \\ a_{22}-\lambda \\ a_{23} \end{pmatrix}$ & $\begin{pmatrix} x \\ y \\ z \end{pmatrix} \perp \begin{pmatrix} a_{31} \\ a_{32} \\ a_{33}-\lambda \end{pmatrix}.$

Hence, by cross multiplying any of these vectors, we can obtain $\begin{pmatrix} x \\ y \\ z \end{pmatrix}$ (the eigenvector).

e.g. Find the eigenvalues of the matrix $A = \begin{pmatrix} 1 & 1 & -2 \\ -1 & 2 & 1 \\ 0 & 1 & -1 \end{pmatrix}$, and the corresponding eigenvectors in parametric form.

$\Rightarrow \det(A-\lambda I) = 0$ is the characteristic eqn.

$\Rightarrow \det\begin{pmatrix} 1-\lambda & 1 & -2 \\ -1 & 2-\lambda & 1 \\ 0 & 1 & -1-\lambda \end{pmatrix} = 0.$

$(1-\lambda)[(2-\lambda)(-1-\lambda)-1] - (1)[(-1)(-1-\lambda)] + (-2)(-1-0) = 0.$

$(1-\lambda)(2-\lambda)(-1-\lambda) - (1-\lambda) + 1(-1-\lambda) + 2 = 0$

$(1-\lambda)(2-\lambda)(-1-\lambda) - 1 + \lambda - 1 - \lambda + 2 = 0$

$\Rightarrow (1-\lambda)(2-\lambda)(-1-\lambda) = 0 \quad \therefore \lambda = 1, \lambda = 2, \lambda = -1.$

$\lambda = 1 \Rightarrow \begin{pmatrix} 0 & 1 & -2 \\ -1 & 1 & 1 \\ 0 & 1 & -2 \end{pmatrix}\underset{\sim}{x} = \underset{\sim}{0}.$

$\therefore \underset{\sim}{x} = \begin{pmatrix} 0 \\ 1 \\ -2 \end{pmatrix} \times \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 3 \\ 2 \\ 1 \end{pmatrix}.$

$\lambda = -1 \Rightarrow \begin{pmatrix} 2 & 1 & -2 \\ -1 & 3 & 1 \\ 0 & 1 & 0 \end{pmatrix}\underset{\sim}{x} = \underset{\sim}{0}.$

$\therefore \underset{\sim}{x} = \begin{pmatrix} 2 \\ 1 \\ -2 \end{pmatrix} \times \begin{pmatrix} -1 \\ 3 \\ 1 \end{pmatrix} = \begin{pmatrix} 7 \\ 0 \\ 7 \end{pmatrix}.$

$\lambda = 2 \Rightarrow \begin{pmatrix} -1 & 1 & -2 \\ -1 & 0 & 1 \\ 0 & 1 & -3 \end{pmatrix}\underset{\sim}{x} = \underset{\sim}{0}.$

$\therefore \underset{\sim}{x} = \begin{pmatrix} -1 \\ 1 \\ -2 \end{pmatrix} \times \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 3 \\ 1 \end{pmatrix}.$

A SHORTCUT TO FIND λ AND $\underset{\sim}{x}$ OF 3x3 MATRICES.

e.g.[1] find λ & $\underset{\sim}{x}$ of $A = \begin{pmatrix} -3 & 5 & 5 \\ -4 & 6 & 5 \\ 4 & -4 & -3 \end{pmatrix}$.

$\Rightarrow \det(A - \lambda I) = 0$

ie $\det\begin{pmatrix} -3-\lambda & 5 & 5 \\ -4 & 6-\lambda & 5 \\ 4 & -4 & -3-\lambda \end{pmatrix}$

> ☆ easier method to find eigenvalues of a 3x3 matrix
> ∵ expansion is tedious.

Add row2 & row3.

$$\Rightarrow \det\begin{pmatrix} -3-\lambda & 5 & 5 \\ 0 & 2-\lambda & 2-\lambda \\ 4 & -4 & -3-\lambda \end{pmatrix}.$$

using 1st column:

$$(-3-\lambda)\left[(2-\lambda)(-3-\lambda) + 4(2-\lambda)\right] - 0 + \cancel{4(5(2-\lambda) - 5(2-\lambda))} = 0$$

$$\Rightarrow (-3-\lambda)(2-\lambda)(-3-\lambda+4) = 0$$

$$(-3-\lambda)(2-\lambda)(1-\lambda) = 0$$

ie $\lambda = -3$, $\lambda = 2$ & $\lambda = 1$.

e.g.[2] Find eigenvalues of $A = \begin{pmatrix} 1 & 2 & 3 \\ -2 & -3 & 6 \\ 2 & 2 & -4 \end{pmatrix}$.

Characteristic eq^{n} of A

$$\Rightarrow \det(A - \lambda I) = 0$$

or $\det\begin{pmatrix} 1-\lambda & 2 & 3 \\ -2 & -3-\lambda & 6 \\ 2 & 2 & -4-\lambda \end{pmatrix} = 0$.

add $\boxed{r3 \text{ to } r2.}$

(r2+=r3) $\Rightarrow \det\begin{pmatrix} 1-\lambda & 2 & 3 \\ 0 & -1-\lambda & 2-\lambda \\ 2 & 2 & -4-\lambda \end{pmatrix} = 0$.

using 1st column:

$$(1-\lambda)\left[(-1-\lambda)(-4-\lambda) - 2(2-\lambda)\right] - 0 + 2\left(2(2-\lambda) - 3(-1-\lambda)\right) = 0$$

$$(1-\lambda)\left[(-1-\lambda)(-4-\lambda) - 2(2-\lambda)\right] + 2(7+\lambda) = 0$$

$$\Rightarrow (1-\lambda)\left((4+5\lambda+\lambda^2-4+2\lambda)\right) + 2(7+\lambda) = 0$$

$$(1-\lambda)(\lambda^2+7\lambda) + 2(7+\lambda) = 0$$

$$(1-\lambda)(\lambda)(\lambda+7) + 2(7+\lambda) = 0$$

$$\Rightarrow (\lambda+7)(\lambda(1-\lambda)+2) = 0$$

$$(\lambda+7)(2+\lambda-\lambda^2) = 0$$

$$(\lambda+7)(\lambda^2-\lambda-2) = 0$$

$$\Rightarrow (\lambda+7)(\lambda+1)(\lambda-2) = 0$$

$$\Rightarrow \lambda = -7,\ \lambda = 2,\ \lambda = -1.$$

DIAGONAL MATRIX OF EIGENVALUES

The diagonalisation of a matrix A, D, is given by

$$\boxed{D = \begin{pmatrix}\lambda_1 & 0 & 0\\ 0 & \lambda_2 & 0\\ 0 & 0 & \lambda_3\end{pmatrix}},$$

where λ_1, λ_2 & λ_3 are the eigenvalues of the matrix.

Let $\boxed{P = (\underset{\sim}{e}_1, \underset{\sim}{e}_2, \underset{\sim}{e}_3)}$. ($\underset{\sim}{e} \Leftrightarrow$ eigenvectors of A corresponding to resp λ)

$\therefore$ $PD = (\underset{\sim}{e}_1\lambda_1, \underset{\sim}{e}_2\lambda_2, \underset{\sim}{e}_3\lambda_3)$.

But $AP = A(\underset{\sim}{e}_1, \underset{\sim}{e}_2, \underset{\sim}{e}_3)$

$= (A\underset{\sim}{e}_1, A\underset{\sim}{e}_2, A\underset{\sim}{e}_3)$

$= (\lambda\underset{\sim}{e}_1, \lambda\underset{\sim}{e}_2, \lambda\underset{\sim}{e}_3)$ $(= PD)$.

Hence, $AP = PD$, ie

$$\boxed{D = P^{-1}AP.}$$

or

$$\boxed{A = PDP^{-1}}$$

However, it can be proven that

$$\boxed{D^n = \begin{pmatrix}\lambda_1^n & 0 & 0\\ 0 & \lambda_2^n & 0\\ 0 & 0 & \lambda_3^n\end{pmatrix}.}$$

Proof by induction.

$n=1$ trivial.

$n=k$: $D^k = \begin{pmatrix}\lambda_1^k & 0 & 0\\ 0 & \lambda_2^k & 0\\ 0 & 0 & \lambda_3^k\end{pmatrix}$.

$D^{k+1} = DD^k$

$= \begin{pmatrix}\lambda_1 & 0 & 0\\ 0 & \lambda_2 & 0\\ 0 & 0 & \lambda_3\end{pmatrix}\begin{pmatrix}\lambda_1^k & 0 & 0\\ 0 & \lambda_2^k & 0\\ 0 & 0 & \lambda_3^k\end{pmatrix}$

$= \begin{pmatrix}\lambda_1^{k+1} & 0 & 0\\ 0 & \lambda_2^{k+1} & 0\\ 0 & 0 & \lambda_3^{k+1}\end{pmatrix}$ as req.

CAYLEY-HAMILTON THEOREM

The Cayley-Hamilton theorem states that a square matrix always satisfies its own characteristic equation.

i.e. if the characteristic eqn of a square matrix, A, is $p(\lambda) = 0$,

then $p(A) = 0$.

i.e. if $\lambda^n + c_1\lambda^{n-1} + c_2\lambda^{n-2} \cdots + c_n = 0$,

then $A^n + c_1A^{n-1} + c_2A^{n-2} \cdots + c_nI = 0$.

CALCULATING A^n USING EIGENVALUES & VECTORS

Since $A = PDP^{-1}$,

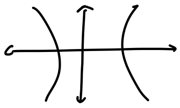
$\therefore$ $A^n = (PDP^{-1})^n$

$= \underbrace{(PDP^{-1})(PDP^{-1})(PDP^{-1})\cdots(PDP^{-1})}_{n}$

$\therefore$ $\boxed{A^n = PD^nP^{-1}.}$

Chapter 2: Hyperbolic Functions

The <u>hyperbolic functions</u> are named because they are related to the <u>hyperbola</u>.



DEFINITIONS

Hyperbolic sine of x, sinh x (pronounced "shine x"), is defined by

$$\boxed{\sinh x = \frac{1}{2}(e^x - e^{-x})}$$

and similarly, hyperbolic cosine of x, cosh x (pronounced "cosh x") is defined by

$$\boxed{\cosh x = \frac{1}{2}(e^x + e^{-x})}$$

☆ sinh x is "smaller" than cosh x for all x

The other hyperbolic functions are hence defined as follows:

① $\tanh x = \frac{\sinh x}{\cosh x}$

$$\boxed{\therefore \tanh x = \frac{e^x - e^{-x}}{e^x + e^{-x}}.}$$

② $\coth x = \frac{1}{\tanh x}$

$$\boxed{\therefore \coth x = \frac{e^x + e^{-x}}{e^x - e^{-x}}}$$

③ $\operatorname{sech} x = \frac{1}{\cosh x}$

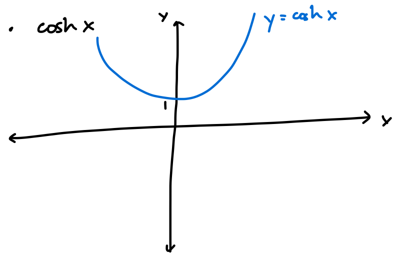
$$\boxed{\operatorname{sech} x = \frac{2}{e^x + e^{-x}}}$$

④ $\operatorname{cosech} x = \frac{1}{\sinh x}$

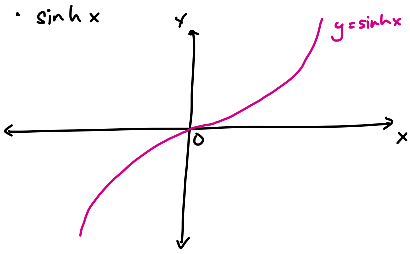
$$\boxed{\operatorname{cosech} x = \frac{2}{e^x - e^{-x}}}$$

GRAPHS OF HFUNCTIONS

• cosh x



• sinh x



• tanh x

$|\tanh x| < 1$.

Proof

$\tanh x = \frac{e^x - e^{-x}}{e^x + e^{-x}} \; (= y)$

$\Rightarrow y(e^x + e^{-x}) = e^x - e^{-x}$

$ye^x + ye^{-x} = e^x - e^{-x}$

$\Rightarrow ye^x + \frac{y}{e^x} = e^x - \frac{1}{e^x}$

$(\cdot e^x)$ $ye^{2x} + y = e^{2x} - 1$

$1 + y = e^{2x} - ye^{2x}$

$\therefore e^{2x} = \frac{1+y}{1-y} \; (>0)$

*can't cross multiply ∵ 1-y can be -ve or +ve

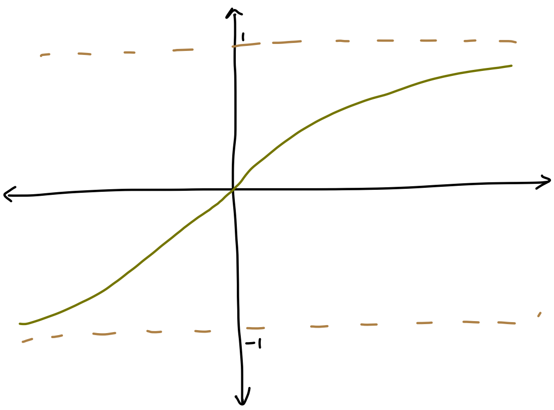
$\Rightarrow \frac{1+y}{1-y} \cdot \frac{1-y}{1-y} > 0$

$\frac{1-y^2}{(1-y)^2} > 0 \Rightarrow 1 - y^2 > 0.$

$(1-y)(1+y) > 0$

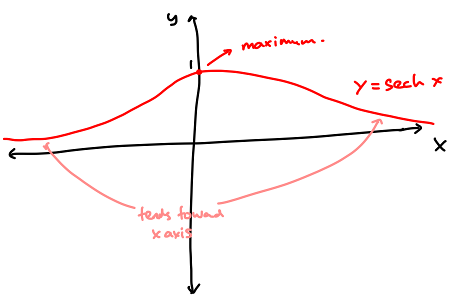
$\Rightarrow -1 < y < 1.$

Dwy follows.

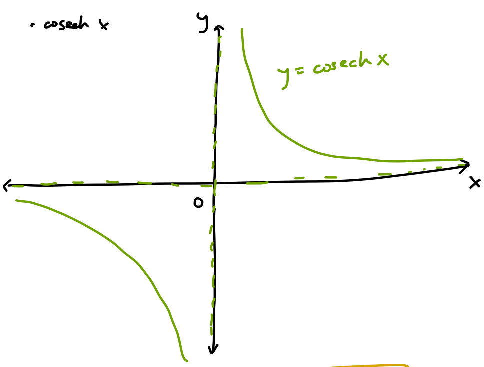


• sech x

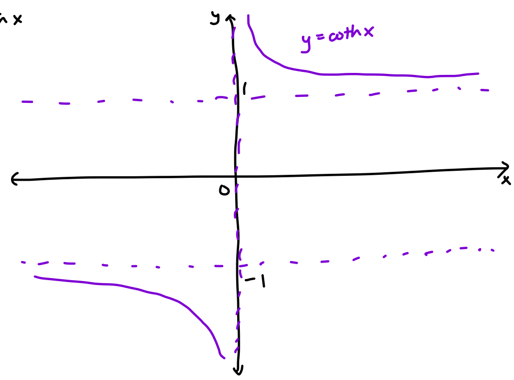
*we can use the graph of cosh x to plot sech x.



• cosech x



• coth x



$\boxed{\text{☆ Hfunctions are NOT periodic whereas circular ones are}}$

HYPERBOLIC IDENTITIES

① $\cosh^2 x - \sinh^2 x \equiv 1$

Proof: $\cosh^2 x - \sinh^2 x$

$\equiv \left[\frac{1}{2}(e^x+e^{-x})\right]^2 - \left[\frac{1}{2}(e^x-e^{-x})\right]^2$

$\equiv \left[\frac{1}{2}(e^x+e^{-x}) - \frac{1}{2}(e^x-e^{-x})\right]\left[\frac{1}{2}(e^x+e^{-x}) + \frac{1}{2}(e^x-e^{-x})\right]$

$\equiv \frac{1}{4}\left[2e^{-x}\right]\left[2e^{x}\right]$

$\equiv 1.$ $\quad\therefore \cosh^2 x - \sinh^2 x \equiv 1.$

② $\cosh^2 x + \sinh^2 x \equiv \cosh 2x$

Proof. $\cosh^2 x + \sinh^2 x$

$\equiv \left[\frac{1}{2}(e^x+e^{-x})\right]^2 + \left[\frac{1}{2}(e^x-e^{-x})\right]^2$

$\equiv \frac{1}{4}\left[e^{2x} + 2 + e^{-2x} + e^{2x} - 2 + e^{-2x}\right]$

$\equiv \frac{1}{4}\left[2e^{2x} + 2e^{-2x}\right] \equiv \frac{1}{2}(e^{2x}+e^{-2x})$

$\equiv \cosh 2x.$ $\quad\therefore \cosh^2 x + \sinh^2 x \equiv \cosh 2x.$ ※

③ $\sinh 2x \equiv 2\sinh x \cosh x$

Proof: $2\sinh x \cosh x$

$\equiv 2\left[\frac{1}{2}(e^x-e^{-x})\right]\left[\frac{1}{2}(e^x+e^{-x})\right]$

$\equiv \frac{1}{2}\left[e^{2x} - e^{-2x}\right]$

$\equiv \sinh 2x.$ ※

④ $\sinh(x+y) \equiv \sinh x \cosh y + \cosh x \sinh y$

Proof: $\sinh x \cosh y + \cosh x \sinh y$

$\equiv \frac{1}{4}(e^x-e^{-x})(e^y+e^{-y}) + \frac{1}{4}(e^x+e^{-x})(e^y-e^{-y})$

$\equiv \frac{1}{4}\left[e^{x+y} + e^{x-y} - e^{y-x} - e^{-x-y} + e^{x+y} - e^{x-y} + e^{y-x} - e^{-x-y}\right] \equiv \frac{1}{4}\left[2e^{x+y} - 2e^{-x-y}\right]$

$\equiv \frac{1}{2}\left[e^{x+y} - e^{-x-y}\right]$

$\equiv \sinh(x+y).$ ※

⑤ $\cosh(x+y) \equiv \cosh x \cosh y + \sinh x \sinh y$

Proof: $\cosh x \cosh y + \sinh x \sinh y$

$= \frac{1}{4}\left[(e^x+e^{-x})(e^y+e^{-y}) + (e^x-e^{-x})(e^y-e^{-y})\right]$

$= \frac{1}{4}\left[e^{x+y} + e^{y-x} + e^{x-y} + e^{-x-y} + e^{x+y} - e^{-x+y} - e^{x-y} + e^{-x-y}\right]$

$= \frac{1}{4}\left[2e^{x+y} + 2e^{-x-y}\right]$

$= \cosh(x+y).$ ※

OSBORN'S RULE

A way of remembering hfunct identities easily by relating them to circfunct identities.

The rule is to change each trigonometrical ratio into the comparative hyperbolic functions. Whenever a product of two sines occurs, change the sign of the term. This rule should be used with care as there are a number of ways in which a product of two sine ratios can be disguised, e.g. in $\tan^2 x$, $\cot^2 x$, $\mathrm{cosec}^2 x$.

eg $1 - \sin^2 x \equiv \cos^2 x \Rightarrow 1 + \sinh^2 x \equiv \cosh^2 x$

change sign

☆ does not ALWAYS work!

☆ only as an aid; NOT a concrete answer.

SOLVING HYPERBOLIC FUNCTIONS

Examples

eg¹ $4\cosh x - \sinh x = 5.$

$\Rightarrow 4\left[\frac{1}{2}(e^x+e^{-x})\right] - \frac{1}{2}(e^x-e^{-x}) = 5.$

$(\cdot 2)$ $\quad 4e^x + 4e^{-x} - e^x + e^{-x} = 10.$

Let $e^x = u.$ $\Rightarrow 3u + \frac{5}{u} = 10.$

$\Rightarrow 3u^2 + 5 = 10u.$ ie $3u^2 - 10u + 5 = 0$

$\Rightarrow u = \frac{10 \pm \sqrt{100 - 4(3)(5)}}{2(3)}$

$\therefore x = \ln\left(\frac{10 \pm \sqrt{100-60}}{6}\right)$ etc etc

eg³ $\cosh 2x - 7\cosh x + 7 = 0.$

$(\cosh^2 x + \sinh^2 x) - 7\cosh x + 7 = 0$

$\Rightarrow \cosh^2 x + (\cosh^2 x - 1) - 7\cosh x + 7 = 0.$

$\Rightarrow 2\cosh^2 x - 7\cosh x + 6 = 0$

$(2\cosh x - 3)(\cosh x - 2) = 0$

$\cosh x = \frac{3}{2}, \quad \cosh x = 2$

$\therefore x = \cosh^{-1}\left(\frac{3}{2}\right), \quad x = \cosh^{-1}(2)$

eg² $12\cosh^2 x + 7\sinh x = 24.$

$\Rightarrow 12(1+\sinh^2 x) + 7\sinh x = 24.$

$\Rightarrow 12\sinh^2 x + 7\sinh x - 12 = 0$

$(4\sinh x - 3)(3\sinh x + 4) = 0$

$\therefore \sinh x = \frac{3}{4}, \quad \sinh x = -\frac{4}{3}.$

$\therefore x = \sinh^{-1}\left(\frac{3}{4}\right), \quad x = \sinh^{-1}\left(-\frac{4}{3}\right).$

$= 0.693 \qquad = -1.10$

eg⁴ $\sinh 2x + \cosh x = 0$

$\Rightarrow 2\sinh x \cosh x + \cosh x = 0$

$\therefore \cosh x \,(2\sinh x + 1) = 0$

$\cosh x \neq 0$ $\quad(\because \cosh x \geq 1 \ \forall x \in \mathbb{R})$

$\therefore \sinh x = -\frac{1}{2}$

$\therefore x = \sinh^{-1}\left(-\frac{1}{2}\right)$

$= \ldots$

eg⁵ (i) Prove $\tanh(x+y) = \dfrac{\tanh x + \tanh y}{1 + \tanh x \tanh y}.$

$\tanh(x) = \dfrac{e^x - e^{-x}}{e^x + e^{-x}}$ $\quad\therefore \tanh(x+y) = \dfrac{e^{x+y} - e^{-x-y}}{e^{x+y} + e^{-x-y}}$

$$\frac{\tanh x + \tanh y}{1 + \tanh x \tanh y} = \frac{\left(\frac{e^x - e^{-x}}{e^x + e^{-x}}\right) + \left(\frac{e^y - e^{-y}}{e^y + e^{-y}}\right)}{1 + \left(\frac{e^x - e^{-x}}{e^x + e^{-x}}\right)\left(\frac{e^y - e^{-y}}{e^y + e^{-y}}\right)}$$

$\left(\cdot \dfrac{(e^x+e^{-x})(e^y+e^{-y})}{(e^x+e^{-x})(e^y+e^{-y})}\right)$

$$\equiv \frac{(e^x-e^{-x})(e^y+e^{-y}) + (e^x+e^{-x})(e^y-e^{-y})}{(e^x+e^{-x})(e^y+e^{-y}) + (e^x-e^{-x})(e^y-e^{-y})}$$

$$\equiv \frac{e^{x+y} - e^{-x+y} + e^{x-y} - e^{-x-y} + e^{x+y} - e^{-y+x} + e^{-x+y} - e^{-x-y}}{e^{x+y} + e^{-x+y} + e^{x-y} + e^{-x-y} + e^{x+y} - e^{y-x} - e^{x-y} + e^{-x-y}}$$

$$\equiv \frac{2e^{x+y} - 2e^{-x-y}}{2e^{x+y} + 2e^{-x-y}}$$

$$\equiv \frac{e^{x+y} - e^{-x-y}}{e^{x+y} + e^{-x-y}} \quad (\equiv \tanh(x+y)).$$

$\therefore \tanh(x+y) \equiv \dfrac{\tanh x + \tanh y}{1 + \tanh x \tanh y}.$

(ii) Deduce $\tanh(3x) = \dfrac{3\tanh x + \tanh^3 x}{1 + 3\tanh^2 x}.$

Let $y = 2x$,

then $\tanh(3x) = \tanh(x + 2x)$

$= \dfrac{\tanh x + \tanh 2x}{1 + \tanh x \tanh 2x}$

Since $\tanh 2x = \dfrac{2\tanh x}{1+\tanh^2 x}$,

$$\therefore \tanh 3x = \frac{\tanh x + \frac{2\tanh x}{1+\tanh^2 x}}{1 + \tanh x \frac{2\tanh x}{1+\tanh^2 x}}$$

$$\left(\cdot \frac{1+\tanh^2 x}{1+\tanh^2 x}\right) = \frac{\tanh x\,(1+\tanh^2 x) + 2\tanh x}{1 + \tanh^2 x + 2\tanh^2 x}$$

$$= \frac{\tanh^3 x + 3\tanh x}{3\tanh^2 x + 1}. \quad \text{QED.}$$

(iii) Hence solve the eqn $\tanh 3x = 2\tanh x.$

$\Rightarrow \dfrac{3\tanh x + \tanh^3 x}{1 + 3\tanh^2 x} = 2\tanh x$

$3\tanh x + \tanh^3 x = 2\tanh x + 6\tanh^3 x$

$\Rightarrow 0 = 5\tanh^3 x - \tanh x$

$= \tanh x\,(5\tanh^2 x - 1)$

$\therefore \tanh x = 0, \quad \tanh x = \frac{1}{\sqrt{5}}, \quad \tanh x = \frac{-1}{\sqrt{5}}$

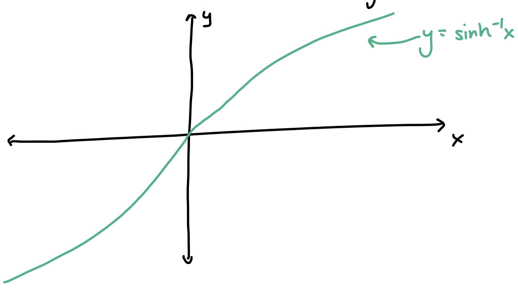
$\Rightarrow x = 0, \quad x = \tanh^{-1}\left(\frac{1}{\sqrt{5}}\right), \quad x = \tanh^{-1}\left(\frac{-1}{\sqrt{5}}\right)$

$x = 0, \quad x = \pm 0.481$

Inverse Hyperbolic Functions

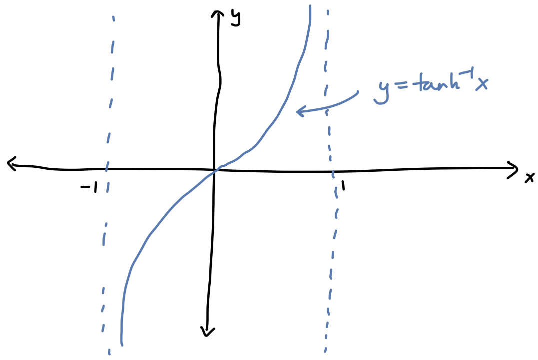
$\sinh x$ has domain, range $\in \mathbb{R}$ (and 1-1)

$\therefore \sinh^{-1} x$ also has dom, range $\in \mathbb{R}$.



$\tanh x$ has domain $\in \mathbb{R}$ & range $-1<x<1$.

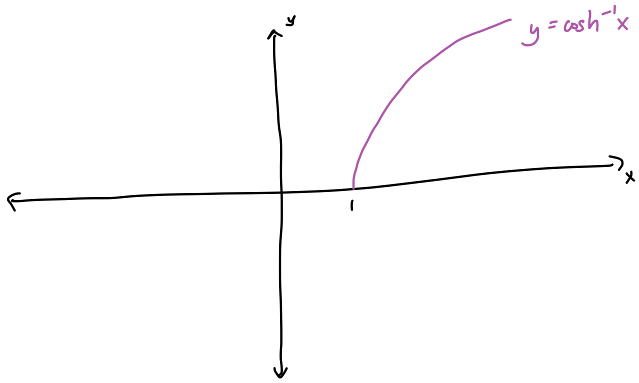
$\therefore \tanh^{-1} x$ has domain $-1<x<1$ & range $\in \mathbb{R}$



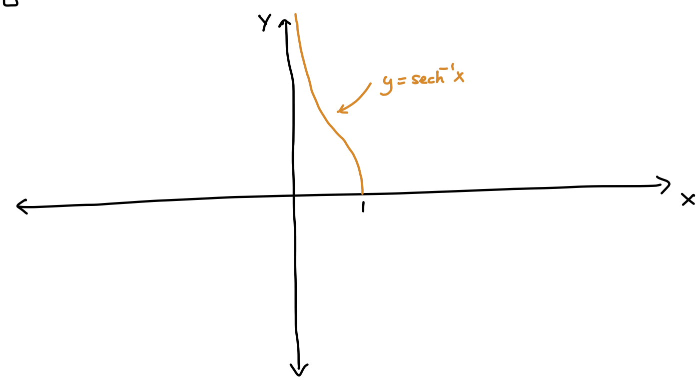
Since $\cosh x$ is NOT a 1-1 function,
we can only take the inverse for
only a certain subset of its domain.

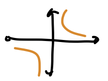
→ Let the domain be $\boxed{x \geq 0}$ (& hence range $\Rightarrow y \geq 1$).

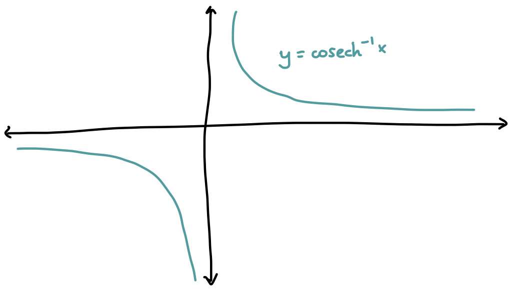
→ Then $\cosh^{-1} x$ has domain $x \geq 1$ & range $y \geq 0$.

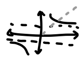


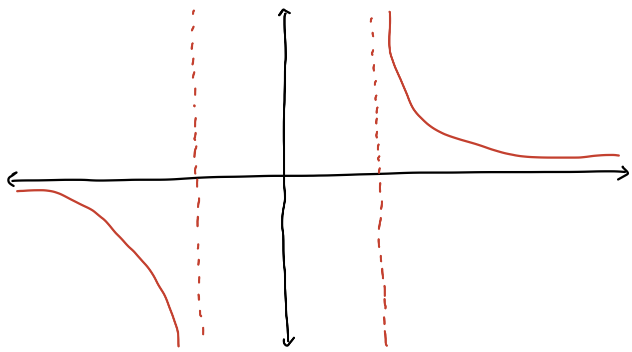
$\operatorname{sech}^{-1} x$ (domain of $\operatorname{sech} x \Rightarrow x \geq 0$, to be 1-1)



$\operatorname{cosech}^{-1} x$ ()



$\coth^{-1} x$ ()



INVERSE HYPERBOLIC FUNCTIONS IN TERMS OF LOGARITHMS

* given in MF19

① $\cosh^{-1} x$

Let $y = \cosh^{-1} x$. Inter△ x & y

$\therefore x = \cosh y$

$x = \frac{1}{2}(e^y + e^{-y})$

$2x = e^y + e^{-y}$

$(\cdot e^y)\quad 2xe^y = e^{2y} + 1$

$0 = e^{2y} - 2xe^y + 1$

$= u^2 - 2xu + 1 \quad (u = e^y)$

$\Rightarrow u\ (=e^y) = \dfrac{2x \pm \sqrt{4x^2 - 4(1)(1)}}{2}$

$e^y = x \pm \sqrt{x^2-1}$.

$\therefore y = \ln(x + \sqrt{x^2-1})$ or $y = \ln(x - \sqrt{x^2-1})$

$= \ln\left(\dfrac{(x-\sqrt{x^2-1})(x+\sqrt{x^2-1})}{(x+\sqrt{x^2-1})}\right)$

$= \ln\left(\dfrac{x^2-(x^2-1)}{(x+\sqrt{x^2-1})}\right)$

$= -\ln(x+\sqrt{x^2-1})$

$$\boxed{\text{Hence } \cosh^{-1} x = (\pm)\ln(x+\sqrt{x^2-1}).}$$

② $\sinh^{-1} x$

Let $y = \sinh^{-1} x$

$\Rightarrow x = \sinh y$

$= \frac{1}{2}(e^y - e^{-y})$

$2x = e^y - e^{-y}$

$\Rightarrow 2xe^y = e^{2y} - 1$

$0 = e^{2y} - 2xe^y - 1$

$\therefore e^y = \dfrac{2x \pm \sqrt{4x^2+4}}{2}$

$= x \pm \sqrt{x^2+1}$.

$\therefore y = \ln(x + \sqrt{x^2+1}) \quad \because \sqrt{x^2+1} > x \ \forall x \geq 1$

Hence $\boxed{\sinh^{-1} x = \ln(x + \sqrt{x^2+1})}$

③ $\tanh^{-1} x$

Let $y = \tanh^{-1} x$

$\Rightarrow x = \tanh y$

$= \dfrac{e^y - e^{-y}}{e^y + e^{-y}}$

$\Rightarrow xe^y + xe^{-y} = e^y - e^{-y}$

$(x-1)e^y = (-x-1)e^{-y}$

$\therefore e^{2y} = \dfrac{-x-1}{x-1}$

$\Rightarrow 2y = \ln(-x-1) - \ln(x-1)$

$\therefore y = \frac{1}{2}(\ln(-x-1) - \ln(x-1))$

$= \frac{1}{2}\ln\left(\frac{1+x}{1-x}\right)$. (def for $-1<x<1$)

Hence $\boxed{\tanh^{-1} x = \frac{1}{2}\ln\left(\frac{1+x}{1-x}\right)}$

④ $\coth^{-1} x$

Let $y = \coth^{-1} x$

$\Rightarrow x = \coth y$

$x = \dfrac{e^y + e^{-y}}{e^y - e^{-y}}$

$xe^y - xe^{-y} = e^y + e^{-y}$

$(x-1)e^y = (x+1)e^{-y}$

$\therefore e^{2y} = \dfrac{x+1}{x-1}$

$\therefore y = \frac{1}{2}\ln\left(\frac{x+1}{x-1}\right)$

Hence $\boxed{\coth^{-1} x = \frac{1}{2}\ln\left(\frac{x+1}{x-1}\right)}$

SOLVING EQNS INVOLVING INV HFUNC

eg[1] $\sinh^{-1}x = \ln(2-x)$

$\Rightarrow e^{\sinh^{-1}x} = 2-x$

$\sinh^{-1}x = \ln(x+\sqrt{x^2+1})$

$\therefore e^{\sinh^{-1}x} = x+\sqrt{x^2+1}$.

Hence eqn reduces to

$x+\sqrt{x^2+1} = 2-x$

$\Rightarrow \sqrt{x^2+1} = 2-2x$

$x^2+1 = 4-8x+4x^2$

$0 = 3x^2-8x+3$

$\therefore x = \dfrac{8 \pm \sqrt{64-4(3)(3)}}{6}$

$= \dfrac{8 \pm \sqrt{28}}{6}$

$\therefore x = 2.215$ or 0.451

since $2-2.215 < 0$

$\Rightarrow \boxed{x = 0.451}$

eg[2] $\sinh A\cosh B - \cosh A \sinh B$

$\equiv \frac{1}{2}(e^{A}-e^{-A})\frac{1}{2}(e^{B}+e^{-B}) - \frac{1}{2}(e^{A}+e^{-A})\frac{1}{2}(e^{B}-e^{-B})$

$\equiv \frac{1}{4}(e^{A+B}-e^{B-A}+e^{A-B}-e^{-A-B}-e^{A+B}-e^{B-A}+e^{A-B}+e^{-A-B})$

$\equiv \frac{1}{4}(2e^{A-B}-2e^{B-A})$

$\equiv \frac{1}{2}(e^{A-B}-e^{B-A}) \equiv \sinh(A-B)$

$\sinh^{-1}(2x) = \sinh^{-1}(\frac{3}{4}) - \sinh^{-1}x$

let $C = \sinh^{-1}(2x)$, $A = \sinh^{-1}(\frac{3}{4})$, $B = \sinh^{-1}(x)$

$\cosh^2 x - \sinh^2 x \equiv 1$

$\therefore \cosh^2 x \equiv 1+\sinh^2 x$

$\Rightarrow C = A-B$

$\therefore \sinh C = \sinh(A-B)$

$= \sinh A \cosh B - \sinh B \cosh A$

$\Rightarrow 2x = \frac{3}{4}\sqrt{1+x^2} - x\sqrt{1+(\frac{3}{4})^2}$

$2x = \frac{3}{4}\sqrt{1+x^2} - x(\frac{5}{4})$

$(\cdot 4)$ $8x = 3\sqrt{1+x^2} - 5x$

$13x = 3\sqrt{1+x^2}$

$\therefore 169x^2 = 9(1+x^2)$

$160x^2 = 9 \quad \therefore x^2 = \frac{9}{160}$

$\boxed{\therefore x = \pm\frac{3}{\sqrt{160}}}$.

eg[3] $\tanh^{-1}(2x) = \tanh^{-1}(x) + \tanh^{-1}(\frac{3}{7})$.

$\tanh^{-1}(x) = \frac{1}{2}\ln(\frac{1+x}{1-x})$

$\therefore$ eqn simplifies to

$\frac{1}{2}\ln(\frac{1+2x}{1-2x}) = \frac{1}{2}\ln(\frac{1+x}{1-x}) + \frac{1}{2}\ln\left(\frac{(\frac{10}{7})}{(\frac{4}{7})}\right)$

$(\cdot 2)$ $\ln(\frac{1+2x}{1-2x}) = \ln\left(\frac{1+x}{1-x}\cdot\frac{10}{4}\right)$

$(e^{\cdot})$ $\dfrac{1+2x}{1-2x} = \frac{5}{2}\left(\frac{1+x}{1-x}\right)$

$2(1+2x)(1-x) = 5(1+x)(1-2x)$

$\Rightarrow -4x^2+2x+2 = -10x^2-5x+5$

$\therefore 6x^2+7x-3 = 0$

$(3x-1)(2x+3) = 0$

$\therefore x = +\frac{1}{3}$, $x = -\frac{3}{2}$ reject $\because x \in (-1,1)$

$\therefore x = \frac{1}{3}$.

eg[4] (Prove $\operatorname{cosech}^{-1}x = \ln(\frac{1+\sqrt{x^2+1}}{x})$).

Let $y = \operatorname{cosech}^{-1}x$

$\Rightarrow \operatorname{cosech} y = x$

$\frac{1}{\sinh y} = x \quad \therefore \sinh y = \frac{1}{x}$.

$\Rightarrow y = \sinh^{-1}(\frac{1}{x})$

$= \ln(\frac{1}{x}+\sqrt{\frac{1}{x^2}+1})$

$= \ln\left(\frac{1+\sqrt{x^2+1}}{x}\right)$

$\Rightarrow \operatorname{cosech}^{-1}x = \ln\left(\frac{1+\sqrt{x^2+1}}{x}\right)$

$\ln(n) = \sinh^{-1}(\frac{3}{4}) + \operatorname{cosech}^{-1}(\frac{3}{4})$

$\ln(n) = \ln\left(\frac{3}{4}+\sqrt{(\frac{3}{4})^2+1}\right) + \ln\left(\frac{1+\sqrt{(\frac{3}{4})^2+1}}{(\frac{3}{4})}\right)$

$= \ln(\frac{3}{4}+\frac{5}{4}) + \ln(1+\frac{5}{4}) - \ln(\frac{3}{4})$

$= \ln(2) + \ln(9) - \cancel{\ln(4)} - \ln(3) + \cancel{\ln(4)}$

$= \ln(6)$

$\therefore n = 6$.

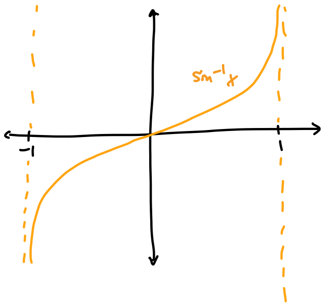
Chapter 3: Differentiation

DIFF OF $\sin^{-1}$ & $\cos^{-1}$

*domain: $-1 \leq x \leq 1$

Similar to diff. of $\tan^{-1}x$.

① $\sin^{-1}x$



OBSERVE gradient $= \frac{dy}{dx} > 0 \ \forall x$ in domain.

Let $y = \sin^{-1}x$.

$\Rightarrow \sin y = x$.

$(\cdot \frac{d}{dy}) \quad \cos y = \frac{dx}{dy}$

$\cos^2 y = 1 - \sin^2 y$
$= 1 - x^2$
$\therefore \cos y = \sqrt{1-x^2} \quad \because \frac{d}{dx} > 0$

$\therefore \frac{dy}{dx} = \frac{1}{\cos y}$

$\therefore \frac{dy}{dx} = \frac{1}{\sqrt{1-x^2}}$.

$$\boxed{\therefore \frac{d}{dx}(\sin^{-1}x) = \frac{1}{\sqrt{1-x^2}}.}$$ in MF19

ie $\frac{d}{dx}(\sin^{-1}f(x)) = \frac{f'(x)}{\sqrt{1-[f(x)]^2}}$

eg¹ $\frac{d}{dx}\sin^{-1}\left(\frac{x}{3}\right)$

$= \frac{1}{\sqrt{1-\left(\frac{x}{3}\right)^2}}\left(\frac{1}{3}\right)$

$= \frac{1}{3\sqrt{1-\frac{x^2}{9}}} = \frac{1}{3\sqrt{\frac{9-x^2}{9}}}$

$= \frac{1}{\sqrt{9-x^2}}$

eg² $\frac{d}{dx}\cos^{-1}\sqrt{x}$

$= \frac{-1}{\sqrt{1-(\sqrt{x})^2}}\frac{d}{dx}(x^{\frac{1}{2}})$

$= \frac{-1}{\sqrt{1-x}}\left(\frac{1}{2}x^{-\frac{1}{2}}\right)$

$= \frac{-1}{2\sqrt{1-x}\sqrt{x}}$

$= \frac{-1}{2\sqrt{x-x^2}}$

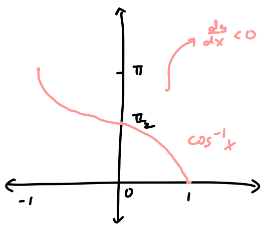
eg³ $\frac{d}{dx}(\sin^{-1}x)^2$

$= 2\sin^{-1}(x)\frac{d}{dx}\sin^{-1}x$

$= 2\sin^{-1}(x)\frac{1}{\sqrt{1-x^2}}$

$= \frac{2\sin^{-1}(x)}{\sqrt{1-x^2}}$

② $\cos^{-1}x$ (domain: -1 to 1)



Let $y = \cos^{-1}x$, ie $\cos y = x$.

$(\cdot \frac{d}{dy}) \therefore -\sin y = \frac{dx}{dy}$

$\sin^2 y = 1 - \cos^2 y$
$= 1 - x^2$
$\therefore \sin y = \sqrt{1-x^2} \quad \because \frac{d}{dx} < 0$

$\Rightarrow \frac{dy}{dx} = \frac{-1}{\sin y}$

$= \frac{-1}{\sqrt{1-x^2}}$

$$\boxed{\therefore \frac{d}{dx}(\cos^{-1}(x)) = \frac{-1}{\sqrt{1-x^2}}}$$

ie $\frac{d}{dx}(\cos^{-1}f(x)) = \frac{-f'(x)}{\sqrt{1-[f(x)]^2}}$

DIFF OF HFUNCTIONS

$\frac{d}{dx}(\cosh x) = \frac{d}{dx}\frac{1}{2}(e^x + e^{-x})$

$= \frac{1}{2}(e^x - e^{-x})$

$$\boxed{\therefore \frac{d}{dx}(\cosh x) = \sinh x.}$$

$\frac{d}{dx}(\sinh x) = \frac{d}{dx}\frac{1}{2}(e^x - e^{-x})$

$= \frac{1}{2}(e^x + e^{-x})$

$$\boxed{\therefore \frac{d}{dx}(\sinh x) = \cosh x.}$$

$\frac{d}{dx}(\tanh x) = \frac{d}{dx}\left(\frac{\sinh x}{\cosh x}\right)$

$= \frac{\frac{d}{dx}(\sinh x)\cosh x - \sinh x \frac{d}{dx}\cosh x}{\cosh^2 x}$

$= \frac{\cosh^2 x - \sinh^2 x}{\cosh^2 x}$

$= \frac{1}{\cosh^2 x} = \text{sech}^2 x$

$$\therefore \boxed{\frac{d}{dx}(\tanh x) = \text{sech}^2 x}$$

DIFF OF INV HFUNCTIONS

① $\sinh^{-1}x$

Let $y = \sinh^{-1}x$

$\Rightarrow x = \sinh y$

$(\cdot \frac{d}{dy})$ $\cosh y = \frac{dx}{dy}$

$\cosh^2 y = 1 + \sinh^2 y$
$= 1 + x^2$

$\therefore \frac{dy}{dx} = \frac{1}{\cosh y}$.

$\Rightarrow \frac{dy}{dx} = \frac{1}{\sqrt{1+x^2}}$.

$$\boxed{\therefore \frac{d}{dx}(\sinh^{-1}x) = \frac{1}{\sqrt{1+x^2}}.}$$

or

$\sin^{-1}(x) = \ln(x + \sqrt{1+x^2})$

$\Rightarrow \frac{d}{dx}(\sin^{-1}(x)) = \frac{1}{x+\sqrt{1+x^2}}\left(1 + \frac{1}{2}(1+x^2)^{-\frac{1}{2}}(2x)\right)$

$= \frac{1}{x+\sqrt{1+x^2}}\left(1 + \frac{x}{\sqrt{1+x^2}}\right)$

$= \frac{1 + \frac{x}{\sqrt{1+x^2}}}{x + \sqrt{1+x^2}}$ $\left(\cdot \frac{\sqrt{1+x^2}}{\sqrt{1+x^2}}\right)$

$= \frac{\sqrt{1+x^2} + x}{\sqrt{1+x^2}(x + \sqrt{1+x^2})}$

$= \frac{1}{\sqrt{1+x^2}}$.

② $\cosh^{-1}x$

Let $y = \cosh^{-1}x$

$\Rightarrow x = \cosh y$.

$(\cdot \frac{d}{dy})$ $\frac{dx}{dy} = \sinh y$

$\sinh^2 y = \cosh^2 y - 1$
$\sinh y = \sqrt{x^2-1}$

$\Rightarrow \frac{dy}{dx} = \frac{1}{\sinh y}$.

$= \frac{1}{\sqrt{x^2-1}}$

$$\boxed{\therefore \frac{d}{dx}(\cosh^{-1}(x)) = \frac{1}{\sqrt{x^2+1}}.}$$

③ $\tanh^{-1}x$

Let $y = \tanh^{-1}x \rightarrow \tanh y = x$

$(\cdot \frac{d}{dy})$ $\operatorname{sech}^2 y = \frac{dx}{dy}$

$\therefore \frac{dy}{dx} = \frac{1}{\operatorname{sech}^2 y}$

$= \frac{1}{1 - \tanh^2 y}$

$\frac{dy}{dx} = \frac{1}{1-x^2}$.

$$\boxed{\therefore \frac{d}{dx}(\tanh^{-1}x) = \frac{1}{1-x^2}}$$

Notice that, if $y = \frac{1}{2}\ln\left(\frac{1+x}{1-x}\right)$

$\frac{dy}{dx} = \frac{d}{dx}\frac{1}{2}(\ln(1+x) - \ln(1-x))$

$= \frac{1}{2}\left(\frac{1}{1+x} - \frac{1}{1-x}\right)$

$= \frac{1}{1-x^2}$. $\left(= \frac{d}{dx}(\tanh^{-1}x)\right)$

$\left(\therefore \tanh^{-1}(x) = \frac{1}{2}\ln\left(\frac{1+x}{1-x}\right)\right).$

eg' diff wrt x $y = \sinh^{-1}x + \tanh^{-1}x$.

Show $\frac{dy}{dx} = 0$ only for one value of $x \in (-1, 1)$

$\Rightarrow \frac{dy}{dx} = \frac{1}{\sqrt{x^2+1}} + \frac{1}{1-x^2}$ $(= 0)$

$(\cdot (1-x^2)(\sqrt{x^2+1}))$

$1 - x^2 + \sqrt{x^2+1} = 0$

$\sqrt{x^2+1} = x^2 - 1$

$(\cdot 2)$ $x^2 + 1 = x^4 - 2x^2 + 1$

$0 = x^4 - 3x^2$

$= x^2(x^2-3)$

$x \neq \pm 3$ $\because$ outside specified domain

$\therefore x = 0$ (if $\frac{dy}{dx} = 0$).

OBTAINING $\frac{d^2y}{dx^2}$ WHERE Y & X IS DEFINED IMPLICITLY

eg1 Find $\frac{d^2y}{dx^2}$ at $(1,2)$ on the curve

$$2x^3 + 2y^3 - 9xy = 0.$$

$$(\cdot \tfrac{d}{dx}) \quad 6x^2 + 6y^2 \frac{dy}{dx} - 9\left[y + x\frac{dy}{dx}\right] = 0$$

$$\Rightarrow (6y^2 - 9x)\frac{dy}{dx} + (6x^2 - 9y) = 0.$$

$$(\div 3) \quad 2y^2\frac{dy}{dx} - 3x\frac{dy}{dx} + 2x^2 - 3y = 0$$ → obtain $\frac{dy}{dx}$ in terms of x & y.

☆ $\frac{d}{dx}\left(2y^2\frac{dy}{dx}\right)$

$= 4y\frac{dy}{dx} \cdot \frac{dy}{dx}$

$= 4y\left(\frac{dy}{dx}\right)^2$!!

$$(\cdot \tfrac{d}{dx}) \left[4y\frac{dy}{dx}\frac{dy}{dx} + 2y^2\frac{d^2y}{dx^2}\right] - \left[3\frac{dy}{dx} + 3x\frac{d^2y}{dx^2}\right] + 4x - 3\frac{dy}{dx} = 0 \quad \text{—(2)}$$ → diff this eq2 again wrt x. (find $\frac{d^2y}{dx^2}$ in terms of $\frac{dy}{dx}$, y & x).

at $(1,2)$, $$\frac{dy}{dx} = \frac{3y - 2x^2}{2y^2 - 3x} = \frac{3(2) - 2(1)^2}{2(2)^2 - 3(1)} = \frac{4}{5}.$$ → find $\frac{dy}{dx}$.

$$(2) \rightarrow 4y\left(\frac{dy}{dx}\right)^2 + 2y^2\frac{d^2y}{dx^2} - 3\frac{dy}{dx} - 3x\frac{d^2y}{dx^2} + 4x - 3\frac{dy}{dx} = 0.$$

$$4(2)\left(\frac{4}{5}\right)^2 + 2(2)^2\frac{d^2y}{dx^2} - 3\left(\frac{4}{5}\right) - 3(1)\frac{d^2y}{dx^2} + 4(1) - 3\left(\frac{4}{5}\right) = 0.$$

$$\Rightarrow \left(2(2)^2 - 3\right)\frac{d^2y}{dx^2} = -\frac{108}{25}$$

$$\therefore \frac{d^2y}{dx^2} = \frac{-108}{125}$$ → find $\frac{d^2y}{dx^2}$.

OBTAINING $\frac{d^2y}{dx^2}$ WHERE Y & X ARE DEFINED PARAMETRICALLY

If $x = f(t)$ & $y = g(t)$,

then $$\frac{dy}{dx} = \frac{\left(\frac{dy}{dt}\right)}{\left(\frac{dx}{dt}\right)}.$$

Subsequently, $$\frac{d^2y}{dx^2} = \frac{d}{dx}\frac{dy}{dx}$$

$$= \frac{\left(\frac{d}{dt}\right)}{\left(\frac{dx}{dt}\right)}\left(\frac{dy}{dx}\right)$$

$$\boxed{\frac{d^2y}{dx^2} = \frac{d}{dt}\left[\frac{dt}{dx}\frac{dy}{dx}\right].}$$

$$= \frac{d}{dt}\left[\frac{dt}{dx}\frac{dy}{dt}\frac{dt}{dx}\right]$$

$$= \frac{d}{dt}\left[\left(\frac{dt}{dx}\right)^2\frac{dy}{dt}\right]$$

eg1 $x = t^2 - 2\ln t$, $y = 4(t-1)$. Find $\frac{d^2y}{dx^2}$.

$$\frac{dy}{dx} = \frac{\left(\frac{dy}{dt}\right)}{\left(\frac{dx}{dt}\right)} = \frac{4}{2t - \frac{2}{t}} \left(\cdot \frac{t}{t}\right) = \frac{4t}{2t^2 - 2}$$

$$= \frac{2t}{t^2 - 1}.$$

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right)$$

$$= \frac{d}{dt}\frac{dt}{dx}\left(\frac{dy}{dx}\right)$$

$$= \frac{d}{dt}\left[\frac{1}{\left(2t - \frac{2}{t}\right)}\right]\left(\frac{2t}{t^2-1}\right)$$

$$= \frac{d}{dt}\left[(2t - 2t^{-1})^{-1}\right]\left(\frac{2t}{t^2-1}\right)$$

$$= -(2t - 2t^{-1})^{-2}(2 + 2t^{-2})\left(\frac{2t}{t^2-1}\right)$$

$$= \frac{-2}{\left(t - \frac{1}{t}\right)^2}\, 2\left(1 + \frac{1}{t^2}\right)\frac{2}{\left(t - \frac{1}{t}\right)}$$

$$= -8\left(1 + \frac{1}{t^2}\right)\left(t - \frac{1}{t}\right)^3.$$

$$= -8\left(1 + \frac{1}{t^2}\right)(t^2 - 1)^3 t^{-3}$$

etc

MACLAURIN SERIES

For any given $f(x)$, it can be shown that

$$\boxed{\begin{aligned} f(x) &= f(0) + xf'(0) + \frac{x^2}{2!}f''(0) + \frac{x^3}{3!}f'''(0) \cdots \\ &= \sum_{k=0}^{\infty} \frac{x^k}{k!} f^{(k)}(0). \end{aligned}}$$

Examples

① e^x. · It is known that $\frac{d^k}{dx^k}e^x = e^x$. $\left(x=0,\ \frac{d^k}{dx^k}e^x = 1\right)$

Hence

$$e^x = \sum_{k=0}^{\infty} \frac{x^k}{k!}(1) = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} \cdots$$

→ in MF19

② $\ln(1+x)$. ($\ln(x)$ is undef at $x=0$ so series undef for it)

→ for $x \geq 1$

- $f(x) = \ln(1+x) \longrightarrow f(0) = \ln(0) = 0$
- $f'(x) = (1+x)^{-1} \longrightarrow f'(0) = 1^{-1} = 1$
- $f''(x) = -(1+x)^{-2} \longrightarrow f''(0) = -1$
- $f'''(x) = 2(1+x)^{-3} \longrightarrow f'''(0) = 2$

etc

Hence, by the Maclaurin series,

$$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3!}(3) \cdots$$

$$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} \cdots$$

→ in MF19

③ $\tanh^{-1}x$

$f(x) = \tanh^{-1}x \longrightarrow f(0) = \tanh^{-1}0 = 0$

$f'(x) = \frac{1}{1-x^2} = (1-x^2)^{-1} \longrightarrow f'(0) = 1^{-1} = 1$

$f''(x) = -(1-x^2)^{-2}(-2x)$

$\quad = 2x(1-x^2)^{-2} \longrightarrow f''(0) = 0$

$f'''(x) = 2(1-x^2)^{-2} + 2x(-2(1-x^2)^{-3}(-2x))$

$\quad = 2(1-x^2)^{-2} + 8x^2(1-x^2)^{-3} \longrightarrow f'''(0) = 2$

$$\therefore \tanh^{-1}x = 0 + 1(x) + 0\frac{x^2}{2!} + 2\frac{x^3}{3!} \cdots$$

$$\tanh^{-1}x = x + \frac{x^3}{3} \cdots$$

or

$y = \tanh^{-1}x$

$\therefore \frac{dy}{dx} = \frac{1}{1-x^2}$

$(1-x^2)\frac{dy}{dx} = 1$

By setting $x=0$, can also obtain $f(0)$, $f'(0)$, $f''(0)$ etc

$\left(\cdot\frac{d}{dx}\right)\ (1-x^2)\frac{d^2y}{dx^2} - 2x\frac{dy}{dx} = 0$

$\left(\cdot\frac{d}{dx}\right)\ (1-x^2)\frac{d^3y}{dx^3} - 2x\frac{d^2y}{dx^2} - 2\frac{dy}{dx} - 2x\frac{d^2y}{dx^2} = 0$

Chapter 4: Integration

SOME IDENTITIES

① $$\int f'(x)\, e^{f(x)}\, dx = e^{f(x)} + C.$$

eg1 $\int x e^{x^2} dx = \frac{1}{2}\int 2x e^{x^2} dx = \frac{1}{2} e^{x^2} + C.$

eg2 $\int x^2 e^{-x^3} dx = -\frac{1}{3}\int -3x^2 e^{-x^3} dx = -\frac{1}{3} e^{-x^3} + C.$

② $$\int f'(x)\, [f(x)]^n\, dx = \frac{(f(x))^{n+1}}{n+1} + C$$

eg1 $\int \cos x \sin^n x \, dx = \frac{\sin^{n+1} x}{n+1} + C$

eg2 $\int x\sqrt{1+x^2}\, dx = \int x(1+x^2)^{\frac{1}{2}} dx$

$= \frac{1}{2}\int 2x(1+x^2)^{\frac{1}{2}} dx$

$= \frac{1}{2}\left[\frac{(1+x^2)^{\frac{3}{2}}}{(\frac{3}{2})}\right] + C$

$= \frac{1}{3}(1+x^2)^{\frac{3}{2}} + C.$

eg3 $\int \frac{\ln x}{x} dx = \int \frac{1}{x}\ln x \, dx$

$= \frac{[\ln x]^2}{2} + C.$

INTEGRATION OF HFUNCTIONS

It can be shown easily that

$$\int \sinh mx \, dx = \frac{1}{m}\cosh mx + C$$
$$\int \cosh mx \, dx = \frac{1}{m}\sinh mx + C$$
$$\int \operatorname{sech}^2 mx \, dx = \frac{1}{m}\tanh mx + C.$$

given in MF19.

⊕ Some hfunct integrals are best evaluated by using their exponential-based defns.

eg1 $\int \sinh^3 x \, dx.$ Let $u = \cosh x \Rightarrow du = \sinh x \, dx.$

$= \int \sinh^2 x \sinh x \, dx$

$= \int (\cosh^2 x - 1)\sinh x \, dx$

$= \int (u^2 - 1)\, du$

$= \frac{u^3}{3} - u + C$

$= \frac{\cosh^3 x}{3} - \cosh x + C.$

eg2 $\int \cosh^3 2x \, dx$ $\quad u = \sinh 2x \Rightarrow du = 2\cosh 2x \, dx$

$= \int \cosh^2 2x \cosh 2x \, dx$

$= \int (1 + \sinh^2 2x)\cosh 2x \, dx$

$= \int (1+u^2)\frac{1}{2} du$

$= \frac{1}{2}\left(u + \frac{u^3}{3}\right) + C$

$= \frac{\sinh 2x}{2} + \frac{\sinh^3 2x}{6} + C.$

eg3 $\int \tanh x \, dx$

$= \int \frac{\sinh x}{\cosh x} dx$ $\quad \left(\int \frac{f'}{f} dx = \ln f\right)$

$= \ln|\cosh x| + C$

eg4 $\int \sinh^2 mx \, dx$

$= \int \frac{1}{2}(\cosh 2x - 1)\, dx$

$= \frac{1}{4}\sinh 2x - x + C.$

eg5 $\int \cosh^2 2x \, dx$

$= \frac{1}{2}\int (\cosh 4x + 1)\, dx$

$= \frac{1}{2}\left(\frac{\sinh 4x}{4} + x\right) + C$

eg6 $\int \tanh^2 x \, dx = \int (1 - \operatorname{sech}^2 x)\, dx$

$= x - \tanh x + C.$

eg7 $\int \operatorname{cosech} x \, dx$

$= \int \frac{1}{\sinh x} dx$

$= \int \frac{2}{e^x - e^{-x}} dx$

$= 2\int \frac{1}{e^x - e^{-x}} dx.$

$= 2\int \frac{e^x}{e^{2x} - 1} dx.$ $\quad$ Let $u = e^x$, $du = e^x dx$

$\Rightarrow 2\int \frac{1}{u^2 - 1} du$

$= -2\tanh^{-1}(e^x) + C.$

eg8 $\int \operatorname{sech} x \, dx$

$= 2\int \frac{1}{e^x + e^{-x}} dx$

$= 2\int \frac{e^x dx}{e^{2x} + 1} dx$

$= 2\int \frac{du}{u^2+1}, \quad u = e^x$

$= 2\tan^{-1}(e^x) + C.$

INTEGRATION OF $\dfrac{1}{\sqrt{ax^2+bx+c}}$

Involves completing the square.

special case → $b = 0$. Let $a, c > 0$.

1) $$\int \frac{1}{\sqrt{a^2-x^2}}\,dx = \sin^{-1}\left(\frac{x}{a}\right) + c.$$

2) $$\int \frac{1}{\sqrt{a^2+x^2}}\,dx = \sinh^{-1}\left(\frac{x}{a}\right) + c.$$

3) $$\int \frac{1}{\sqrt{x^2-a^2}}\,dx = \cosh^{-1}\left(\frac{x}{a}\right) + c.$$

eg[1] $$\int \frac{1}{\sqrt{1-4x^2}}\,dx$$
$$= \frac{1}{2}\int \frac{1}{\sqrt{\frac{1}{4}-x^2}}\,dx$$
$$= \frac{1}{2}\sin^{-1}\left(\frac{x}{\frac{1}{2}}\right) + c$$
$$= \frac{1}{2}\sin^{-1}(2x) + c.$$

or $$= \int \frac{1}{\sqrt{1-(2x)^2}}\,d(2x)\cdot\frac{1}{2}$$
$$= \frac{1}{2}\sin^{-1}(2x) + c.$$

eg[2] $$\int \frac{1}{\sqrt{4-9x^2}}\,dx$$
$$= \frac{1}{2}\int \frac{1}{\sqrt{1-\frac{9}{4}x^2}}\,dx$$
$$= \frac{1}{2}\int \frac{1}{\sqrt{1-\left(\frac{3}{2}x\right)^2}}\,d\left(\frac{3}{2}x\right)\cdot\frac{2}{3}$$
$$= \frac{1}{2}\sin^{-1}\left(\frac{3}{2}x\right)\cdot\frac{2}{3} + c$$
$$= \frac{1}{3}\sin^{-1}\left(\frac{3}{2}x\right) + c.$$

eg[3] $$\int \frac{1}{\sqrt{2x^2-1}}\,dx$$
$$= \frac{1}{\sqrt{2}}\int \frac{1}{\sqrt{(\sqrt{2}x)^2-1}}\,d(\sqrt{2}x)$$
$$= \frac{1}{\sqrt{2}}\cosh^{-1}(\sqrt{2}x) + c$$

eg[4] $$\int \frac{1}{\sqrt{1+4x^2}}\,dx$$
$$= \frac{1}{2}\int \frac{1}{\sqrt{1+(2x)^2}}\,d(2x)$$
$$= \frac{1}{2}\sinh^{-1}(2x) + c.$$

When $b \neq 0$. We need to complete the square.

1) $$\int \frac{1}{\sqrt{(x\pm p)^2-q^2}}\,dx = \cosh^{-1}\left(\frac{x\pm p}{q}\right) + c.$$

2) $$\int \frac{1}{\sqrt{(x\pm p)^2+q^2}}\,dx = \sinh^{-1}\left(\frac{x\pm p}{q}\right) + c$$

3) $$\int \frac{1}{\sqrt{q^2-(x\pm p)^2}}\,dx = \sin^{-1}\left(\frac{x\pm p}{q}\right) + c.$$

*for def integ: $-\frac{\pi}{2}$ to $\frac{\pi}{2}$

eg[5] $$\int_3^5 \frac{1}{\sqrt{15+2x-x^2}}\,dx$$
$$= \int \frac{1}{\sqrt{16-(x-1)^2}}\,dx$$
$$= \int \frac{1}{\sqrt{4^2-(x-1)^2}}\,d(x-1)$$
$$= \left[\sin^{-1}\left(\frac{x-1}{4}\right)\right]_3^5$$
$$= \sin^{-1}(1) - \sin^{-1}\left(\frac{1}{2}\right)$$
$$= \frac{\pi}{2} - \frac{\pi}{6} = \frac{\pi}{3}.$$

$15+2x-x^2$
$= -(x^2-2x)+15$
$= -((x-1)^2-1)+15$
$= 16-(x-1)^2.$

eg[1] $$\int \frac{1}{\sqrt{2x-x^2}}\,dx$$
$$= \int \frac{1}{\sqrt{-(x^2-2x)}}\,dx$$
$$= \int \frac{1}{\sqrt{-[(x-1)^2-1]}}\,dx$$
$$= \int \frac{1}{\sqrt{1-(x-1)^2}}\,dx$$
$$= \int \frac{1}{\sqrt{1-(x-1)^2}}\,d(x-1)$$
$\because dx = d(x-1)$

must be the same! (ie $u = x-1$)
$$= \sin^{-1}(x-1) + c.$$

eg[2] $$\int \frac{1}{\sqrt{x^2-6x+5}}\,dx$$
$$= \int \frac{1}{\sqrt{(x-3)^2-4}}\,dx$$
$$= \int \frac{1}{\sqrt{(x-3)^2-2^2}}\,d(x-3)$$
$$= \cosh^{-1}\left(\frac{x-3}{2}\right) + c.$$

eg[3] $$\int \frac{1}{\sqrt{x^2+6x+10}}\,dx$$
$$= \int \frac{1}{\sqrt{(x+3)^2+1}}\,dx$$
$$= \int \frac{1}{\sqrt{(x+3)^2+1^2}}\,d(x-3)$$
$$= \sinh^{-1}(x+3) + c.$$

eg[4] $$\int_{-1}^{4} \frac{1}{\sqrt{4x^2-12x+34}}\,dx$$
$$= \int \frac{1}{\sqrt{(2x-3)^2+25}}\,dx$$
$$= \frac{1}{2}\int \frac{1}{\sqrt{(2x-3)^2+5^2}}\,d(2x-3)$$
$$= \frac{1}{2}\left[\sinh^{-1}\left(\frac{2x-3}{5}\right)\right]_{-1}^{4}$$
$$= \frac{1}{2}\left(\sinh^{-1}(1) - \sinh^{-1}(-1)\right)$$
$$= \frac{1}{2}\ln\left(\frac{1+\sqrt{2}}{-1+\sqrt{2}}\right)$$
$$= \frac{1}{2}\ln\left(\frac{(1+\sqrt{2})(-1-\sqrt{2})}{1-2}\right)$$
$$= \frac{1}{2}\ln\left(\frac{-(1+\sqrt{2})^2}{-1}\right)$$
$$= \ln(1+\sqrt{2}).$$

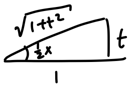
$4x^2-12x+34$
$= 4(x^2-3x)+34$
$= 4\left[\left(x-\frac{3}{2}\right)^2-\frac{9}{4}\right]+34$
$= 4\left(x-\frac{3}{2}\right)^2-9+34$
$= (2x-3)^2+25.$

INTEGRATION USING THE SUBST$^{\underline{N}}$ $t = \tan\frac{1}{2}x$

$t = \tan\frac{1}{2}x \Rightarrow \frac{dt}{dx} = \frac{1}{2}\sec^2\frac{1}{2}x$.

$= \frac{1}{2}\left(1+\tan^2\frac{1}{2}x\right)$

$= \frac{1}{2}\left(1+t^2\right) \longrightarrow \boxed{dx = \frac{2}{1+t^2}\,dt}$.



$\left(\tan\frac{1}{2}x = \frac{t}{1}\right)$

$\Rightarrow \sin\frac{1}{2}x = \frac{t}{\sqrt{1+t^2}}$ and

$\cos\frac{1}{2}x = \frac{1}{\sqrt{1+t^2}}$.

Then, $\sin x = 2\cos\frac{1}{2}x\sin\frac{1}{2}x$

$\boxed{\sin x = \frac{2t}{1+t^2}}$.

$\cos x = \cos^2\frac{x}{2} - \sin^2\frac{x}{2}$

$= \frac{1}{1+t^2} - \frac{t^2}{1+t^2}$

$\boxed{\cos x = \frac{1-t^2}{1+t^2}}$.

in MF19.

eg' $I = \int_0^{\frac{\pi}{2}} \frac{1}{2+\sin x}\,dx$

$t = \tan\frac{1}{2}x \Rightarrow \frac{dt}{dx} = \frac{1}{2}\sec^2\frac{1}{2}x$

$= \frac{1}{2}(1+t^2)$.

$\Rightarrow dx = \frac{2}{1+t^2}\,dt$. $\quad \sin x = \frac{2t}{1+t^2}$.

$\Rightarrow I = \int_0^1 \frac{1}{2+\frac{2t}{1+t^2}}\,\frac{2}{1+t^2}\,dt$

$= \int \frac{2}{2(1+t^2)+2t}\,dt$ $\quad x=\frac{\pi}{2} \to t=1$, $x=0 \to t=0$

$= \int \frac{1}{1+t^2+t}\,dt$

$= \int \frac{1}{\left(t+\frac{1}{2}\right)^2-\frac{1}{4}+1}\,dt$

$= \int \frac{1}{\left(t+\frac{1}{2}\right)^2+\frac{3}{4}}\,dt$

$= \left[\frac{1}{\left(\frac{\sqrt{3}}{2}\right)}\tan^{-1}\left(\frac{t+\frac{1}{2}}{\left(\frac{\sqrt{3}}{2}\right)}\right)\right]_0^1$

$= \frac{2}{\sqrt{3}}\left[\tan^{-1}\sqrt{3} - \tan^{-1}\frac{1}{\sqrt{3}}\right]$

$= \frac{2}{\sqrt{3}}\left(\frac{\pi}{3}-\frac{\pi}{6}\right)$

$= \frac{\sqrt{3}}{9}\pi$.

eg² $I = \int \frac{1}{3-\cos x}\,dx$.

Let $t = \tan\frac{1}{2}x$

$\Rightarrow \frac{dt}{dx} = \frac{1}{2}\sec^2\frac{1}{2}x$

$= \frac{1}{2}(1+t^2)$

$\Rightarrow dx = \frac{2}{1+t^2}\,dt$.

then $\cos x = \frac{1-t^2}{1+t^2}$.

$\Rightarrow I = \int \frac{1}{\left(3-\left(\frac{1-t^2}{1+t^2}\right)\right)}\,\frac{2}{1+t^2}\,dt$

$= \int \frac{2}{3(1+t^2)-(1-t^2)}\,dt$

$= \int \frac{2}{4t^2+2}\,dt$

$= \int \frac{1}{2t^2+1}\,dt$

$= \frac{1}{\sqrt{2}}\int \frac{1}{(\sqrt{2}t)^2+1^2}\,d(\sqrt{2}t)$

$= \frac{1}{\sqrt{2}}\tan^{-1}(\sqrt{2}t) + c$

$= \frac{1}{\sqrt{2}}\tan^{-1}\left(\sqrt{2}\tan\frac{1}{2}x\right) + c$.

APPLICATIONS OF INTEGRATION

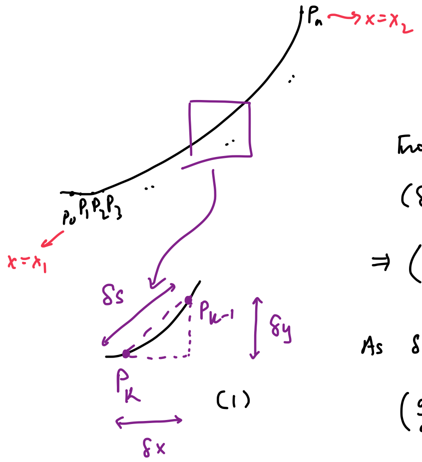
ARC LENGTH OF A CURVE

CARTESIAN

Arc length of curve

$= \overline{P_0P_1} + \overline{P_1P_2} \cdots + \overline{P_{n-1}P_n}$. Let this be "$s$".

$(n \to \infty)$



From (1):

$(\delta s)^2 = (\delta x)^2 + (\delta y)^2$.

$\Rightarrow \left(\frac{\delta s}{\delta x}\right)^2 = 1 + \left(\frac{\delta y}{\delta x}\right)^2$.

As $\delta x \to 0$, the eq^{n} reduces to

$$\left(\frac{ds}{dx}\right)^2 = 1 + \left(\frac{dy}{dx}\right)^2$$

or $$\frac{ds}{dx} = \sqrt{1 + \left(\frac{dy}{dx}\right)^2}.$$

Then $s = \int_{x_1}^{x_2} \frac{ds}{dx}\, dx$,

or $$\boxed{s = \int_{x_1}^{x_2} \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx.}$$

Solving: either

(a) $1 + \left(\frac{dy}{dx}\right)^2$ is square

(b) $\sqrt{1 + \left(\frac{dy}{dx}\right)^2} = \sqrt{ax+b}$

(c) use a subst^{n}

PARAMETRIC

$$\begin{cases} x = f(t) \\ y = g(t). \end{cases}$$

From earlier, we have proved that

$$(\delta s)^2 = (\delta x)^2 + (\delta y)^2.$$

Hence $(\div (\delta t)^2)$ $\Rightarrow \left(\frac{\delta s}{\delta t}\right)^2 = \left(\frac{\delta x}{\delta t}\right)^2 + \left(\frac{\delta y}{\delta t}\right)^2$.

As $\delta x \to 0$ the expression simplifies to

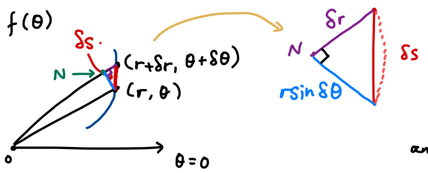
$$\left(\frac{ds}{dt}\right)^2 = \left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2$$

and hence

$$\boxed{s = \int_{t_1}^{t_2} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt.}$$

POLAR

$r = f(\theta)$



$\Rightarrow (\delta s)^2 \approx (\delta r)^2 + (r\sin\delta\theta)^2$

As $\delta\theta \to 0$, $\sin\delta\theta \approx \delta\theta$.

$\Rightarrow (\delta s)^2 = (\delta r)^2 + r^2(\delta\theta)^2$

$\Rightarrow \left(\frac{\delta s}{\delta\theta}\right)^2 = \left(\frac{\delta r}{\delta\theta}\right)^2 + r^2$.

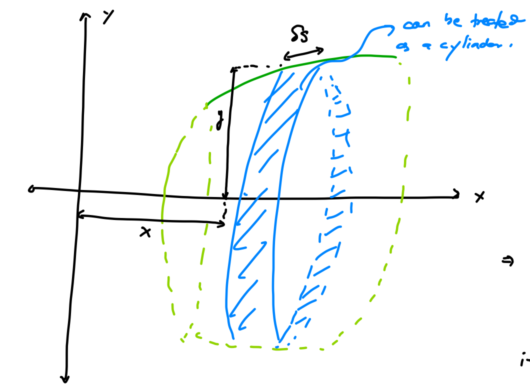
$\Rightarrow \left(\frac{ds}{d\theta}\right)^2 = \left(\frac{dr}{d\theta}\right)^2 + r^2$

or $\frac{ds}{d\theta} = \sqrt{\left(\frac{dr}{d\theta}\right)^2 + r^2}$

and hence

$$\boxed{s = \int_{\theta_1}^{\theta_2} \sqrt{\left(\frac{dr}{d\theta}\right)^2 + r^2}.}$$

AREA OF SURFACE OF REVOLUTION



Surface area, $\delta S \approx 2\pi y\, \delta s$.

$\Rightarrow \dfrac{\delta S}{\delta s} = 2\pi y$.

as $\delta s \to 0$, $\dfrac{\delta S}{\delta s} \to \dfrac{dS}{ds}$.

$\Rightarrow \dfrac{dS}{ds} = 2\pi y$

$dS = 2\pi y\, ds$

ie $S = \int 2\pi y\, ds$.

☆ replace y by x when the curve is rotated about the y-axis.

CARTESIAN

$\dfrac{ds}{dx} = \sqrt{1 + \left(\dfrac{dy}{dx}\right)^2}$ $\therefore$ $ds = \sqrt{1 + \left(\dfrac{dy}{dx}\right)^2}\, dx$.

$$\Rightarrow \boxed{S = \int_{x_1}^{x_2} 2\pi y \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx.}$$

POLAR

$\dfrac{ds}{dt} = \sqrt{\left(\dfrac{dx}{dt}\right)^2 + \left(\dfrac{dy}{dt}\right)^2}$ $\therefore$ $ds = \sqrt{\left(\dfrac{dx}{dt}\right)^2 + \left(\dfrac{dy}{dt}\right)^2}\, dt$.

$$\Rightarrow \boxed{S = \int_{t_1}^{t_2} 2\pi y \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt.}$$

REDUCTION FORMULAE

Certain methods of integration which are applicable to functions involving a power n where n is a positive integer, are viable only when n is relatively small. For instance $\int cos^n x dx$ can be found when $n = 2$ or 4 by using the identity $cos^2 A \equiv \frac{1}{2}(1 + cos 2A)$ as often as necessary, but the same method applied to $\int cos^{10} dx$ would be extremely unwieldy. In cases like this, a means of systematically reducing the value of n is useful, and is called a reduction method. By far the most common method of establishing reduction formulae is by integration by parts.

Throughout this section, let $I_n = \int (f(x))^n dx$.

$\int \sin^n x\, dx$

Proof

$I_n = \int_0^{\frac{\pi}{2}} \sin^n x\, dx$

$= \int_0^{\frac{\pi}{2}} \sin^{n-1} x \cdot \sin x\, dx$.

$u = \sin^{n-1} x$
$\Rightarrow u' = (n-1)\sin^{n-2} x \cos x\, dx$
$v' = \sin x\, dx$
$\therefore v = -\cos x$.

$= \left[-\cos x \sin^{n-1} x\right]_0^{\frac{\pi}{2}} - \int (-\cos x)(n-1)\sin^{n-2} x\, (\cos x)\, dx$

$= 0 + (n-1)\int \cos^2 x \sin^{n-2} x\, dx$

$= 0 + (n-1)\int (1 - \sin^2 x) \sin^{n-2} x\, dx$

$I_n = (n-1)\int \sin^{n-2} x - \sin^n x\, dx$

$= (n-1) I_{n-2} - (n-1) I_n$.

$\therefore nI_n = (n-1) I_{n-2}$

$I_n = \dfrac{(n-1)}{n} I_{n-2}$.

or $I_{n+2} = \dfrac{n+1}{n+2} I_n$. ⟿ the reduction formula.

Example of application.

Suppose we want to find I_5.

Then $I_5 = \frac{4}{5} I_3$ & $I_3 = \frac{2}{3} I_1$,

and $I_1 = \int_0^{\frac{\pi}{2}} \sin x\, dx$

$= \left[-\cos x\right]_0^{\frac{\pi}{2}}$

$= 1$.

Hence $I_5 = \frac{4}{5} \cdot \frac{2}{3} \cdot 1$

$= \frac{8}{15}$.

* similar derivation for $\int \cos^n dx$.

$\int \cosh^n x \, dx$

$I_n = \int \cosh^n x$

$= \int \cosh^{n-1} x \cdot \cosh x$

$u = \cosh^{n-1} x$
$\therefore u' = (n-1)\cosh^{n-2} x \sinh x$
$v' = \cosh x$
$\therefore v = \sinh x.$

$= \sinh x \cosh^{n-1} x$

$- \int \sinh^2 x \ (n-1). \cosh^{n-2} x \, dx$

$I_n = \sinh x \cosh^{n-1} x - (n-1)\int (\cosh^2 x - 1)\cosh^{n-2} x \, dx$

$= \sinh x \cosh^{n-1} x + (n-1)\int \cosh^{n-2} x - \cosh^n x \, dx$

$\Rightarrow I_n = \sinh x \cosh^{n-1} x + (n-1) I_{n-2} - (n-1) I_n$

$\Rightarrow n I_n = \sinh x \cosh^{n-1} x + (n-1) I_{n-2}.$

Application: Find value of $\int_0^{\ln 2} \cosh^5 x \, dx. \quad (= I_5).$

$n I_n = \left[\sinh x \cosh^{n-1} x\right]_0^{\ln 2} + (n-1) I_{n-2}$

$\Rightarrow n I_n = \left[\left(\frac{3}{4}\right)\left(\frac{5}{4}\right)^{n-1} - 0\right] + (n-1) I_{n-2}.$

$\therefore n=5: \quad 5I_5 = \left(\frac{3}{4}\right)\left(\frac{5}{4}\right)^4 + 4I_3$

or $\quad I_5 = \frac{375}{1024} + \frac{4}{5} I_3$ — ①

$n=3: \quad 3I_3 = \left(\frac{3}{4}\right)\left(\frac{5}{4}\right)^2 + 2I_1.$

or $\quad I_3 = \frac{25}{64} + \frac{2}{3} I_1$ — ②

$I_1 = \int_0^{\ln 2} \cosh x \, dx = \left[\sinh x\right]_0^{\ln 2}$

$= \left(\frac{3}{4} - 0\right) = \frac{3}{4}.$

$(\to ②) \quad \therefore I_3 = \frac{25}{64} + \frac{2}{3}\left(\frac{3}{4}\right) = \frac{57}{64}.$

$(\to ①) \quad \therefore I_5 = \frac{375}{1024} + \frac{4}{5}\frac{57}{64}$

$I_5 = \frac{5523}{5120}.$

$\int \cot^n x \, dx$

Let $I_n = \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \cot^n x \, dx$

$= \int \cot^{n-2} x \cdot \cot^2 x \, dx$

split into $\cot^2$
(compared to sin, cosh, where you split into $\cdot^1$)

$= \int \cot^{n-2} x \cdot \left[\operatorname{cosec}^2 x - 1\right] dx$

$= \int \cot^{n-2} x \operatorname{cosec}^2 x - \int \cot^{n-2} x \, dx$

$\Rightarrow I_n = -\int (-\operatorname{cosec}^2 x)(\cot x)^{n-2} dx - \int \cot^{n-2} x \, dx$

$\Rightarrow I_n = -\left[\frac{(\cot x)^{n-1}}{n-1}\right]_{\frac{\pi}{4}}^{\frac{\pi}{2}} - I_{n-2}.$

$I_n = -\left(0 - \frac{1}{n-1}\right) - I_{n-2}$

ie $\quad I_n = \frac{1}{n-1} - I_{n-2}.$

$\int \tanh^n x \, dx$

$\frac{c^2 - s^2 = 1}{c^2 \quad c^2 \quad c^2}$

Let $I_n = \int_0^{\ln 2} \tanh^n x \, dx.$

$\Rightarrow I_n = \int \tanh^{n-2} x \cdot \tanh^2 x \, dx$

$= \int \tanh^{n-2} x \cdot (1 - \operatorname{sech}^2 x) \, dx$

$= \int \tanh^{n-2} x \, dx - \int \tanh^{n-2} x \operatorname{sech}^2 x \, dx$

$= \int \tanh^{n-2} x \, dx - \int (\tanh x)^{n-2} \operatorname{sech}^2 x \, dx$

derivative of this

$\Rightarrow I_n = I_{n-2} - \left[\frac{(\tanh x)^{n-1}}{n-1}\right]_0^{\ln 2}$

$= I_{n-2} - \left(\frac{\left(\frac{3}{5}\right)^{n-1}}{(n-1)} - 0\right)$

ie $\quad I_n = I_{n-2} - \frac{1}{n-1}\left(\frac{3}{5}\right)^{n-1}.$ QED.

$\int \operatorname{cosec}^n x \, dx$

$\frac{c^2 + s^2 = 1}{s^2 \quad s^2 \quad sc}$

$I_n = \int \operatorname{cosec}^n x \, dx$

$= \int \operatorname{cosec}^{n-2} x \cdot \operatorname{cosec}^2 x \, dx$

$u = \operatorname{cosec}^{n-2} x \qquad v' = \operatorname{cosec}^2 x$
$u' = (n-2)\operatorname{cosec}^{n-3} x \, (-\operatorname{cosec} x \cot x)$
$v = -\cot x$

$\Rightarrow I_n = \operatorname{cosec}^{n-2} x \cdot -\cot x - \int (n-2)(-\operatorname{cosec}^{n-2} x)(-\cot^2 x) \, dx$

$= -\cot x \operatorname{cosec}^{n-2} x - (n-2)\int \operatorname{cosec}^{n-2} x \cot^2 x \, dx$

$= -\cot x \operatorname{cosec}^{n-2} x - (n-2)\int \operatorname{cosec}^{n-2} x \, (\operatorname{cosec}^2 x - 1) \, dx$

$= -\cot x \operatorname{cosec}^{n-2} x - (n-2)\int \operatorname{cosec}^n x \, dx + (n-2)\int \operatorname{cosec}^{n-2} x \, dx$

$\Rightarrow I_n = -\cot x \operatorname{cosec}^{n-2} x - (n-2) I_n + (n-2) I_{n-2}$

$\Rightarrow (n-1) I_n = -\cot x \operatorname{cosec}^{n-2} x + (n-2) I_{n-2}.$

$\int \operatorname{sech}^n x \, dx$

$\frac{d}{dx}\operatorname{sech} x = \frac{d}{dx}(\cosh x)^{-1}$
$= -(\cosh x)^{-2}(\sinh x)$
$= -\operatorname{sech} x \tanh x$

$I_n = \int_0^{\ln 2} \operatorname{sech}^n x \, dx$

$u = \operatorname{sech}^{n-2} x$
$u' = (n-2)\operatorname{sech}^{n-3} x \left[-\operatorname{sech} x \tanh x\right]$
$v' = \operatorname{sech}^2 x$
$v = \tanh x$

$= \int \operatorname{sech}^{n-2} x \cdot \operatorname{sech}^2 x \, dx$

$\frac{c^2 - s^2 = 1}{c^2 \quad c^2 \quad c^2}$
$1 - \tanh^2 = \operatorname{sech}^2$

$I_n = \tanh x \operatorname{sech}^{n-2} x - \int (n-2)(-\operatorname{sech}^{n-2} x)(\tanh^2 x) \, dx$

$\Rightarrow I_n = \left[\tanh x \operatorname{sech}^{n-2} x\right]_0^{\ln 2} + (n-2)\int \operatorname{sech}^{n-2} x \tanh^2 x \, dx$

$= \left\{\left(\frac{4}{5}\right)^{n-2}\left(\frac{3}{5}\right) - 0\right\} + (n-2)\int \operatorname{sech}^{n-2} x \left[1 - \operatorname{sech}^2 x\right] dx$

$= \left(\frac{3}{5}\right)\left(\frac{4}{5}\right)^{n-2} + (n-2)\int \operatorname{sech}^{n-2} x - (n-2)\int \operatorname{sech}^n x \, dx$

$\Rightarrow I_n = \left(\frac{3}{5}\right)\left(\frac{4}{5}\right)^{n-2} + (n-2) I_{n-2} - (n-2) I_n$

or $\quad (n-1) I_n = \left(\frac{3}{5}\right)\left(\frac{4}{5}\right)^{n-2} + (n-2) I_{n-2}.$

OTHER CASES

Ex1. Let $I_n = \int_0^1 t^n e^{-t}dx$, $n \geq 0$. Prove that for $n \geq 1$, $I_n = -e^{-1} + nI_{n-1}$.

(i) For a general (x, y) on a curve C it is known that $\frac{dx}{dt} = t^4e^{-t}\cos t$ and $\frac{dy}{dt} = t^4e^{-t}\sin t$

where t is a real parameter. Show that the length of that part of C from the point where $t = 0$ to the point where $t = 1$ is equal $24 - 65e^{-1}$.

$I_n = \int_0^1 t^n e^{-t} dx$

$u = t^n \quad v' = e^{-t}$
$u' = nt^{n-1} \quad v = -e^{-t}$

$\Rightarrow I_n = \left[-te^{-t}\right]_0^1 - \int -e^{-t} \cdot nt^{n-1}$

$= (0 - e^{-1}) + nI_{n-1}$

$\Rightarrow I_n = -e^{-1} + nI_{n-1}.$

(i) $s = \int_0^1 \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$

$= \int \sqrt{(t^4e^{-t})^2(\cos^2 t + \sin^2 t)}\, dt$

$= \int t^4 e^{-t} dt$

$= I_4.$

Then,

$(n=4) \quad I_4 = -e^{-1} + 4I_3$ — ④

$I_3 = -e^{-1} + 3I_2$ — ③

$I_2 = -e^{-1} + 2I_1$ — ②

$I_1 = -e^{-1} + I_0$ — ①

and $I_0 = \int_0^1 e^{-t}dt = \left[-e^{-t}\right]_0^1 = 1 - e^{-1}$

(resubstitution eventually gives $I_4 = s = 24 - 65e^{-1}$.

Ex2. Given that $I_n = \int_0^1 (1 - x^2)^n dx, n \geq 0$.

Show that $I_n = \frac{2n}{2n+1} I_{n-1}, n \geq 1$.

(i) Hence evaluate I_5, leaving your answer as a fraction in its lowest terms.

$I_n = \int_0^1 (1-x^2)^n dx \qquad u = (1-x^2)^n \quad v' = 1$
$u' = n(1-x^2)^{n-1}(-2x) \quad v = x.$
$= -2nx(1-x^2)^{n-1}$

$\Rightarrow I_n = \left[x(1-x^2)^n\right]_0^1 - \int_0^1 x(-2nx(1-x^2)^{n-1})\, dx$

$= \left[x(1-x^2)^n\right]_0^1 + 2n\int x^2(1-x^2)^{n-1} dx$

$= (0) + 2n\int (x^2-1)(1-x^2)^{n-1} + (1-x^2)^{n-1} dx$

$= 2n\int -(1-x^2)^n + (1-x^2)^{n-1} dx$

$I_n = -2nI_n + 2nI_{n-1}.$

ie $(2n+1)I_n = 2nI_{n-1}$

or $I_n = \frac{2n}{2n+1} I_{n-1}.$

(i) $I_5 = \left(\frac{10}{11}\right)\left(\frac{8}{9}\right)\left(\frac{6}{7}\right)\left(\frac{4}{5}\right)\left(\frac{2}{3}\right)I_0$

$I_0 = \int_0^1 1\, dx = 1$

$\therefore I_5 = \frac{10 \times 8 \times 6 \times 4 \times 2}{11 \times 9 \times 7 \times 5 \times 3}$

$= \frac{256}{693}.$

Ex3. If $I_n = \int_0^1 x^n e^{-x^2} dx$, prove that $2I_n = (n-1)I_{n-2} - e^{-1}$, $n \geq 2$.

Hence, find I_5, leaving your answer in terms of e.

$I_n = \int_0^1 x^n e^{-x^2} dx \qquad u = e^{-x^2} \quad v' = x^n$
$u' = -2xe^{-x^2} \quad v = \frac{x^{n+1}}{n+1}$

$\Rightarrow I_n = \left[\frac{e^{-x^2}x^{n+1}}{n+1}\right]_0^1 - \int_0^1 \frac{-2x^{n+2}e^{-x^2}}{n+1} dx$

$I_n = \left(\frac{1}{n+1}e^{-1} - 0\right) + \frac{2}{n+1} I_{n+2}.$

$(\cdot (n+1))$

$(n+1)I_n = e^{-1} + 2I_{n+2}$

$n = n-2 \quad \Rightarrow (n-1)I_{n-2} = e^{-1} + 2I_n$

ie $2I_n = (n-1)I_{n-2} - e^{-1}.$

$I_1 = \int_0^1 xe^{-x^2}$

$= -\frac{1}{2}\int_0^1 (-2x)e^{-x^2}$

$= -\frac{1}{2}\left[e^{-x^2}\right]_0^1$

$= -\frac{1}{2}(e^{-1} - 1).$

Can use successive subst^n to solve (i)

OBTAINING THE REDUCTION FORMULA BY INTEGRATING BY PARTS TWICE

Ex1. Given that $I_n = \int_0^{\frac{\pi}{2}} e^{-x}\sin^n x\, dx$,

show that show that $(n^2 + 1)\, I_n = -e^{-\frac{\pi}{2}} + n(n-1)I_{n-2},\ n \geq 2$.

Hence show that $I_4 = \frac{1}{85}\left(24 - 41e^{-\frac{\pi}{2}}\right)$.

$I_n = \int_0^{\frac{\pi}{2}} e^{-x}\sin^n x\, dx \qquad u = \sin^n x \quad v' = e^{-x}$
$u' = n\sin^{n-1}x\cos x \quad v = -e^{-x}$

$I_n = \left[-e^{-x}\sin^n x\right]_0^{\frac{\pi}{2}} - \int_0^{\frac{\pi}{2}} -e^{-x} \cdot n\sin^{n-1}x \cdot \cos x\, dx$

$\Rightarrow I_n = \left(0 - e^{-\frac{\pi}{2}}\right) + n\int_0^{\frac{\pi}{2}} e^{-x}\sin^{n-1}x\cos x\, dx$

$u = \sin^{n-1}x\cos x$

$u' = (n-1)\sin^{n-2}x\cos^2 x + \sin^{n-1}x(-\sin x)$

$= (n-1)\sin^{n-2}x\cos^2 x - \sin^n x$

$= (n-1)\sin^{n-2}x(1 - \sin^2 x) - \sin^n x$

$= (n-1)\sin^{n-2}x + (1-n)\sin^n x - \sin^n x$

$= (n-1)\sin^{n-2}x - n\sin^n x$

$v' = e^{-x}$

$v = -e^{-x}$

$I_n = -e^{-\frac{\pi}{2}}$

$+ n\left\{\left[-e^{-x}\sin^{n-1}x\cos x\right]_0^{\frac{\pi}{2}} - \int_0^{\frac{\pi}{2}} -e^{-x}\left[(n-1)\sin^{n-2}x - n\sin^n x\right] dx\right\}$

$\Rightarrow I_n = -e^{-\frac{\pi}{2}} + n(0 - 0) + n\int_0^{\frac{\pi}{2}} (n-1)e^{-x}\sin^{n-2}x - n\int_0^{\frac{\pi}{2}} ne^{-x}\sin^n dx$

$\Rightarrow I_n = -e^{-\frac{\pi}{2}} + n(n-1)I_{n-2} - n^2 I_n$

ie $(n^2+1)I_n = -e^{-\frac{\pi}{2}} + n(n-1)I_{n-2}.$ QED.

$n=4 \Rightarrow (17)I_4 = -e^{-\frac{\pi}{2}} + 4(3)I_2$ —①

$n=2 \Rightarrow (5)I_2 = -e^{-\frac{\pi}{2}} + 2(1)I_0$ —②

$I_0 = \int_0^{\frac{\pi}{2}} e^{-x}dx$

$= \left[-e^{-x}\right]_0^{\frac{\pi}{2}}$

$= 1 - e^{-\frac{\pi}{2}}.$

$(\to ②) \quad 5I_2 = -e^{-\frac{\pi}{2}} + 2\left(1 - e^{-\frac{\pi}{2}}\right).$

$(\to ①) \quad 17I_4 = -e^{-\frac{\pi}{2}} + \frac{12}{5}\left[e^{-\frac{\pi}{2}} + 2 - 2e^{-\frac{\pi}{2}}\right]$

$17I_4 = -e^{-\frac{\pi}{2}} + \frac{12}{5}\left(2 - e^{-\frac{\pi}{2}}\right)$

$= \frac{24}{5} - \frac{17}{5}e^{-\frac{\pi}{2}}$

ie $I_4 = \frac{24}{85} - \frac{1}{5}e^{-\frac{\pi}{2}}.$

Ex2. Given that $I_n=\int_0^1(1-x)^n\sinh x\,dx$,

Show that, for $n\geq 2$, $I_n=-1+n(n-1)I_{n-2}$.

Hence find the exact value of I_4.

$$I_n=\int_0^1(1-x)^n\sinh x\,dx.$$

$u=(1-x)^n$ $\quad v'=\sinh x$

$u'=n(1-x)^{n-1}(-1)$ $\quad v=\cosh x$

$$\Rightarrow I_n=\left[(1-x)^n\cosh x\right]_0^1-\int_0^1\cosh x\left[-n(1-x)^{n-1}\right]dx$$

$$=(0-1)+n\int_0^1\cosh x\,(1-x)^{n-1}dx$$

$u=(1-x)^{n-1}$ $\quad v'=\cosh x$

$u'=(n-1)(1-x)^{n-2}(-1)$ $\quad v=\sinh x$

$$\Rightarrow I_n=-1+n\left\{\left[\sinh x(1-x)^{n-1}\right]_0^1-\int_0^1\sinh x\cdot(-(n-1)(1-x)^{n-2})\,dx\right\}$$

$$=-1+n\left\{0+(n-1)\int_0^1\sinh x\,(1-x)^{n-2}dx\right\}$$

$$I_n=-1+n(n-1)I_{n-2}.\quad \underline{QED}.$$

$(n=4)$ $I_4=-1+12I_2$ — ②

$(n=2)$ $I_2=-1+2I_0$ — ①

and $I_0=\int_0^1\sinh x\,dx$

$=[\cosh x]_0^1$

$=\cosh 1-\cosh 0$

$=\cosh 1-1.$ (Let this be "a")

$(\to①)$ $I_2=-1+2a$

$(\to②)$ $I_4=-1+12(-1+2a)$

$=-13+24a$

$=-13+24(\cosh 1-1)$

$=-37+24\cosh 1$

$=-37+24\left(\frac12(e^1+e^{-1})\right)$

$I_4=-37+12e+12e^{-1}.$

OTHER METHODS.

Ex1. It is given that $I_n=\int_0^1x^ne^{-x^3}dx$, $n\geq 0$. By considering $\frac{d}{dx}(x^{n+1}e^{-x^3})$, show that $I_{n+3}=\frac13(n+1)I_n-\frac13e^{-1}$. Hence evaluate I_8, leaving your answer in terms of e.

$$\frac{d}{dx}(x^{n+1}e^{-x^3})$$

$$=(n+1)x^ne^{-x^3}+x^{n+1}(-3x^2)e^{-x^3}$$

$$\frac{d}{dx}(x^{n+1}e^{-x^3})=(n+1)x^ne^{-x^3}-3x^{n+3}e^{-x^3}.$$

$$\Rightarrow\int_0^1\frac{d}{dx}(x^{n+1}e^{-x^3})dx=\int_0^1(n+1)x^ne^{-x^3}-3x^{n+3}e^{-x^3}dx$$

or $\left[x^{n+1}e^{-x^3}\right]_0^1=(n+1)I_n-3I_{n+3}$

$\Rightarrow$ $(e^{-1}-0)=(n+1)I_n-3I_{n+3}$

or $I_{n+3}=\frac13(n+1)I_n-\frac13e^{-1}.$

$(n=5)$ $I_8=\frac13(5+1)I_5-\frac13e^{-1}$

$I_8=2I_5-\frac13e^{-1}.$ — ②

$(n=2)$ $I_5=\frac13(2+1)I_2-\frac13e^{-1}$

$I_5=I_2-\frac13e^{-1}$ — ①

and $I_2=\int_0^1x^2e^{-x^3}dx$

$=-\frac13\int_0^1(-3x^2)e^{(-x^3)}dx$

$=-\frac13\left[e^{-x^3}\right]_0^1$

$=-\frac13(e^{-1}-1).$

Let $-\frac13(e^{-1}-1)=x.$

$\Rightarrow I_5=x-\frac13e^{-1}.$

$\Rightarrow I_8=2(x-\frac13e^{-1})-\frac13e^{-1}$

$=2x-e^{-1}$

$=2(-\frac13(e^{-1}-1))-e^{-1}$

$=-\frac23(e^{-1}-1)-e^{-1}$

$=\frac23-\frac53e^{-1}.$

Ex2. Given that $I_n=\int_1^e(1+\ln x)^ndx$. By considering $\frac{d}{dx}(x(1+\ln x)^{n+1})$ or otherwise, show that $I_{n+1}=(2^{n+1})e-1-(n+1)I_n$. Hence evaluate $\int_1^e(5+\ln x)(1+\ln x)^2dx$.

$$\frac{d}{dx}x(1+\ln x)^{n+1}$$

$$=(1+\ln x)^{n+1}+x(n+1)(1+\ln x)^n\frac1x$$

$$=(1+\ln x)^{n+1}+(n+1)(1+\ln x)^n.$$

Hence $I_{n+1}+(n+1)I_n=\int_1^e(1+\ln x)^{n+1}+(n+1)(1+\ln x)^n\,dx$

$=\left[x(1+\ln x)^{n+1}\right]_1^e$

$I_{n+1}+(n+1)I_n=\left(e(2)^{n+1}-1\right)$

ie $I_{n+1}=e\cdot2^{n+1}-1-(n+1)I_n.$

$I=\int_1^e(5+\ln x)(1+\ln x)^2dx$

$=\int_1^e4(1+\ln x)^2dx+\int_1^e(1+\ln x)^3dx$

$=4I_2+I_3.$

$n=2\to I_3=e\cdot2^3-1-3I_2$ — ②

$n=1\to I_2=e\cdot2^2-1-2I_1$ — ①

$n=0\to I_1=e\cdot2^1-1-I_0$ — ⓪

$I_0=\int_1^e(1)\,dx=e-1.$

$(\to⓪)$ $I_1=2e-1-(e-1)$

$=e.$

$(\to①)$ $I_2=4e-1-2e$

$=2e-1$

$(\to②)$ $I_3=8e-1-3(2e-1)$

$=2e+2.$

$\therefore I=4(2e-1)+(2e+2)$

$=10e-2.$

Ex3. Given that $I_n = \int_0^1 \left(1+x^2\right)^{-n} dx \quad n \geq 1$,

show by considering $\frac{d}{dx}\left\{x(1+x^2)^{-n}\right\}$, or otherwise, that

$2nI_{n+1} = 2^{-n} + (2n-1)I_n$.

Hence, or otherwise, find the value of I_3, leaving your answer in terms of π.

$$\frac{d}{dx} x(1+x^2)^{-n}$$
$$= (1+x^2)^{-n} + x(-n)(1+x^2)^{-n-1}(2x)$$
$$= (1+x^2)^{-n} - 2nx^2(1+x^2)^{-n-1}$$
$$= (1+x^2)^{-n} - 2n\left[x^2(1+x^2)^{-n-1}\right]$$
$$= (1+x^2)^{-n} - 2n\left[(x^2+1)(1+x^2)^{-n-1} - 1(1+x^2)^{-n-1}\right]$$
$$= (1+x^2)^{-n} - 2n\left[+(1+x^2)^{-n} - (1+x^2)^{-n-1}\right]$$
$$= (1+x^2)^{-n} - 2n(1+x^2)^{-n} + 2n(1+x^2)^{-n-1}$$
$$= (1-2n)(1+x^2)^{-n} + 2n(1+x^2)^{-n-1}.$$

Hence $(1-2n)I_n + 2nI_{n+1}$
$$= \int_0^1 (2n+1)(1+x^2)^{-n} - 2n(1+x^2)^{-n-1}\, dx$$
$$= \left[x(1+x^2)^{-n}\right]_0^1$$

ie $(1-2n)I_n + 2nI_{n+1} = (2^{-n} - 0)$

or $2nI_{n+1} = (2n-1)I_n + 2^{-n}$.

$n=2 \rightarrow 4I_3 = 3I_2 + 2^{-2}$ — ②

$n=1 \rightarrow 2I_2 = I_1 + 2^{-1}$ — ①

$$I_1 = \int_0^1 \frac{1}{1+x^2}\, dx$$
$$= \left[\tan^{-1}x\right]_0^1$$
$$= \left(\frac{\pi}{4} - 0\right) = \frac{\pi}{4}$$

(→①) $2I_2 = \frac{\pi}{4} + \frac{1}{2}$

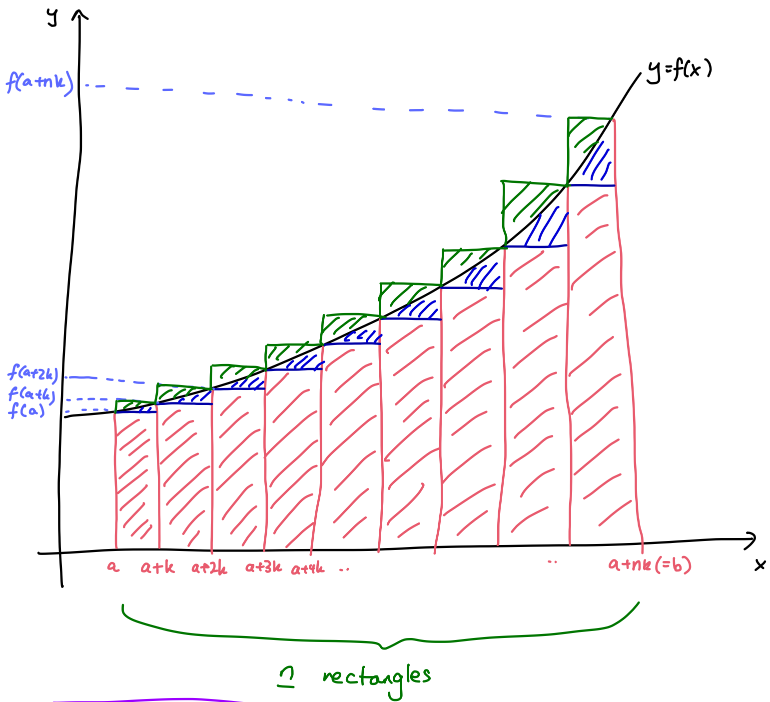
$\therefore I_2 = \frac{\pi}{8} + \frac{1}{4}$.

(→②) $4I_3 = 3\left(\frac{\pi}{8} + \frac{1}{4}\right) + \frac{1}{4}$

$= \frac{3\pi}{8} + 1$.

$$\boxed{\therefore I_3 = \frac{3\pi}{32} + \frac{1}{4}.}$$

RECTANGLE APPROXIMATION TO THE AREA UNDER THE CURVE



The area of the + is

$$\int_a^b f(x)\,dx.$$

The combined area of the " " rectangles is equal to

$$\sum_{i=0}^{n-1} k \cdot f(a+ik), \quad \text{where } k = \frac{b-a}{n}.$$

The combined area of the + + rectangles is equal to

$$\sum_{i=1}^{n} k \cdot f(a+ik), \quad \text{where } k = \frac{b-a}{n}.$$

< + < + +

hence

$$\sum_{i=0}^{n-1} f(a+ik) < \int_a^b f(x)\,dx < \sum_{i=0}^{n-1} f(a+(i+1)k).$$

$k = \frac{b-a}{n}$ so

$$\boxed{\sum_{i=0}^{n-1} f\left(a + i\cdot\frac{b-a}{n}\right) < \int_a^b f(x)\,dx < \sum_{i=0}^{n-1} f\left(a + (i+1)\frac{b-a}{n}\right).}$$

↳ as $n \to \infty$, by the squeeze theorem, both bounds approach the definite integral.

Chapter 5: Differential Equations

1st ORDER LINEAR DIFFERENTIAL EQNS

A "differential eqn" in x, y & derivatives of y is linear if y and its derivatives all appear in linear form, w/ no products or powers.

↳ ie it takes the form

$$F(x)\frac{dy}{dx} + G(x)y = H(x).$$

ie $\boxed{\dfrac{dy}{dx} + \dfrac{G(x)}{F(x)}y = \dfrac{H(x)}{F(x)}}$

or $\boxed{\dfrac{dy}{dx} + [P(x)]y = [Q(x)].}$

WORKED EXAMPLE

solve the differential equation $x\dfrac{dy}{dx}+y=e^x$.

step 1 Find $R(x) = e^{\int P dx}$;

$x\dfrac{dy}{dx} + y = e^x.$

$(\div x)$ $\therefore \dfrac{dy}{dx} + \dfrac{y}{x} = \dfrac{1}{x}e^x.$

$\therefore$ IF $= e^{\int P dx} = e^{\int \frac{1}{x} dx} = e^{\ln x}$
$= x.$

step 2 multiply both sides by $R(x)$; integrate both sides wrt x.

$\boxed{e^{\int P dx}\cdot y = \int Q\cdot e^{\int P dx}\, dx.}$ → general eqn.

$(x)\cdot y = \int \frac{1}{x}\cdot x\, dx$
$= x + c$

$\Rightarrow xy = x + c$ ie $y = 1 + \frac{c}{x}$.

SOLVING 1st ORDER LINEAR DIFF EQNS.

We can find a function R(x) such that $R(x)\cdot \text{LHS} = \frac{d}{dx}(R(x)y)$;

ie $R(x)\left[\dfrac{dy}{dx} + Py\right] = \dfrac{d}{dx}(R(x)y)$

$R(x)\dfrac{dy}{dx} + Py\,R(x) = \dfrac{d}{dx}(R(x)y)$
$= R(x)\dfrac{dy}{dx} + R'(x)\cdot y$

$\Rightarrow Py\,R(x) = R'(x)y$

$\Rightarrow R(x)\cdot P = R'(x)$

or $P\left(= \dfrac{G(x)}{F(x)}\right) = \dfrac{R'(x)}{R(x)}.$

Thus $\int P\,dx = \int \dfrac{R'(x)}{R(x)}dx = \ln(R(x)),$

or $R(x) = e^{\int P dx}.$

↳ we denote R(x) as the "integrating factor".

By equating the LHS and RHS;

$\dfrac{d}{dx}(R(x)y) = R(x)\cdot Q$

$\Rightarrow R(x)\cdot y = \int Q\cdot R(x)\,dx$

ie $\boxed{e^{\int P dx}\cdot y = \int e^{\int P dx}\cdot Q\,dx}$

USING A GIVEN SUBSTN TO REDUCE A DIFF EQN TO A 1st ORDER LINEAR EQN OR TO 1st ORDER EQN WITH SEPARABLE VARIABLES

Ex 1 Find the general solution of $(y-x)\frac{dy}{dx} = y$, using the substn $v = y - x$. (express as $x = f(y)$)

$v = y - x$

$\therefore \dfrac{dv}{dx} = \dfrac{dy}{dx} - 1.$

ie $\dfrac{dy}{dx} = \dfrac{dv}{dx} + 1.$

$\Rightarrow v\left(\dfrac{dv}{dx} + 1\right) = (v + x).$

$\Rightarrow v\dfrac{dv}{dx} + v = v + x.$

$v\dfrac{dv}{dx} = x$

or $\int v\,dv = \int x\,dx$

$\dfrac{v^2}{2} = \dfrac{x^2}{2} + c,\ c\in\mathbb{R}.$

$\Rightarrow v^2 = x^2 + c_1,\ c_1 = 2c.$

$(y-x)^2 = x^2 + c_1$

$y^2 - 2xy + x^2 = x^2 + c_1.$

$y^2 - c_1 = 2xy$

ie $x = \dfrac{y^2 - c_1}{2y}.$

Ex 2 Using the substn $u = y^2$, find the general solution of the differential eqn $2xy\frac{dy}{dx} = y^2 - 4x^2$.

$u = y^2 \quad \therefore \dfrac{du}{dx} = 2y\dfrac{dy}{dx}.$

$\Rightarrow x\dfrac{du}{dx} = u - 4x^2.$

$\dfrac{du}{dx} = \left(\dfrac{1}{x}\right)u - 4x.$

$\Rightarrow \dfrac{du}{dx} + \left(-\dfrac{1}{x}\right)u = (-4x). \text{—} (*)$

$\Rightarrow$ integrating factor, $F_i = e^{\int P dx}$
$= e^{\int\left(-\frac{1}{x}\right)dx}$
$= e^{-\ln x}$
$= \frac{1}{x}.$

multiply both sides of (*) by F_i and integrate both sides;

$F_i\cdot u = \int F_i\cdot Q\,dx$

$\frac{1}{x}\cdot u = \int \frac{1}{x}(-4x)\,dx$
$= \int -4\,dx$
$= -4x + c.$

$\Rightarrow u = -4x^2 + cx.$

$\Rightarrow \underline{\underline{y^2 = -4x^2 + cx,\ c\in\mathbb{R}.}}$

SECOND ORDER LINEAR DIFF EQNS WITH CONSTANT COEFF

FIRST ORDER LINEAR HOMOGENOUS DE

These are of the form

$$\boxed{a\frac{dy}{dx} + by = 0.}$$

$\hookrightarrow \frac{dy}{dx} = -\frac{b}{a}y$

$\Rightarrow \int \frac{1}{y}\,dy = \int -\frac{b}{a}\,dx$

$\ln|y| = -\frac{b}{a}x + c$

$\Rightarrow y = \pm e^{c} e^{-\frac{b}{a}x}$

$\Rightarrow \boxed{y = Ae^{-\frac{b}{a}x}, \quad A = \pm e^{c}.}$ → soln to the diff eqn.

⇒ This shows the solution to any 1st order eqn of this type will contain an exponential function and an unknown constant.

This is known as the auxiliary equation.

If we know the form of the solution, we can find it without doing any integration.

Example. Solve $a\frac{dy}{dx} + by = 0$.

We know $y = Ae^{mx}$, $A, m \in \mathbb{R}$.

$\Rightarrow a(Ame^{mx}) + b(Ae^{mx}) = 0.$

$(\div Ae^{mx})$ $\quad am + b = 0$

or $m = -\frac{b}{a}$. (as above).

2ND ORDER LINEAR DIFF HOMOGENOUS EQNS WITH CONSTANT COEFFICIENTS.

1. We want to find the general solution to the diff eqn

$$\boxed{a\frac{d^2y}{dx^2} + b\frac{dy}{dx} + cy = 0.}$$

Using the auxiliary method, we can assume that $y = Ae^{mx}$. $\rightarrow \frac{dy}{dx} = Ame^{mx}$

$\frac{d^2y}{dx^2} = Am^2e^{mx}$.

Hence,

$a(Am^2e^{mx}) + b(Ame^{mx}) + c(Ae^{mx}) = 0$

$(\div Ae^{mx}) \quad \Rightarrow \quad \boxed{am^2 + bm + c = 0.}$ → this is the "auxiliary quadratic eqn" (AQE).

2. The AQE could have been written down immediately without any intermediate working, because it is very similar to the original DE.

3. The two solutions can be written as

$y = Ae^{m_1x}$ and $y = Be^{m_2x}$.

4. For any second order linear homogenous DE, the general solution of the DE is the sum of the two solutions; ie

$$\boxed{y = Ae^{m_1x} + Be^{m_2x}.}$$

(the complementary function)

case #1: $m_1, m_2 \in \mathbb{R}$ $(\Delta > 0)$

$$\Rightarrow \boxed{y = Ae^{m_1x} + Be^{m_2x}.}$$

case #2: $m_1 = m_2$ $(\Delta = 0)$

The general soln is NOT $Ae^{m_1x} + Be^{m_1x}$. Why? $\Rightarrow (A+B)e^{m_1x}$. One soln only.

instead:

Ex consider $\frac{d^2y}{dx^2} - 4\frac{dy}{dx} + 4y = 0$.

AQE $\Rightarrow m^2 - 4m + 4 = 0$

$(m-2)^2 = 0$

$\Rightarrow m = 2.$

$\Rightarrow$ a solution is $y = Ae^{2x}$.

Now consider whether $y = Bxe^{2x}$ is a soln?

$\Rightarrow \frac{dy}{dx} = B(e^{2x} + 2xe^{2x})$

$\frac{dy}{dx} = Be^{2x} + 2y$

$\Rightarrow \frac{dy}{dx} - 2y = Be^{2x}.$

$(\cdot \frac{d}{dx}) \quad \Rightarrow \quad \frac{d^2y}{dx^2} - 2\frac{dy}{dx} = 2Be^{2x}$

$= 2(\frac{dy}{dx} - 2y).$

$\Rightarrow \frac{d^2y}{dx^2} - 4\frac{dy}{dx} + 4y = 0.$ (this matches the DE).

$\Rightarrow$ the general solution is

$$\boxed{y = Ae^{m_1x} + Bxe^{m_1x}.}$$

case #3: $m_1, m_2 \notin \mathbb{R}$ $(\Delta < 0)$

We let $m_1 = p + qi$. (then $m_2 = p - qi$).

$\Rightarrow$ the complementary function is

$y = Ce^{m_1x} + De^{m_2x}$

$= Ce^{(p+qi)x} + De^{(p-qi)x}$

$\Rightarrow e^{px}(Ce^{qix} + De^{-qix})$

$\Rightarrow y = e^{px}(C\cos(qx) + Ci\sin qx + D\cos qx - Di\sin qx)$

$= e^{px}((C+D)\cos qx + i(C-D)\sin qx)$

★ $A, B \in \mathbb{R}$.

$\rightarrow$ so $C+D \in \mathbb{R}$ and $C-D$ is purely imaginary.

$\Rightarrow$ C & D are conjugates.

$\Rightarrow$ general solution is

$$\boxed{y = e^{px}[A\cos qx + B\sin qx], \quad A = C+D, \quad B = i(C-D).}$$

NON-HOMOGENOUS LINEAR 2ND ORDER LINEAR DIFFERENTIAL EQNS

These have the form

$$\boxed{a\frac{d^2y}{dx^2} + b\frac{dy}{dx} + cy = f(x).}$$

Consider first the 1st order homogeneous linear diff eqn,

$$\frac{dy}{dx} + 2y = 3x - 1.$$

$F_i = e^{\int P dx} = e^{\int 2 dx} = e^{2x}.$

$\Rightarrow F_i \cdot y = \int F_i \cdot Q\, dx.$

$e^{2x} \cdot y = \int e^{2x}(3x-1)\, dx$

$= \frac{3}{2}xe^{2x} - \frac{5}{4}e^{2x} + A.$

$\Rightarrow \boxed{y = \left(\frac{3}{2}x - \frac{5}{4}\right) + Ae^{-2x}.}$

→ the solution consists of 2 parts:

1. Ae^{-2x}.
   - ↳ consists of an arbitrary constant;
   - ↳ satisfies the DE $\frac{dy}{dx} + 2y = 0$.
2. $\frac{3}{2}x - \frac{5}{4}$
   - ↳ does not contain any arbitrary constants.

* This suggests part of the solution only depends on the LHS of the DE, and another part depends on the full eqn.

Consider the DE

$$\frac{dy}{dx} + 2y = e^{3x}.$$

$\Rightarrow$ soln is $\boxed{y = Ae^{-2x} + \frac{1}{5}e^{3x}.}$

the same complementary function!

We observe that, in both cases, the solution consists of 2 parts:

1. It contains the complementary function, which can be found by solving the corresponding homogenous eqn (LHS).
2. The second part is the same "sort" of function as the RHS of the DE.

We denote this as the "particular integral".

* The general solution of any linear non-homogenous DE is

$$\boxed{y = \text{complementary function} + \text{particular integral.}}$$

To find the particular integral

We can write down a "trial function", whose form is determined by the RHS of the DE.

Take the second example.

Let the particular integral,

$I_p = ce^{3x}$. ($y = ce^{3x}$ is a soln to the DE).

$\Rightarrow y = ce^{3x}$, $\frac{dx}{dx} = 3ce^{3x}$.

$\Rightarrow 3ce^{3x} + 2ce^{3x} = e^{3x}$

$5ce^{3x} = e^{3x}$ ie $c = \frac{1}{5}$.

$\Rightarrow I_p = \frac{1}{5}e^{3x}$,

Hence the gen soln is

y = comp. func + particular integ.

$y = \underbrace{Ae^{-2x}}_{\text{from earlier.}} + \frac{1}{5}e^{3x}.$

Which trial function to use to solve the I_p

f(x)	trial function
Linear	$ax + b$
Polynomial of order n	$a + bx + cx^2 \cdots + kx^n$
Trig function	$a\sin(px) + b\cos(px)$
Exponential function	ae^{px}

If the function on the RHS has exactly the same form as one of the complementary functions, you multiply the usual trial function by the independent variable, x, to get a new trial function.

USING A SUBST$^{\underline{N}}$ TO REDUCE A DE TO A 2ND ORDER LINEAR DE WITH CONSTANT COEFFICIENTS

$\underline{Ex}$ Solve $x^2\frac{d^2y}{dx^2} + 3x\frac{dy}{dx} + 10y = 10\ln x + 42$, using the subst$^{\underline{n}}$ $x = e^u$.

$x = e^u \Rightarrow \frac{dx}{du} = e^u = x.$

$\therefore \frac{dy}{dx} = \frac{dy}{du}\cdot\frac{du}{dx}$

$= \frac{1}{x}\frac{dy}{du}.$ Hence $x\frac{dy}{dx} = \frac{dy}{du}.$

$(\frac{d}{dx})$ $\frac{d^2y}{dx^2} = \frac{1}{x}\frac{d}{dx}\left(\frac{dy}{du}\right) + \left(-\frac{1}{x^2}\right)\left(\frac{dy}{du}\right)$

$= \frac{1}{x}\frac{d}{du}\left(\frac{dy}{du}\cdot\frac{du}{dx}\right) - \frac{1}{x^2}\frac{dy}{du}$

$= \frac{1}{x}\cdot\frac{1}{x}\frac{d^2y}{du^2} - \frac{1}{x^2}\frac{dy}{du}$

$\Rightarrow x^2\frac{d^2y}{dx^2} = \frac{d^2y}{du^2} - \frac{dy}{du}$

DE reduces to

$\left(\frac{d^2y}{du^2} - \frac{dy}{du}\right) + 3\left(\frac{dy}{du}\cdot\cancel{\frac{dx}{dy}}\right)\left(\cancel{\frac{dy}{dx}}\right) + 10y = 10u + 42$

$\Rightarrow \frac{d^2y}{du^2} + 2\frac{dy}{du} + 10y = 10u + 42.$

AQE is $m^2 + 2m + 10 = 0.$

$\hookrightarrow m = \frac{-2 \pm \sqrt{4 - 4(1)(10)}}{2}$

$= -1 \pm 3i.$

complementary function is $e^{-u}[A\cos 3u + B\sin 3u].$

Let a trial Pi be $y = au + b.$ ⟶ This satisfies the DE, hence

$(0) + 2a + 10(au + b) = 10u + 42.$

(coeff of u) $10a = 10$ $\therefore a = 1$

(coeff of u^0) $2a + 10b = 42$ $\therefore b = 4.$

$\therefore$ Pi is $y = u + 4.$

$\underline{\text{Hence}}$ the general sol$^{\underline{n}}$ is

$y = e^{-u}(A\cos 3u + B\sin 3u) + u + 4$

$\Rightarrow \boxed{y = \frac{1}{x}(A\cos(3\ln x) + B\sin(3\ln x)) + \ln x + 4.}$

Chapter 6: Complex Numbers

DE MOIVRE'S THEOREM.

$$\boxed{(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta.}$$

⊕ when n is a rational fraction,

$(\cos\theta + i\sin\theta)^{\frac{p}{q}} = \cos\frac{p}{q}\theta + i\sin\frac{p}{q}\theta.$

However there are further values this can take.

AN EASIER METHOD TO SIMPLIFY A COMPLEX NUMBER.

If $w = \dfrac{(a+bi)}{c+z}$, where $z = \cos\theta + i\sin\theta$. (easier to simplify).

we can multiply the num & den by $c + \frac{1}{z}$;

$$w = \frac{(a+bi)(c+\frac{1}{z})}{(c^2+1)+c(z+\frac{1}{z})} = \frac{(a+bi)(c+\frac{1}{dz})}{c^2+1+c(2\cos\theta)}. ..$$

APPLICATIONS

EXPRESS TRIG RATIOS OF MULTIPLE ANGLES IN TERMS OF POWERS OF TRIG RATIOS OF THE FUNDAMENTAL ANGLE.

$\cos n\theta + i\sin n\theta = (\cos\theta + i\sin\theta)^n$

$= (c+is)^n$, $c = \cos\theta$, $s = \sin\theta$.

$= c^n + {}^nC_1 c^{n-1}(is) + {}^nC_2 c^{n-2}(is)^2 .. + {}^nC_n (is)^n.$

(Coeff of Re) $\boxed{\cos n\theta = c^n - {}^nC_2 c^{n-2}s^2 + {}^nC_4 c^{n-4}s^4 ..}$

(coeff of Im) $\boxed{\sin n\theta = {}^nC_1 sc^{n-1} - {}^nC_3 s^3c^{n-3} + {}^nC_5 s^5 n^{-5} ..}$

Relationship b/w trig & algebraic eqn:

$\cos n\theta$ = polynomial of $\cos\theta$.

$\sin n\theta$ = polynomial of $\sin\theta$.

$\tan n\theta = \dfrac{\text{poly of } \tan\theta}{\text{poly of } \tan\theta}$.

they form a polynomial eqn in x, where $x = \cos\theta$, $\sin\theta$ or $\tan\theta$.

coeff are related to the coeff of the poly of $\cos\theta$, $\sin\theta$, $\tan\theta$.

Ex

I Use de Moivre's theorem to show that

$$\cos 4\theta = 8\cos^4\theta - 8\cos^2\theta + 1. \quad [3]$$

II Without using a calculator, verify that $\cos 4\theta = -\cos 3\theta$ for each of the values $\theta = \frac{1}{7}\pi, \frac{3}{7}\pi, \frac{5}{7}\pi, \pi$. [2]

III Using the result $\cos 3\theta = 4\cos^3\theta - 3\cos\theta$, show that the roots of the equation

$$8c^4 + 4c^3 - 8c^2 - 3c + 1 = 0$$

are $\cos\frac{1}{7}\pi, \cos\frac{3}{7}\pi, \cos\frac{5}{7}\pi, -1$. [2]

IV Deduce that $\cos\frac{1}{7}\pi + \cos\frac{3}{7}\pi + \cos\frac{5}{7}\pi = \frac{1}{2}$. [2]

(I) $\cos 4\theta + i\sin 4\theta = (c+is)^4$

$= c^4 + 4c^3(is) + 6c^2(is)^2 + 4c(is)^3 + (is)^4$

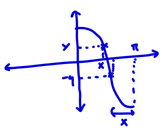
(Re) $\Rightarrow \cos 4\theta = (c^4 - 6c^2s^2 + s^4)$

$= (c^4 - 6c^2(1-c^2) + (1-c^2)^2)$

$= (c^4 - 6c^2 + 6c^4 + 1 - 2c^2 + c^4$

$\therefore \cos 4\theta = (8\cos^4\theta - 8\cos^2\theta + 1).$

(II)



$\cos(\theta) = -\cos(\pi - \theta).$

$\theta = \frac{\pi}{7}$ $\quad \cos 4\theta = \cos\frac{4\pi}{7}$

$-\cos 3\theta = -\cos\frac{3\pi}{7} = -\cos(\pi - \frac{4\pi}{7})$

$= \cos\frac{4\pi}{7}$. $\quad \therefore \cos 4\theta = -\cos 3\theta.$

Similarly, $\theta = \frac{3\pi}{7} \rightarrow \cos 4\theta = \cos\frac{12\pi}{7} = \cos(-\frac{2\pi}{7})$

and $-\cos 3\theta = -\cos(\frac{9\pi}{7})$

$= -\cos(\pi - (-\frac{2\pi}{7}))$

$= -\cos(-\frac{2\pi}{7}) = \cos 4\theta.$

$\theta = \frac{5\pi}{7} \rightarrow \cos 4\theta = \cos(\frac{20\pi}{7}) = \cos(\frac{6\pi}{7}).$

$-\cos 3\theta = -\cos(\frac{15\pi}{7}) = -\cos(\frac{\pi}{7})$

$= -\cos(\pi - \frac{6\pi}{7})$

$= \cos(\frac{6\pi}{7}) = \cos 4\theta.$

$\theta = \pi \rightarrow \cos 4\theta = \cos 4\pi = 1$

$-\cos(3\pi) = -(-1) = 1 (= \cos 4\theta).$

$\therefore \theta = \frac{\pi}{7}, \theta = \frac{3\pi}{7}, \theta = \frac{5\pi}{7}$ and $\theta = \pi$ satisfies $\cos 4\theta = -\cos 3\theta.$

(III) $8c^4 + 4c^3 - 8c^2 - 3c + 1 = 0.$

$(8c^4 - 8c^2 + 1) + (4c^3 - 3c) = 0.$

Let $c = \cos\theta.$

$\Rightarrow$ the eqn reduces to

$\cos 4\theta + \cos 3\theta = 0$ or

$\cos 4\theta = -\cos 3\theta.$

By (II), we see that $\theta = \frac{\pi}{7}$, $\theta = \frac{3\pi}{7}$, $\theta = \frac{5\pi}{7}$ and $\theta = \pi$ satisfies this eqn.

$\Rightarrow c = \cos\frac{\pi}{7}$, $c = \cos\frac{3\pi}{7}$, $c = \cos\frac{5\pi}{7}$ and $c = \cos\pi = -1.$

(IV) $\alpha + \beta + \gamma + \delta = -\frac{b}{a} = \frac{-4}{8} = -\frac{1}{2}.$ (by polynomial roots)

$\cos\frac{\pi}{7} + \cos\frac{3\pi}{7} + \cos\frac{5\pi}{7} + (-1) = -\frac{1}{2}.$

$\therefore \cos\frac{\pi}{7} + \cos\frac{3\pi}{7} + \cos\frac{5\pi}{7} = \frac{1}{2}.$

EXPRESS POWERS OF $\sin\theta$ & $\cos\theta$ IN TERMS OF MULTIPLE ANGLES.

Recall, if $z = \cos\theta + i\sin\theta$,

then $z^n = \cos n\theta + i\sin n\theta$.

$\Rightarrow z^{-n} = \cos(-n\theta) + i\sin(-n\theta)$

$\Rightarrow \frac{1}{z^n} = \cos(n\theta) - i\sin(n\theta)$.

$\therefore$ ① $z^n + \frac{1}{z^n} = 2\cos(n\theta)$;

② $z^n - \frac{1}{z^n} = 2i\sin(n\theta)$.

To express $\cos^n\theta$ in terms of multiple angles

* $(n=1)$ $z + \frac{1}{z} = 2\cos\theta$.

$\therefore \left(z+\frac{1}{z}\right)^n = (2\cos\theta)^n$.

$\therefore 2^n\cos^n\theta = z^n + \binom{n}{1}z^{n-2} + \binom{n}{2}z^{n-4} \cdots + z^{-n}$

$= \left(z^n + \frac{1}{z^n}\right) + \binom{n}{1}\left(z^{n-2} + \frac{1}{z^{n-2}}\right) \cdots$

$$\boxed{2^n\cos^n\theta = 2\cos n\theta + \binom{n}{1}(2\cos(n-2)\theta) + \binom{n}{2}(2\cos(n-4)\theta)}$$

eg[1] Show $\sin^7\theta$ can be expressed in the form $a\sin\theta + b\sin 3\theta + c\sin 5\theta + d\sin 7\theta$.

$(2i\sin\theta)^7 = \left(z - \frac{1}{z}\right)^7$, $z = \cos\theta + i\sin\theta$.

$\Rightarrow (-128i)\sin^7\theta = z^7 - 7z^5 + 21z^3 - 35z^1 + 35z^{-1} - 21z^{-3} + 7z^{-5} - z^{-7}$

$= (z^7 - z^{-7}) - 7(z^5 - z^{-5}) + 21(z^3 - z^{-3}) - 35(z^1 - z^{-1})$.

$(-128i)\sin^7\theta = 2i\sin 7\theta - 7(2i\sin 5\theta) + 21(2i\sin 3\theta) - 35(2i\sin\theta)$

$(\div 2i)$ $(-64)\sin^7\theta = \sin 7\theta - 7\sin 5\theta + 21\sin 3\theta - 35\sin\theta$.

$(\div -64)$ $\therefore \sin^7\theta = \frac{1}{64}(35\sin\theta - 21\sin 3\theta + 7\sin 5\theta - \sin 7\theta)$.

To express $\sin^n\theta$ in terms of multiple angles

* $z - \frac{1}{z} = 2i\sin\theta$.

$\Rightarrow \left(z - \frac{1}{z}\right)^n = (2i\sin\theta)^n$.

$\Rightarrow (2i)^n\sin^n\theta = \left(z^n - \frac{1}{z^n}\right) - \binom{n}{1}\left(z^{n-2} - \frac{1}{z^{n-2}}\right) \cdots$

$$\boxed{\therefore (2i)^n\sin^n\theta = (2i\sin n\theta) - \binom{n}{1}(2i\sin(n-2)\theta) \cdots}$$

eg[2] Use dM theorem to show that $\cos^6\theta + \sin^6\theta = \frac{1}{8}(3\cos 4\theta + 5)$.

$(2\cos\theta)^6 - (2i\sin\theta)^6 = \left(z + \frac{1}{z}\right)^6 - \left(z - \frac{1}{z}\right)^6$

$\Rightarrow 64\cos^6\theta + 64\sin^6\theta = (z^6 + 6z^4 + 15z^2 + 20 + 15z^{-2} + 6z^{-4} + z^{-6})$
$\quad - (z^6 - 6z^4 + 15z^2 - 20 + 15z^{-2} - 6z^{-4} + z^{-6})$

$= 2(6z^4 + 20 + 6z^{-4})$

$= 40 + 12(z^4 + z^{-4})$

$= 40 + 12(2\cos 4\theta)$.

$(\div 64)$ $\therefore \cos^6\theta + \sin^6\theta = \frac{1}{64}(40 + 24\cos 4\theta)$

$\Rightarrow \cos^6\theta + \sin^6\theta = \frac{1}{8}(5 + 3\cos 4\theta)$. *

TO FIND THE nTH ROOTS OF UNITY OR ANY COMPLEX NUMBER.

nth roots of unity $\Leftrightarrow$ roots of the eq^n $\boxed{z^n = 1}$.

eg $z = 1 \to z = 1$

$z^2 = 1 \to z = \pm 1$

$z^4 = 1 \to z = \pm 1, \pm i$

$z^n = 1$

$\Rightarrow z^n = \cos 0 + i\sin 0$.

$= \cos(k2\pi + 0) + i\sin(k2\pi + 0)$.

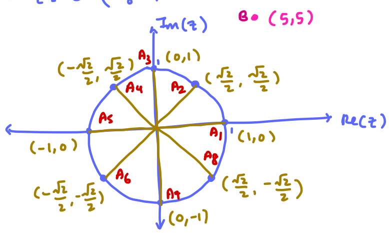
$\therefore z = [\cos(k2\pi + 0) + i\sin(k2\pi)]^{\frac{1}{n}}$

$$\Rightarrow \boxed{z = \cos\left(\frac{k}{n}2\pi\right) + i\sin\left(\frac{k}{n}2\pi\right) = e^{i\frac{k2\pi}{n}} \quad k = 0, 1, 2 \cdots, n-1.}$$

Ex[1] Find, in any form, the 8 complex numbers which satisfy the eq^n $z^8 - 1 = 0$, and represent them in an Argand diagram.

$z^8 = 1$.

$\therefore z = \cos\left(\frac{k2\pi}{8}\right) + i\sin\left(\frac{k2\pi}{8}\right)$ for $z = 0, 1 \cdots 7$.



$|z| = \sqrt{\cos^2\left(\frac{k2\pi}{8}\right) + \sin^2\left(\frac{k2\pi}{8}\right)}$

$= 1$.

$\arg z = \frac{1}{4}k\pi$, $k = 0, 1 \cdots 7$.

Use a graphical argument to find the greatest & least value of $|z - 5 - 5i|$.

$|z_1 - z_2|$ = distance between the points representing z_1 & z_2.

$\therefore |z - 5 - 5i| = |z - (5 + 5i)|. \to AB$.

(A = pt rep by z, $B = (5,5)$).

Greatest value of AB:

$= A_6B$

$= OA_6 + OB$

$= 1 + \sqrt{5^2 + 5^2}$

$= 1 + 5\sqrt{2}$.

least value of AB:

$= A_2B$

$= OB - OA_2$

$= \sqrt{5^2 + 5^2} - 1$

$= 5\sqrt{2} - 1$.

Ex[a]2 The complex number ω is a root of the eq^n $z^3 - 1 = 0$.

(i) Show $1 + \omega + \omega^2 = 0$.

ω is a root $\to \omega^3 - 1 = 0$.

$\Rightarrow (\omega - 1)(\omega^2 + \omega + 1) = 0$

$\omega \neq 1$ $\therefore \omega^2 + \omega + 1 = 0$. *

(ii) Hence show that $1 + \omega^k + \omega^{2k}$, where $k \in \mathbb{Z}^+$, $\neq 0$ when k is a multiple of 3, but $= 0$ when k is not a multiple of 3.

k is a multiple of 3 $\to k = 3m$, $m \in \mathbb{Z}^+$.

$\therefore 1 + \omega^k + \omega^{2k} = 1 + (\omega^3)^m + (\omega^3)^{2m}$

$= 1 + (1)^m + (1)^{2m} = 3.$ $(\neq 0)$.

k is not a multiple of 3

$\to k = 3m+1$ or $k = 3m+2$, $m \in \mathbb{Z}^+$.

$k = 3m+1 \to 1 + \omega^k + \omega^{2k}$

$= 1 + \omega^{3m+1} + \omega^{2(3m+1)}$

$= 1 + (\omega^3)^m\omega + (\omega^3)^{2m}\omega^2$

$= 1 + (1)\omega + (1)\omega^2$

$= 0$.

$k = 3m+2 \to 1 + \omega^k + \omega^{2k}$

$= 1 + \omega^{3m+2} + \omega^{6m+4}$

$= 1 + (\omega^3)^m\omega^2 + (\omega^3)^{2m}\omega^4$

$= 1 + \omega^2 + \omega^4$

$= 1 + \omega^2 + (\omega^3)\omega$

$= 1 + \omega + \omega^2$

$= 0$.

(Hence the statement is proven.)

TO FIND THE REAL FACTORS OF z^n-1.

$n=2 \rightarrow z^2-1=0$

$\Rightarrow (z-1)(z+1)=0 \rightarrow z=\pm 1.$

$n=3 \rightarrow z^3-1=0.$

$\Rightarrow (z-1)(z^2+z+1)=0$

$n=4 \rightarrow z^4-1=0$

$\Rightarrow (z-1)(z+1)(z^2+1)=0.$

Steps

① Solve the eqⁿ $z^n-1=0.$

$z^n=1 \rightarrow z=e^{i\frac{2k\pi}{n}}, \quad k=0,1 .. n-1$

Hence, the factors of z^n-1 are $(z-e^{\frac{ik\pi}{n}})$, $(z-e^{\frac{i2k\pi}{2}})$ etc.

② Use the following result:

$$(z-e^{i\theta})(z-e^{-i\theta}) = z^2-ze^{i\theta}-ze^{-i\theta}+1$$

$$= z^2-z(\cos\theta+i\sin\theta+\underbrace{\cos(-\theta)}_{\cos\theta}+\underbrace{i\sin(-\theta)}_{-i\sin\theta})+1$$

$$= z^2-z(2\cos\theta)+1$$

$$\therefore \boxed{(z-e^{i\theta})(z-e^{-i\theta}) = z^2-(2\cos\theta)z+1.}$$

eg¹ Factorise z^3-1.

$z^3=1 \rightarrow z=e^{i\frac{2k\pi}{3}}, \quad k=0,\pm1.$

$= 1, e^{i\frac{2\pi}{3}}, e^{i(-\frac{2\pi}{3})}.$

$\therefore z^3-1 = (z-1)(z-e^{i\frac{2\pi}{3}})(z-e^{i(-\frac{2\pi}{3})})$

By the result in step ②:

$z^3-1=(z-1)\left[z^2-(2\cos\frac{2\pi}{3})z+1\right]$

$=(z-1)\left[z^2-2(-\frac{1}{2})z+1\right]$

$=(z-1)\left[z^2+z+1\right]$

eg² Express z^8-1 as a product of 2 linear & 3 quadratic factors, where all coeff are real and expressed in a non trigometric form.

$z^8=1 \rightarrow z=e^{i\frac{2k\pi}{8}}, \quad k=0,\pm1,\pm2,\pm3,4.$

$$\Rightarrow z^8-1=(z-e^{i(0)})(z-e^{i\frac{2\pi}{8}})(z-e^{i(-\frac{2\pi}{8})})(z-e^{i\frac{4\pi}{8}})(z-e^{i(-\frac{4\pi}{8})})(z-e^{i(\frac{6\pi}{8})})(z-e^{i(-\frac{6\pi}{8})})(z-e^{i(\pi)}).$$

$\frac{\pi}{4}$ $\frac{\pi}{2}$ $\frac{3\pi}{4}$

$$=(z-1)\left[z^2-(2\cos\frac{\pi}{4})z+1\right]\left[z^2-(2\cos\frac{\pi}{2})z+1\right]\left[z^2-(2\cos\frac{3\pi}{4})z+1\right](z+1).$$

$$=(z-1)(z+1)(z^2-2(\tfrac{\sqrt2}{2})z+1)(z^2+1)(z^2-2(-\tfrac{\sqrt2}{2})z+1)$$

$$\therefore z^8-1=(z^2-\sqrt2 z+1)(z^2+\sqrt2 z+1)(z^2+1)(z-1)(z+1).$$

TO FIND THE n^{th} ROOTS OF A COMPLEX NUMBER.

$z^n = a+bi = re^{i\theta}.$

$\Rightarrow z=(re^{i\theta})^{\frac1n}\cdot 1^{\frac1n}$

$=(re^{i\theta})^{\frac1n}\cdot(e^{i\frac{2k\pi}{n}}), \quad k=0,1 .. n-1.$

$=r^{\frac1n}e^{\frac{i\theta}{n}}e^{i\frac{2k\pi}{n}}$

$=r^{\frac1n}e^{i(\frac{\theta+2k\pi}{n})}.$

eg¹ Find the roots of the eqⁿ $z^4=8(\sqrt3-i).$

$8(\sqrt3-i)=re^{i\theta}$

$r=\sqrt{(8\sqrt3)^2+(-8)^2}$
$=16.$

$\theta=\tan^{-1}(\frac{-1}{\sqrt3})$
$=-\frac{\pi}{6}.$

$8(\sqrt3-i)=16\cos-\frac{\pi}{6}+i\sin-\frac{\pi}{6}.$

$z^4=8(\sqrt3-i)$

$\Rightarrow z^4=16\cos-\frac{\pi}{6}+i\sin-\frac{\pi}{6}=16e^{i(-\frac{\pi}{6})}.$

$\Rightarrow z=(16e^{i(-\frac{\pi}{6})})^{\frac14}\cdot 1^{\frac14}$

$=2e^{i(-\frac{\pi}{24})}\cdot e^{i\frac{2k\pi}{n}}, \quad n=0,\pm1,2.$

$\Rightarrow z=2\left[\cos(-\frac{\pi}{24}+\frac{2k\pi}{n})+i\sin(-\frac{\pi}{24}+\frac{2k\pi}{n})\right], \quad n=0,\pm1,2.$

SIMPLIFYING A COMPLEX NUMBER (WITH A FRACTION FORM) EASILY

If the denonminator is in the form $a\pm az$ or $a+bz$, it can be simplified easily:

case 1 $\quad \frac{k}{a\pm az^n}=\frac{k}{a(1\pm z^n)}=\frac{k}{az^{\frac n2}(z^{-\frac n2}\pm z^{\frac n2})}$

$$\Rightarrow \boxed{\begin{aligned}\frac{k}{a+az^n}&=\frac{k}{az^{\frac n2}(2\cos\frac n2\theta)}\\ \frac{k}{a-az^n}&=\frac{k}{az^{\frac n2}(2i\sin\frac n2\theta)}\end{aligned}}$$

case 2 $\quad \frac{k}{a+bz^n}=\frac{k}{a+bz^n}\cdot\frac{a+\frac{b}{z^n}}{a+\frac{b}{z^n}}$

$=\frac{k(a+\frac{b}{z^n})}{a^2+b^2+2ab(z^n+\frac{1}{z^n})}$

$$\boxed{\frac{k}{a+bz}=\frac{k(a+\frac bz)}{a^2+b^2-2ab(2\cos n\theta)}}$$

TO FIND THE SUM OF THE FIRST n TERMS OF A TRIGONOMETRIC SERIES.

Use of a geometric series

If $f(k)$ and $g(k)$ are linear functions, and a is a constant:

$(\cos\theta+i\sin\theta)^{f(k)}$

$$\sum_{k=1}^{n}a^{g(k)}z^{f(k)}=\sum_{k=1}^{n}a^{g(k)}\left[\cos[f(k)\cdot\theta]+i\sin[f(k)\cdot\theta]\right], \text{ by dM theorem.}$$

$$\boxed{\begin{aligned}&(Re)\ \therefore \sum_{k=1}^{n}a^{g(k)}\cos(f(k)\theta)=Re\sum_{k=1}^{n}a^{g(k)}z^{f(k)}.\\ &(Im)\ \therefore \sum_{k=1}^{n}a^{g(k)}\sin(f(k)\theta)=Im\sum_{k=1}^{n}a^{g(k)}z^{f(k)}.\end{aligned}}$$

eg¹ Show, by using dM theorem, that $\sum_{k=1}^{10}(-1)^{k-1}\cos(2k-1)\theta=\frac{\sin^2 10\theta}{\cos\theta}$, provided $\cos\theta\neq0.$

$\sum_{k=1}^{10}(-1)^{k-1}\cos(2k-1)\theta = Re\sum_{k=1}^{10}(-1)^{k-1}z^{2k-1}, \quad z=\cos\theta+i\sin\theta.$

$=Re\left[z^1-z^3+z^5-z^7+z^9 ... -z^{19}\right].$ a GP w/ $a=z$, $r=-z^2$, $n=10$

$=Re\left[\frac{z(1-(-z^2)^{10})}{1-(-z^2)}\right]$

$=Re\left[\frac{z(1-z^{20})}{1+z^2}\right]$

$=Re\left[\frac{\cancel{z}(1-z^{20})}{\cancel{z}(z+\frac1z)}\right]$

$=Re\left[\frac{1-(\cos20\theta+i\sin20\theta)}{2\cos\theta}\right]$

$=\frac{1-\cos20\theta}{2\cos\theta}=\frac{1-(1-2\sin^2 10\theta)}{2\cos\theta}$

$=\frac{\sin^2 10\theta}{\cos\theta}.$

Use of a binomial series

$(1+z^k)^n=1+\binom n1 z^k+\binom n2 z^{2k} .. + z^{nk}, \quad z=\cos\theta+i\sin\theta.$

$$\boxed{\begin{aligned}&\therefore Re(1+z^k)^n=1+\tbinom n1\cos k\theta+\tbinom n2\cos2k\theta .. +\cos nk\theta.\\ &\therefore Im(1+z^k)^n=\tbinom n1\sin k\theta+\tbinom n2\sin2k\theta .. +\sin nk\theta.\end{aligned}}$$

eg² By considering $(1+i\tan\theta)^n=\frac{\cos n\theta+i\sin n\theta}{\cos^n\theta}$, and setting $n=2p$ and letting θ be a suitable value, show that $1-\binom{2p}{2}+\binom{2p}{4} .. +(-1)^p\binom{2p}{2p}=2^p\cos(\frac12 p\pi).$

$(1+i\tan\theta)^{2p}=1+\binom{2p}{1}(i\tan\theta)+\binom{2p}{2}(i\tan\theta)^2 .. +\binom{2p}{2p}(i\tan\theta)^{2p}=\frac{\cos2p\theta+i\sin2p\theta}{\cos^{2p}\theta}.$

(Re) $\therefore 1-\binom{2p}{2}\tan^2\theta+\binom{2p}{4}\tan^4\theta .. +(-1)^p\binom{2p}{2p}\tan^{2p}\theta=\frac{\cos2p\theta}{\cos^{2p}\theta}.$ $\quad (i^{2p}=(i^2)^p=(-1)^p)$

let $\theta=\frac{\pi}{4}$ (so $\tan\theta=1$) $\quad \therefore 1-\binom{2p}{2}+\binom{2p}{4} .. +(-1)^p\binom{2p}{2p}=\frac{\cos(p\frac{\pi}{2})}{\cos^{2p}(\frac{\pi}{4})}$ $\quad \cos^{2p}(\frac{\pi}{4})=(\frac{\sqrt2}{2})^{2p}=(\frac12)^p$

$\therefore 1-\binom{2p}{2}+\binom{2p}{4} .. +(-1)^p\binom{2p}{2p}=2^p\cos(\frac12 p\pi).$