MATH 148
Personal Notes

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UW Math '25

Chapter 1: The Riemann Integral

CODE KEY
- D : definition
- N : note
- R : remark
- L : lemma
- E : example
- C : corollary
- T : theorem
- NT : notation

PARTITION OF $[a,b]$ (D1.1)

N1: A "partition" of the closed interval $[a,b]$ is any set $X=\{x_0,x_1,\dots,x_n\}$ such that
$$a=x_0<x_1<\cdots<x_n=b.$$

SUBINTERVAL (D1.1)

N1: Let $X=\{x_0,x_1,\dots,x_n\}$ be a partition of $[a,b]$.

Then a "sub-interval" of $[a,b]$ is any interval of the form $[x_{k-1},x_k]$, where $k\in\{1,2,\dots,n\}$, and denote them by
$$\Delta_k x = x_k - x_{k-1}.$$

N2: Note that
$$\Delta_1 x+\Delta_2 x+\cdots+\Delta_n x=\sum_{k=1}^{n}\Delta_k x = b-a.$$

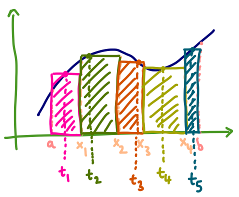
SIZE (D1.1)

N1: Let X be a partition of $[a,b]$. Then the "size" of X, denoted as $|X|$, is defined to be
$$|X| = \max(\{\Delta_k x \mid 1\le k\le n\}).$$

RIEMANN SUM (D1.2)

N1: Let X be a partition of $[a,b]$, and let $f:[a,b]\to\mathbb{R}$ be bounded.

Then, a "Riemann sum" for f on X is a sum of the form
$$S=\sum_{k=1}^{n} f(t_k)\,\Delta_k x,$$
where $t_k\in[x_{k-1},x_k]$ $\forall k\in\{1,2,\dots,n\}$.



S = sum of areas of rectangles

= ▨ + ▨ + ▨ + ▨ + ▨.

* note that t_1,t_2,t_3,t_4 & t_5 are at arbitrary positions in their respective intervals!

(also note distance between consecutive x_n need not be constant).

SAMPLE POINTS (D1.2)

N1: Let $S=\sum_{k=1}^{n} f(t_k)\,\Delta_k x$ be a Riemann sum for some bounded function $f:[a,b]\to\mathbb{R}$ on a partition X of $[a,b]$.

Then we say t_k is a "sample point" of S for any $k\in\{1,2,\dots,n\}$.

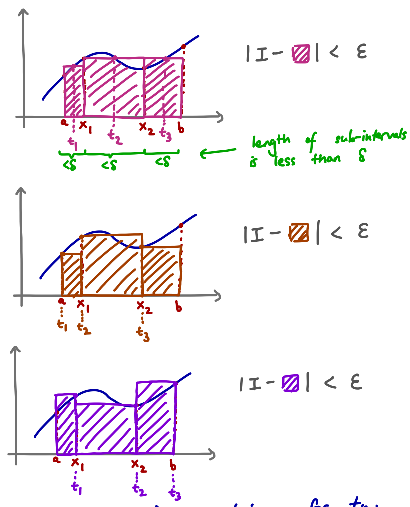
RIEMANN INTEGRAL

(RIEMANN) INTEGRABLE (D1.3)

N1: Let $f:[a,b]\to\mathbb{R}$ be bounded.

Then, we say f is "Riemann integrable", or just "integrable", on $[a,b]$ if there exists an $I\in\mathbb{R}$ such that for any $\varepsilon>0$, there exists a $\delta>0$ such that for any partition X of $[a,b]$ with $|X|<\delta$, we have that
$$|S-I|<\varepsilon$$
for any Riemann sum S for f on X.

N2: In other words, we have that
$$\left|\sum_{k=1}^{n} f(t_k)\,\Delta_k(x) - I\right| < \varepsilon$$
irregardless of our choices for $t_k\in[x_{k-1},x_k]$.



$|I - ▨| < \varepsilon$

$|I - ▨| < \varepsilon$

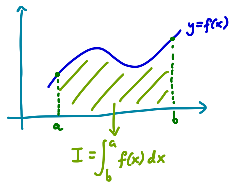
$|I - ▨| < \varepsilon$

* regardless of our choices for t_k, we must always get that $|$sum of areas $- I| < \varepsilon$!

(RIEMANN) INTEGRAL (D1.3)

N1: We say the "Riemann integral" of f on $[a,b]$ is defined to be the number $I\in\mathbb{R}$ described above, and write
$$I=\int_a^b f = \int_a^b f(x)\,dx.$$

* I represents the "area under the graph".



N2: We can prove I is unique.

Proof. Suppose I & J are two such numbers. Let $\varepsilon>0$ be arbitrary. Then, choose a $\delta_1>0$ such that for any partition X with $|X|<\delta_1$, we have $|S-I|<\frac{\varepsilon}{2}$ for every Riemann sum S on X.

Similarly, choose a $\delta_2>0$ such that for any partition X with $|X|<\delta_2$, we have $|S-J|<\frac{\varepsilon}{2}$ for every Riemann sum S on X.

Then, let $\delta=\min(\delta_1,\delta_2)$, and let X be any partition of $[a,b]$ with $|X|<\delta$.

For each $k\in\{1,2,\dots,n\}$, choose a $t_k\in[x_{k-1},x_k]$.

Let $S=\sum_{k=1}^{n} f(t_k)\,\Delta_k x$.

Then, by the Triangle Inequality, we have that
$$|I-J|\le|I-S|+|S-J|<\frac{\varepsilon}{2}+\frac{\varepsilon}{2}=\varepsilon.$$

But since $\varepsilon>0$ was arbitrary, it follows that $I=J$, proving uniqueness. ▨

SOME FUNCTIONS ARE NOT INTEGRABLE (E1.4)

We can show that certain functions are not integrable on a specific closed interval.

Example: $f(x) = \begin{cases} 1, & x \in \mathbb{Q} \\ 0, & x \notin \mathbb{Q} \end{cases}$ is not integrable on $[0,1]$.

Proof. Suppose f is integrable on $[0,1]$. Write $I = \int_0^1 f(x)\,dx$.

Let $\varepsilon = \frac{1}{2}$. Then, by definition, we can choose a $\delta > 0$ such that for every partition X with $|X| < \delta$, we have that $|S - I| < \frac{1}{2}$ for every Riemann sum S for f on X.

Then, choose some partition X with $|X| < \delta$.

Denote $S_1 = \sum_{k=1}^{n} f(t_k)\Delta_k x$ and $S_2 = \sum_{k=1}^{n} f(s_k)\Delta_k x$,

where $t_k \in \mathbb{Q}$ and $s_k \notin \mathbb{Q}$ and $t_k, s_k \in [x_{k-1}, x_k]$ for each $k \in \{1, 2, \dots, n\}$.

Note that $|S_1 - I| < \frac{1}{2}$ and $|S_2 - I| < \frac{1}{2}$, by our previous assumption that $\varepsilon = \frac{1}{2}$.

Subsequently, since $t_k \in \mathbb{Q}$ and $s_k \in \mathbb{R}\setminus\mathbb{Q}$ $\forall k \in \{1, 2, \dots, n\}$, it follows that $f(t_k) = 1$ and $f(s_k) = 0$; hence, we must get that

$$S_1 = \sum_{k=1}^{n} f(t_k)\Delta_k x = \sum_{k=1}^{n} \Delta_k x = 1 - 0 = 1$$

and

$$S_2 = \sum_{k=1}^{n} f(s_k)\Delta_k x = 0.$$

Thus, since $|S_1 - I| < \frac{1}{2}$ and $|S_2 - I| < \frac{1}{2}$, we must finally deduce that

$$|1 - I| < \frac{1}{2} \quad \text{and} \quad |I| < \frac{1}{2},$$

so that

$$\frac{1}{2} < I < \frac{3}{2} \quad \text{and} \quad -\frac{1}{2} < I < \frac{1}{2},$$

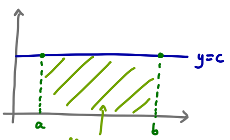
which is clearly impossible.

Therefore f is not integrable on $[0,1]$, which we wanted to show. ∎

THE INTEGRAL OF THE CONSTANT FUNCTION (E1.5)

The constant function $f(x) = c$ is always integrable on any interval $[a,b]$, and

$$\int_a^b c\,dx = c(b-a).$$



$I = \int_a^b c\,dx = c(b-a)$.

Proof. Let S be a Riemann sum for f on a partition X of $[a,b]$.

Then,

$$S = \sum_{k=1}^{n} f(t_k)\Delta_k x, \quad t_k \in [x_{k-1}, x_k] \ \forall k \in \{1, 2, \dots, n\}$$

$$= \sum_{k=1}^{n} c\,\Delta_k x$$

$$= c\sum_{k=1}^{n} \Delta_k x$$

$\therefore S = c(b-a)$.

But since S was arbitrary, it follows that

$$I = \int_a^b c\,dx = c(b-a),$$

as needed. ∎

THE INTEGRAL OF THE IDENTITY FUNCTION (E1.6)

The identity function $f(x) = x$ is also integrable on any interval $[a,b]$, and

$$\int_a^b x\,dx = \frac{1}{2}(b^2 - a^2).$$

Proof. Let $\varepsilon > 0$ be arbitrary, and let $\delta = \frac{2\varepsilon}{b-a}$.

Let X be any partition of $[a,b]$ with $|X| < \delta$.

Then, the Riemann sum S for f on X is equal to

$$S = \sum_{k=1}^{n} f(t_k)\Delta_k x = \sum_{k=1}^{n} t_k \Delta_k x,$$

where $t_k \in [x_{k-1}, x_k]$ $\forall k \in \{1, 2, \dots, n\}$.

Next, notice that

$$\sum_{k=1}^{n} (x_k + x_{k-1})\Delta_k x = \sum_{k=1}^{n} (x_k + x_{k-1})(x_k - x_{k-1})$$

$$= \sum_{k=1}^{n} (x_k^2 - x_{k-1}^2)$$

$$= (\cancel{x_1^2} - x_0^2) + (\cancel{x_2^2} - \cancel{x_1^2}) + \cdots + (x_n^2 - \cancel{x_{n-1}^2})$$

$$= x_n^2 - x_0^2$$

$$\therefore \sum_{k=1}^{n} (x_k + x_{k-1})\Delta_k x = b^2 - a^2.$$

Moreover, $t_k \in [x_{k-1}, x_k]$ implies that

$$\left|t_k - \tfrac{1}{2}(x_k + x_{k-1})\right| \leq \tfrac{1}{2}(x_k - x_{k-1}) = \tfrac{1}{2}\Delta_k x,$$

and consequently it follows that

$$\left|S - \tfrac{1}{2}(b^2 - a^2)\right| = \left|\sum_{k=1}^{n} t_k \Delta_k x - \tfrac{1}{2}\sum_{k=1}^{n} (x_k + x_{k-1})\Delta_k x\right|$$

$$= \left|\sum_{k=1}^{n} \left(t_k - \tfrac{1}{2}(x_k + x_{k-1})\right)\Delta_k x\right|$$

$$\leq \sum_{k=1}^{n} \left|t_k - \tfrac{1}{2}(x_k + x_{k-1})\right| \Delta_k x$$

$$\leq \sum_{k=1}^{n} \left(\tfrac{1}{2}\Delta_k x\right)\Delta_k x$$

$$\leq \sum_{k=1}^{n} \tfrac{1}{2}\delta(b-a) \quad \text{(since } \Delta_k x < \delta \text{ and } \Delta_k x = b-a \text{ by definition)}$$

$$= \varepsilon, \quad \left(\text{since } \delta = \tfrac{2\varepsilon}{b-a}\right)$$

So that $\left|S - \frac{1}{2}(b^2 - a^2)\right| \leq \varepsilon$.

But as $\varepsilon > 0$ was arbitrary, this tells us that

$$I = \int_a^b x\,dx = \frac{1}{2}(b^2 - a^2),$$

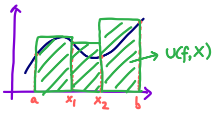
which we wanted to prove. ∎

UPPER & LOWER RIEMANN SUMS (D1·7)

Let X be a partition of $[a,b]$, and let $f:[a,b]\to\mathbb{R}$ be bounded.

Then, the "upper Riemann sum" for f on X, denoted by $U(f,X)$, is defined to be

$$U(f,X)=\sum_{k=1}^{n} M_k\Delta_k x,$$

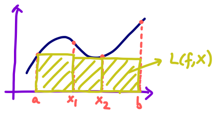


where

$$M_k=\sup(\{f(t):\ t\in[x_{k-1},x_k]\})\quad\forall k\in\{1,2,\dots,n\}.$$

Similarly, the "lower Riemann sum" for f on X, denoted by $L(f,X)$, is defined to be

$$L(f,X)=\sum_{k=1}^{n} m_k\Delta_k x,$$



where

$$m_k=\inf(\{f(t):\ t\in[x_{k-1},x_k]\}\quad\forall k\in\{1,2,\dots,n\}).$$

U(f,X) & L(f,X) ARE NOT ALWAYS RIEMANN SUMS (R1·8)

Note that, in gener $U(f,X)$ and $L(f,X)$ are not always Riemann sums, as we do not always have that $M_k=f(t_k)$ or $m_k=f(s_k)$ $\forall t_k,s_k\in[x_{k-1},x_k]$, where $k\in\{1,2,\dots,n\}$.

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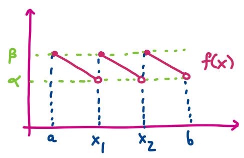


$m_2=\inf(\{f(t):\ t\in[x_1,x_2]\})$

$\therefore m_2=\alpha$

But $f(t)\neq\alpha$ $\forall t\in[x_1,x_2]$!

(So $L(f,X)$ is not a Riemann sum.)

U(f,X) & L(f,X) ARE RIEMANN SUMS IF f IS STRICTLY MONOTONIC (R1·8)

Let X be a partition of $[a,b]$, and $f:[a,b]\to\mathbb{R}$ be bounded.

Assume f is strictly monotonic (ie increasing or decreasing).

Then $U(f,X)$ and $L(f,X)$ are Riemann sums for f on X.

Proof. If f is increasing, then $M_k=f(x_k)$ and $m_k=f(x_{k-1})$ $\forall k\in\{1,2,\dots,n\}$, so that $U(f,X)$ and $L(f,X)$ are indeed Riemann sums.

Similarly, if f is decreasing, then $M_k=f(x_{k-1})$ and $m_k=f(x_k)$ $\forall k\in\{1,2,\dots,n\}$ so that $U(f,X)$ and $L(f,X)$ are indeed Riemann sums. ☒

U(f,X) & L(f,X) ARE RIEMANN SUMS IF f IS CONTINUOUS (R1·8)

Let X be a partition of $[a,b]$, and let $f:[a,b]\to\mathbb{R}$ be bounded.

Assume f is continuous on $[a,b]$.

Then $U(f,X)$ and $L(f,X)$ are Riemann sums for f on X.

Proof. By the Extreme Value Theorem, there necessarily exists some $t_k,s_k\in[x_{k-1},x_k]$ such that $f(s_k)\leqslant f(t)\leqslant f(t_k)$ $\forall t\in[x_{k-1},x_k]$ for each $k\in\{1,2,\dots,n\}$.

Let $m_k=f(s_k)$ and $M_k=f(t_k)$. Then since $t_k,s_k\in[x_{k-1},x_k]$, it follows that $U(f,X)$ and $L(f,X)$ are indeed Riemann sums for f on X. ☒

U(f,X) IS THE LARGEST RIEMANN SUM & L(f,X) IS THE SMALLEST RIEMANN SUM FOR f ON X (N1·9)

Let $\mathcal{T}$ be the set of all Riemann sums for a bounded function $f:[a,b]\to\mathbb{R}$ on a partition X of $[a,b]$.

Then $U(f,X)=\sup(\mathcal{T})$ and $L(f,X)=\inf(\mathcal{T})$.

In particular, we have that

$$L(f,X)\leqslant S\leqslant U(f,X)$$

for every $S\in\mathcal{T}$.

Proof. We prove the former statement, since the proof for the latter is similar.

Then, note for any $S\in\mathcal{T}$, we have that

$$S=\sum_{k=1}^{n} f(t_k)\Delta_k x\leqslant\sum_{k=1}^{n} M_k\Delta_k x=U(f,X),$$

by construction of M_k.

Hence $U(f,X)$ is an upper bound for $\mathcal{T}$, and so necessarily $U(f,X)\geqslant\sup(\mathcal{T})$.

Next, let $\varepsilon>0$ be arbitrary.

Then, since $M_k=\sup(\{f(t)\mid t\in[x_{k-1},x_k]\})$, we can choose a $t_k\in[x_{k-1},x_k]$ with $M_k-f(t_k)<\frac{\varepsilon}{b-a}$, for every $k\in\{1,2,\dots,n\}$.

Hence, it follows that there exists a $S\in\mathcal{T}$ such that

$$\begin{aligned}U(f,X)-S&=\sum_{k=1}^{n} M_k\Delta_k x-\sum_{k=1}^{n} f(t_k)\Delta_k x\\&=\sum_{k=1}^{n}(M_k-f(t_k))\Delta_k x\\&<\sum_{k=1}^{n}\frac{\varepsilon}{b-a}\Delta_k x\\&=\frac{\varepsilon}{b-a}(b-a)\end{aligned}$$

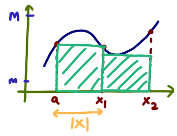
$\therefore U(f,X)-S<\varepsilon$,

and since $\varepsilon>0$ was arbitrary it follows that

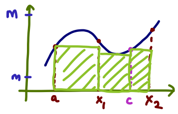
$$\sup(\mathcal{T})=U(f,X),$$

as needed. ☒

$0 \leq L(f, X \cup \{c\}) - L(f,X) \leq (M-m)|X|$,
$0 \leq U(f, X) - U(f, X \cup \{c\}) \leq (M-m)|X|$ (L1.10)



Let $f : [a,b] \to \mathbb{R}$ be bounded with upper bound M and lower bound m.
Let X and $Y = X \cup \{c\}$ be partitions of $[a,b]$, where $c \notin X$.
Then
$$0 \leq L(f,Y) - L(f,X) \leq (M-m)|X|,$$
and
$$0 \leq U(f,X) - U(f,Y) \leq (M-m)|X|.$$



($0 \leq$ [green] $-$ [teal] $\leq (M-m)|X|$)

*similar illustration for second statement

Proof. We prove the first statement, as the proof for the second statement is similar.

Say $X = \{x_0, x_1, \dots, x_n\}$ and $c \in [x_{k-1}, x_k]$ for some $k \in \{1, 2, \dots, n\}$, so that
$$Y = \{x_0, x_1, \dots, x_{k-1}, c, x_k, \dots, x_n\}.$$

Then
$$L(f,Y) - L(f,X) = [r(c - x_{k-1}) + s(x_k - c)] - m_k(x_k - x_{k-1}),$$
where $r = \inf(\{f(t) \mid t \in [x_{k-1}, c]\})$, $s = \inf(\{f(t) \mid t \in [c, x_k]\})$
and $m_k = \inf(\{f(t) \mid t \in [x_{k-1}, x_k]\})$.

Next, since $m_k = \min(r,s)$, it follows that $r \geq m_k$ & $s \geq m_k$ so that
$$L(f,Y) - L(f,X) \geq m_k(c - x_{k-1}) + m_k(x_k - c) - m_k(x_k - x_{k-1}) = 0,$$
establishing the first inequality.

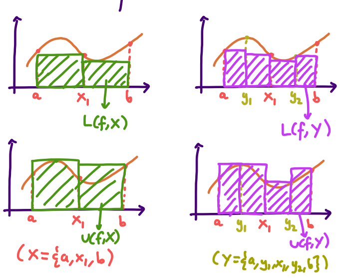
Then, since $r \leq M$ and $s \leq M$, & $m_k \geq m$ by construction, it also follows that
$$\begin{aligned} L(f,Y) - L(f,X) &\leq M(c - x_{k-1}) + M(x_k - c) - m(x_k - x_{k-1}) \\ &= (M-m)(x_k - x_{k-1}) \\ &\leq (M-m)|X|. \end{aligned}$$

Therefore, we have that
$$0 \leq L(f,Y) - L(f,X) \leq (M-m)|X|,$$
which we wanted to prove. ■

$L(f,X) \leq L(f,Y) \leq U(f,Y) \leq U(f,X)$; $X \subseteq Y$ (N1.11)

Let X and Y be partitions of $[a,b]$, such that $X \subseteq Y$.
Let $f : [a,b] \to \mathbb{R}$ be bounded.
Then, we always have that
$$L(f,X) \leq L(f,Y) \leq U(f,Y) \leq U(f,X).$$

Proof. If Y is obtained by adding one point to X, then this follows from the above lemma.
In general, Y can be obtained by adding finitely many points to X, one point at a time. ■



$L(f,X) \leq L(f,Y) \leq U(f,Y) \leq U(f,Y)$

$L(f,X) \leq U(f,Y)$ (N1.12)

Let X and Y be any partitions of $[a,b]$.
Then necessarily $L(f,X) \leq U(f,Y)$.

Proof. Let $Z = X \cup Y$. Then by the above note,
$$L(f,X) \leq L(f,Z) \leq U(f,Z) \leq U(f,Y),$$
and the proof follows from here. ■

UPPER & LOWER INTEGRALS (D1.13)

Let $f:[a,b]\to\mathbb{R}$ be bounded.
Then, the "upper integral" of f on $[a,b]$, denoted by $U(f)$, is defined to be

$$U(f) = \inf(\{U(f,X) \mid X \text{ is a partition of } [a,b]\}).$$

Similarly, the "lower integral" of f on $[a,b]$, denoted by $L(f)$, is defined to be

$$L(f) = \sup(\{L(f,X) \mid X \text{ is a partition of } [a,b]\}).$$

Note that $U(f)$ and $L(f)$ always exist even if f is not integrable. (N1.14)

$L(f) \leq U(f)$ (N1.15)

Let $f:[a,b]\to\mathbb{R}$ be a bounded function.
Then $L(f)\leq U(f)$.

Proof. Let $\varepsilon>0$ be arbitrary.
Choose partitions X_1 and X_2 such that
$L(f) - L(f,X_1) < \frac{\varepsilon}{2}$ and $U(f,X_2) - U(f) < \frac{\varepsilon}{2}$.
Then

$$\begin{aligned} U(f) - L(f) &= (U(f) - U(f,X_2)) + (U(f,X_2) - L(f,X_1)) + (L(f,X_1) - L(f)) \\ &> -\frac{\varepsilon}{2} + 0 - \frac{\varepsilon}{2} \\ &= -\varepsilon. \end{aligned}$$

Hence $L(f) - U(f) < \varepsilon$, and since $\varepsilon>0$ was arbitrary, this in turn implies that $L(f)\leq U(f)$. ☒

EQUIVALENT DEFINITIONS OF INTEGRABILITY (T1.16)

Let $f:[a,b]\to\mathbb{R}$ be bounded.
Then the following statements are equivalent:

① $L(f) = U(f)$;

② For any $\varepsilon>0$, there exists a partition X such that $U(f,X) - L(f,X) < \varepsilon$; and

③ f is integrable on $[a,b]$.

Proof. First, we show ①⇒②.
Suppose $L(f) = U(f)$. Let $\varepsilon>0$ be arbitrary.
Then, choose partitions X_1 and X_2 so that
$L(f) - L(f,X_1) < \frac{\varepsilon}{2}$ and $U(f,X_2) - U(f) < \frac{\varepsilon}{2}$.
Let $X = X_1 \cup X_2$.
Next, since $L(f,X_1) \leq L(f,X) \leq L(f)$ (as $X_1\subseteq X$),
it follows that $L(f) - L(f,X) \leq L(f) - L(f,X_1) < \frac{\varepsilon}{2}$,
and since $U(f)\leq U(f,X)\leq U(f,X_2)$ (as $X_2\subseteq X$), it follows that
$U(f,X) - U(f) < \frac{\varepsilon}{2}$ also.
Hence

$$\begin{aligned} U(f,X) - L(f,X) &= [U(f,X) - U(f)] + [U(f) - L(f)] + [L(f) - L(f,X)] \\ &< \frac{\varepsilon}{2} + 0 + \frac{\varepsilon}{2} \\ &= \varepsilon, \end{aligned}$$

which is sufficient to show that ② is true. ※

Subsequently, we show ②⇒①.
Suppose for any $\varepsilon>0$, there exists a partition X such that
$U(f,X) - L(f,X) < \varepsilon$.
Fix $\varepsilon>0$, and choose X so that $U(f,X) - L(f,X) < \varepsilon$.
Then

$$\begin{aligned} U(f) - L(f) &= [U(f) - U(f,X)] + [U(f,X) - L(f,X)] + [L(f,X) - L(f)] \\ &< 0 + \varepsilon + 0 \\ &= \varepsilon. \end{aligned}$$

Since $0\leq U(f) - L(f) < \varepsilon$ $\forall\varepsilon>0$, this tells us that
$U(f) = L(f)$, proving ①. ※

Next, we show ③⇒②.
Suppose f is integrable on $[a,b]$, with $I = \int_a^b f(x)\,dx$.
Let $\varepsilon>0$. Then, choose a $\delta>0$ such that
for every partition X with $|X|<\delta$, we have that
$|S - I| < \frac{\varepsilon}{4}$ for every Riemann sum S for f on X.
Let S_1 and S_2 be Riemann sums for f on X
such that $|U(f,X) - S_1| < \frac{\varepsilon}{4}$ and $|S_2 - L(f,X)| < \frac{\varepsilon}{4}$.
Then, by the Triangle Inequality,

$$\begin{aligned} |U(f,X) - L(f,X)| &\leq |U(f,X) - S_1| + |S_1 - I| + |I - S_2| + |S_2 - L(f,X)| \\ &< \frac{\varepsilon}{4} + \frac{\varepsilon}{4} + \frac{\varepsilon}{4} + \frac{\varepsilon}{4} \\ &= \varepsilon, \end{aligned}$$

which is sufficient to prove ②. ※

Lastly, we prove ①⇒③.
Suppose $L(f) = U(f)$, and let $I = L(f) = U(f)$.
Then, let $\varepsilon>0$.
Choose a partition X_0 of $[a,b]$ so that
$L(f) - L(f,X_0) < \frac{\varepsilon}{2}$ and $U(f,X_0) - U(f) < \frac{\varepsilon}{2}$.
Say $X_0 = \{x_0, x_1, \dots, x_n\}$, and set $\delta = \frac{\varepsilon}{2(n-1)(M-m)}$,
where M and m are upper and lower bounds for f on $[a,b]$.
Let X be any partition of $[a,b]$ with $|X|<\delta$,
and $Y = X_0 \cup X$.
Note that Y is obtained from X by adding at most $n-1$ points, and that each time we add a point, the size of the new partition is at most $|X|<\delta$.
Hence

$$0 \leq U(f,X) - U(f,Y) \leq (n-1)(M-m)|X| < (n-1)(M-m)\delta = \frac{\varepsilon}{2}$$

and

$$0 \leq L(f,Y) - L(f,X) \leq (n-1)(M-m)|X| < (n-1)(M-m)\delta = \frac{\varepsilon}{2}$$

by the first lemma on the previous page.
Next, let S be any Riemann sum for f on X.
Note that $L(f,X_0) \leq L(f,Y) \leq L(f) = U(f) \leq U(f,Y) \leq U(f,X_0)$
and $L(f,X) \leq S \leq U(f,X)$, so that

$$\begin{aligned} S - I &\leq U(f,X) - I \\ &= U(f,X) - U(f) \\ &= (U(f,X) - U(f,Y)) + (U(f,Y) - U(f)) \\ &\leq (U(f,X) - U(f,Y)) + (U(f,X_0) - U(f)) \\ &< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon, \end{aligned}$$

and

$$\begin{aligned} I - S &\leq I - L(f,X) \\ &= L(f) - L(f,X) \\ &= (L(f) - L(f,Y)) + (L(f,Y) - L(f,X)) \\ &\leq (L(f) - L(f,X_0)) + (L(f,Y) - L(f,X)) \\ &< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon, \end{aligned}$$

and since $\varepsilon>0$ was arbitrary this is sufficient to prove ③. ☒

INTEGRALS OF CONTINUOUS FUNCTIONS

CONTINUOUS FUNCTIONS ARE ALWAYS INTEGRABLE (T1.17)

Let $f : [a,b] \to \mathbb{R}$ be continuous. Then f is integrable on $[a,b]$.

Proof. First, note f is uniformly continuous on $[a,b]$.

Hence, we can choose a $\delta > 0$ so that for all $x, y \in [a,b]$, we have that $|x-y| < \delta$ implies $|f(x) - f(y)| < \frac{\varepsilon}{b-a}$.

Let X be any partition of $[a,b]$ with $|X| < \delta$.

Then, by the Extreme Value Theorem, there exists some $t_k, s_k \in [x_{k-1}, x_k]$ such that $m_k = f(s_k) \leqslant t \leqslant M_k = f(t_k)$ $\forall t \in [x_{k-1}, x_k]$, where $k \in \{1, 2, \dots, n\}$.

Finally, since $|t_k - s_k| \leqslant |x_k - x_{k-1}| \leqslant |X| = \delta$, it follows that $|M_k - m_k| = |f(t_k) - f(s_k)| < \frac{\varepsilon}{b-a}$ (since f is uniformly continuous).

Thus

$$U(f,X) - L(f,X) = \sum_{k=1}^{n} (M_k - m_k) \Delta_k x < \frac{\varepsilon}{b-a} \sum_{k=1}^{n} \Delta_k x = \varepsilon,$$

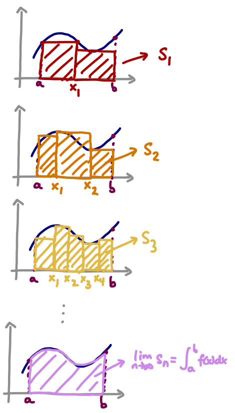
and as $\varepsilon > 0$ was arbitrary this tells us that $U(f,X) = L(f,X)$, which by the equivalent definitions of integrability implies that f is integrable on $[a,b]$. ▨

SEQUENTIAL CHARACTERISATION OF INTEGRATION (N1.18)

Let $f : [a,b] \to \mathbb{R}$ be integrable on $[a,b]$, and let $\{X_n\}$ be a sequence of partitions of $[a,b]$ with $\lim_{n\to\infty} |X_n| = 0$. For any given $n \in \mathbb{N}$, let S_n be any Riemann sum for f on X_n.

Then the sequence $\{S_n\}$ necessarily converges, with

$$\lim_{n\to\infty} S_n = \int_a^b f(x)\,dx.$$



Proof. Denote $I = \int_a^b f(x)\,dx$.

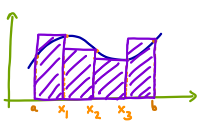
Then, given a $\varepsilon > 0$, choose a $\delta > 0$ so that for every partition X of $[a,b]$ with $|X| < \delta$, we have that $|S - I| < \varepsilon$ for every Riemann sum S for f on X.

Choose a $N \in \mathbb{N}$ so that if $n > N$, $|X_n| < \delta$. (We can do this since $\{|X_n|\} \to 0$.)

It follows that if $n > N$, then $|S_n - I| < \varepsilon$, and as $\varepsilon > 0$ was arbitrary this is sufficient to prove that $\lim_{n\to\infty} S_n = \int_a^b f(x)\,dx$. ▨

Let $f : [a,b] \to \mathbb{R}$ be integrable on $[a,b]$. Then, if we let X_n be the partition of $[a,b]$ into n equal-sized sub-intervals, and S_n be the Riemann sum on X_n using right-endpoints, it follows from the above that

$$\int_a^b f(x)\,dx = \lim_{n\to\infty} \sum_{k=1}^{n} f(x_{n,k}) \Delta_{n,k} x = \lim_{n\to\infty} \left(\frac{b-a}{n}\right) \sum_{k=1}^{n} f\left(a + \frac{b-a}{n} k\right). \quad \text{(N1.19)}$$



Note $x_1 - a = x_2 - x_1 = x_3 - x_2 = b - x_3$, so that $x_n - x_{n-1} = \frac{b-a}{4}$ for each $n \in \{1,2,3,4\}$.

Hence

$$▨ = \left(\frac{b-a}{4}\right)[f(x_1) + f(x_2) + f(x_3) + f(x_4)] = \left(\frac{b-a}{4}\right) \sum_{k=1}^{4} f(x_k).$$

EXAMPLE: INTEGRAL OF $f(x) = 2^x$ (E1.20)

We can use the previous derived results to evaluate integrals of specific continuous functions; eg $\int_0^2 2^x\,dx$.

Let $f(x) = 2^x$. Note f is continuous, and hence integrable (on $[0,2]$).

Then, using the formula from N1.19, we have that

$$\begin{aligned}
\int_0^2 2^x\,dx &= \lim_{n\to\infty} \sum_{k=1}^{n} f(x_{n,k}) \Delta_{n,k} x \\
&= \lim_{n\to\infty} \sum_{k=1}^{n} f\left(\frac{2k}{n}\right)\left(\frac{2}{n}\right) \quad (\text{since } |[0,2]| = 2) \\
&= \lim_{n\to\infty} \sum_{k=1}^{n} 2^{\frac{2k}{n}} \cdot \frac{2}{n} \\
&= \lim_{n\to\infty} \frac{2 \cdot 4^{\frac{1}{n}}}{n} \cdot \frac{4-1}{4^{\frac{1}{n}} - 1} \quad (\text{by the formula for the sum of a geometric sequence}) \\
&= \lim_{n\to\infty} (6 \cdot 4^{\frac{1}{n}}) \lim_{n\to\infty} \frac{1}{n(4^{\frac{1}{n}} - 1)} \\
&= 6 \lim_{n\to\infty} \frac{\left(\frac{1}{n}\right)}{4^{\frac{1}{n}} - 1} \\
&= 6 \lim_{x\to 0^+} \frac{x}{4^x - 1} \\
&= 6 \lim_{x\to 0^+} \frac{1}{\ln(4) \cdot 4^x} \quad (\text{by L'Hôpital's rule, since } \tfrac{x}{4^x - 1} = \tfrac{0}{0} \text{ if } x = 0) \\
&= \frac{6}{\ln(4)}
\end{aligned}$$

$$\therefore \int_0^2 2^x\,dx = \frac{3}{\ln(2)}.$$

SUMMATION FORMULAS (L1.21)

Note that

① $\sum_{i=1}^{n} 1 = n$;

② $\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$;

③ $\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}$; and

④ $\sum_{i=1}^{n} i^3 = \frac{n^2(n+1)^2}{4}$.

Proof. ① is trivial, so we prove ② first.

Consider $\sum_{k=1}^{n} (k^2 - (k-1)^2)$.

On one hand,

$$\sum_{k=1}^{n} (k^2 - (k-1)^2) = (1^2 - 0^2) + (2^2 - 1^2) + \dots + (n^2 - (n-1)^2) = n^2,$$

but on the other hand,

$$\sum_{k=1}^{n} (k^2 - (k-1)^2) = \sum_{k=1}^{n} (k^2 - (k^2 - 2k + 1)) = 2\sum_{k=1}^{n} k - \sum_{k=1}^{n} 1.$$

Hence $n^2 = 2\sum_{k=1}^{n} k - n$,

so that

$$\sum_{k=1}^{n} k = \frac{1}{2}(n^2 + n) = \frac{n(n+1)}{2}. \quad \#$$

Next, we prove ③.

Consider $\sum_{k=1}^{n} (k^3 - (k-1)^3)$.

On one hand,

$$\sum_{k=1}^{n} (k^3 - (k-1)^3) = (1^3 - 0^3) + (2^3 - 1^3) + \dots + (n^3 - (n-1)^3) = n^3.$$

On the other hand,

$$\begin{aligned}
\sum_{k=1}^{n} (k^3 - (k-1)^3) &= \sum_{k=1}^{n} (k^3 - (k^3 - 3k^2 + 3k - 1)) \\
&= 3\sum_{k=1}^{n} k^2 - 3\sum_{k=1}^{n} k + \sum_{k=1}^{n} 1 \\
&= 3\sum_{k=1}^{n} k^2 - 3\frac{n(n+1)}{2} + n.
\end{aligned}$$

Equating these, we get that

$$n^3 = 3\sum_{k=1}^{n} k^2 - 3\frac{n(n+1)}{2} + n$$

which eventually simplifies to

$$\sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6}. \quad \#$$

Lastly, we prove ④.

Consider $\sum_{k=1}^{n} (k^4 - (k-1)^4)$.

On one hand,

$$\sum_{k=1}^{n} (k^4 - (k-1)^4) = (1^4 - 0^4) + (2^4 - 1^4) + \dots + (n^4 - (n-1)^4) = n^4,$$

and on the other hand,

$$\begin{aligned}
\sum_{k=1}^{n} (k^4 - (k-1)^4) &= 4\sum_{k=1}^{n} k^3 - 6\sum_{k=1}^{n} k^2 + 4\sum_{k=1}^{n} k - \sum_{k=1}^{n} 1 \\
&= 4\sum_{k=1}^{n} k^3 - 6\frac{n(n+1)(2n+1)}{6} + 4\frac{n(n+1)}{2} - n.
\end{aligned}$$

Hence

$$n^4 = 4\sum_{k=1}^{n} k^3 - 6\frac{n(n+1)(2n+1)}{6} + 4\frac{n(n+1)}{2} - n,$$

which simplifies to

$$\sum_{k=1}^{n} k^3 = \frac{n^2(n+1)^2}{4}. \quad \#$$

USING SUMMATION FORMULAS TO CALCULATE INTEGRALS (E1.22)

We can use summation formulae to calculate integrals of certain functions;

eg $\int_1^3 (x+2x^3)\,dx$.

Note that

$$\begin{aligned}
\int_1^3 (x+2x^3)\,dx &= \lim_{n\to\infty} \sum_{k=1}^{n} f(x_{n,k})\,\Delta_{n,k}x \\
&= \lim_{n\to\infty} \sum_{k=1}^{n} f(1+\tfrac{2}{n}k)\left(\tfrac{2}{n}\right) \quad (\text{since } |[1,3]|=2) \\
&= \lim_{n\to\infty} \sum_{k=1}^{n} \left((1+\tfrac{2}{n}k) + 2(1+\tfrac{2}{n}k)^3\right)\left(\tfrac{2}{n}\right) \\
&= \lim_{n\to\infty} \sum_{k=1}^{n} \left(\tfrac{6}{n} + \tfrac{28}{n^2}k + \tfrac{48}{n^3}k^2 + \tfrac{32}{n^4}k^3\right) \\
&= \lim_{n\to\infty} \left(\tfrac{6}{n}\sum_{k=1}^{n} 1 + \tfrac{28}{n^2}\sum_{k=1}^{n} k + \tfrac{48}{n^3}\sum_{k=1}^{n} k^2 + \tfrac{32}{n^4}\sum_{k=1}^{n} k^3\right) \\
&= \lim_{n\to\infty} \left(\tfrac{6}{n}\cdot n + \tfrac{28}{n^2}\cdot\tfrac{n(n+1)}{2} + \tfrac{48}{n^3}\cdot\tfrac{n(n+1)(2n+1)}{6} + \tfrac{32}{n^4}\cdot\tfrac{n^2(n+1)^2}{4}\right) \\
&= 6 + \tfrac{28}{2} + \tfrac{48}{6} + \tfrac{32}{4}
\end{aligned}$$

$\therefore \int_1^3 (x+2x^3)\,dx = 44$.

BASIC PROPERTIES OF INTEGRALS

LINEARITY (T1.23)

Let f and g be integrable on $[a,b]$.
Let $c\in\mathbb{R}$ be arbitrary.
Then $(f+g)$ and $c\cdot f$ are both integrable on $[a,b]$, and

$$\int_a^b (f+g) = \int_a^b f + \int_a^b g$$

and

$$\int_a^b cf = c\int_a^b f.$$

Proof. Note that

$$\begin{aligned}
\int_a^b f + \int_a^b g &= \lim_{n\to\infty}\sum_{k=1}^{n} f(x_{n,k})\,\Delta_{n,k}x + \lim_{n\to\infty}\sum_{k=1}^{n} g(x_{n,k})\,\Delta_{n,k}x \\
&= \lim_{n\to\infty} \frac{b-a}{n}\sum_{k=1}^{n}(f+g)(x_{n,k}) \\
&= \int_a^b (f+g),
\end{aligned}$$

and that

$$\begin{aligned}
c\int_a^b f &= c\lim_{n\to\infty}\sum_{k=1}^{n} f(x_{n,k})\,\Delta_{n,k}x \\
&= \lim_{n\to\infty}\frac{b-a}{n}\sum_{k=1}^{n}(cf)(x_{n,k}) \\
&= \int_a^b cf. \quad \blacksquare
\end{aligned}$$

COMPARISON (T1.24)

Let f and g be integrable on $[a,b]$.
Suppose $f(x)\leq g(x)$ $\forall x\in[a,b]$.
Then

$$\int_a^b f \leq \int_a^b g.$$

Proof. Note that

$$\begin{aligned}
\int_a^b f &= \lim_{n\to\infty}\sum_{k=1}^{n} f(x_{n,k})\,\Delta_{n,k}x \\
&\leq \lim_{n\to\infty}\sum_{k=1}^{n} g(x_{n,k})\,\Delta_{n,k}x \quad (\text{since } f(x)\leq g(x)\ \forall x\in[a,b]) \\
&= \int_a^b g. \quad \blacksquare
\end{aligned}$$

ADDITIVITY (T1.25)

Let $a<b<c$, and $f:[a,c]\to\mathbb{R}$ be bounded.
Then f is integrable on $[a,c]$ if and only if f is integrable on both $[a,b]$ and $[b,c]$,
and in this case

$$\int_a^b f + \int_b^c f = \int_a^c f.$$

Proof. First, suppose f is integrable on $[a,c]$.
Choose a partition X of $[a,c]$ such that $U(f,X)-L(f,X)<\varepsilon$.
Say that $b\in[x_{k-1},x_k]$, and let $Y=\{x_0,x_1,\dots,x_{k-1},b\}$ and $Z=\{b,x_k,x_{k+1},\dots,x_n\}$, so that Y and Z are partitions on $[a,b]$ and $[b,c]$ respectively.

Then
$$\begin{aligned}
U(f,Y)-L(f,Y) &\leq U(f,X\cup\{b\}) - L(f,X\cup\{b\}) \quad (\text{by N1.11}) \\
&\leq U(f,X)-L(f,X) \quad (\text{by N1.11 also}) \\
&< \varepsilon,
\end{aligned}$$
and
$$\begin{aligned}
U(f,Z)-L(f,Z) &\leq U(f,X\cup\{b\}) - L(f,X\cup\{b\}) \\
&\leq U(f,X)-L(f,X) \\
&< \varepsilon,
\end{aligned}$$
which is sufficient to show f is integrable on both $[a,b]$ and $[b,c]$.

Conversely, suppose f is integrable on both $[a,b]$ and $[b,c]$.
Choose partitions Y of $[a,b]$ & Z of $[b,c]$ so that $U(f,Y)-L(f,Y)<\frac{\varepsilon}{2}$ and $U(f,Z)-L(f,Z)<\frac{\varepsilon}{2}$.
Then $X=Y\cup Z$ is a partition of $[a,c]$, and

$$U(f,X)-L(f,X) = [U(f,Y)+U(f,Z)] - (L(f,Y)+L(f,Z)) < \varepsilon,$$

which tells us $U(f,X)=L(f,X)$ (since $\varepsilon>0$ was arbitrary) and consequently (by the Equivalent Definitions of Integrability) that f is integrable on $[a,c]$.

Finally, suppose f is integrable on $[a,c]$, and hence also on $[a,b]$ and $[b,c]$.
Let $I_1=\int_a^b f$, $I_2=\int_b^c f$ and $I=\int_a^c f$.
Let $\varepsilon>0$ be arbitrary. Then, choose a $\delta>0$ so that for all partitions X_1, X_2 and X of $[a,b]$, $[b,c]$ and $[a,c]$ respectively, if $|X_1|, |X_2|, |X|<\delta$, then $|S_1-I_1|$, $|S_2-I_2|$, $|S-I|<\frac{\varepsilon}{3}$ for all Riemann sums S_1, S_2, S for f on X_1, X_2 & X respectively.
Choose partitions X_1 and X_2 of $[a,b]$ and $[b,c]$ with $|X_1|<\delta$ and $|X_2|<\delta$.
Choose Riemann sums S_1 and S_2 for f on X_1 and X_2.
Let $X=X_1\cup X_2$, and note that $|X|<\delta$ and $S=S_1+S_2$ is a Riemann sum for f on X.
Then necessarily

$$\begin{aligned}
|I-(I_1+I_2)| &= |(I-S)+(S_1-I_1)+(S_2-I_2)| \\
&\leq |I-S| + |S_1-I_1| + |S_2-I_2| \quad (\text{by the Triangle Inequality}) \\
&\leq \frac{\varepsilon}{3}+\frac{\varepsilon}{3}+\frac{\varepsilon}{3} \\
&= \varepsilon,
\end{aligned}$$

and since $\varepsilon>0$ was arbitrary this is sufficient to prove that $I=I_1+I_2$. $\blacksquare$

PIECEWISE CONTINUOUS FUNCTIONS ARE INTEGRABLE (C1.26)

Let $X = \{x_0, x_1, \cdots, x_n\}$ be a partition of $[a,b]$, and let $g_k : [x_{k-1}, x_k] \to \mathbb{R}$ be continuous $\forall k \in \{1,2,\cdots,n\}$. Let $f : [a,b] \to \mathbb{R}$ be a function with $f(t) = g_k(t)$ $\forall t \in (x_{k-1}, x_k)$. Then f is integrable on $[a,b]$ with

$$\int_a^b f(x)\,dx = \sum_{k=1}^n \int_{x_{k-1}}^{x_k} g_k(x)\,dx.$$

Proof. This follows from the additivity and linearity properties of integrals. ☒

$\int_a^a f = 0$, $\int_b^a f = -\int_a^b f$ (D1.27)

For any function f and $a \in \mathbb{R}$,

$$\int_a^a f = 0.$$

Additionally, if $\int_a^b f(x)\,dx$ exists, then

$$\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx.$$

Note that this definition can be used to extend the scope of the Additivity Theorem to the case where $a,b,c \in \mathbb{R}$ are not in increasing order. (N1.28)

ESTIMATION (T1.29)

Let f be integrable on $[a,b]$. Then $|f|$ is also integrable on $[a,b]$, and

$$\left|\int_a^b f\right| \leq \int_a^b |f|.$$

Proof. Let $\varepsilon > 0$ be arbitrary. Choose a partition X of $[a,b]$ such that $U(f,X) - L(f,X) < \varepsilon$.
Denote $M_k(f) = \sup\{f(t) \mid t \in [x_{k-1}, x_k]\}$ and $M_k(|f|) = \sup\{|f(t)| \mid t \in [x_{k-1}, x_k]\}$ $\forall k \in \{1,2,\cdots,n\}$, with similar definitions for $m_k(f)$ and $m_k(|f|)$.
Then,

① if $0 \leq m_k(f) \leq M_k(f)$, $M_k(|f|) = M_k(f)$ and $m_k(|f|) = m_k(f)$;

② if $m_k(f) \leq 0 \leq M_k(f)$, $M_k(|f|) = \max(\{M_k(f), -m_k(f)\})$ and $m_k(|f|) \geq 0$, so that $M_k(|f|) - m_k(|f|) \leq \max(\{M_k(f), -m_k(f)\}) \leq M_k(f) - m_k(f)$; &

③ if $m_k(f) \leq M_k(f) \leq 0$, $M_k(|f|) = -m_k(f)$ and $m_k(|f|) = -M_k(f)$, so that $M_k(|f|) - m_k(|f|) = M_k(f) - m_k(f)$.

In any one of these cases, we have that

$$M_k(|f|) - m_k(|f|) \leq M_k(f) - m_k(f),$$

and so

$$\begin{aligned} U(|f|,X) - L(|f|,X) &= \sum_{k=1}^n (M_k(|f|) - m_k(|f|))\,\Delta_k x \\ &\leq \sum_{k=1}^n (M_k(f) - m_k(f))\,\Delta_k x \\ &= U(f,X) - L(f,X) \\ &< \varepsilon, \end{aligned}$$

which is sufficient to prove that $|f|$ is integrable on $[a,b]$.

Next, let $\varepsilon > 0$ be arbitrary. Choose a partition X on $[a,b]$ and choose values $t_k \in [x_{k-1}, x_k]$ $\forall k \in \{1,2,\cdots,n\}$ so that

$$\left|\sum_{k=1}^n f(t_k)\Delta_k x - \int_a^b f\right| < \frac{\varepsilon}{2} \quad\text{and}\quad \left|\sum_{k=1}^n |f(t_k)|\Delta_k x - \int_a^b |f|\right| < \frac{\varepsilon}{2}.$$

Note by the Triangle Inequality that

$$\sum_{k=1}^n f(t_k)\Delta_k x \leq \sum_{k=1}^n |f(t_k)|\Delta_k x,$$

so that

$$\begin{aligned} \left|\int_a^b f\right| - \int_a^b |f| &= \left(\left|\int_a^b f\right| - \left|\sum_{k=1}^n f(t_k)\Delta_k x\right|\right) + \left(\left|\sum_{k=1}^n f(t_k)\Delta_k x\right| - \sum_{k=1}^n |f(t_k)|\Delta_k x\right) \\ &\quad + \left(\sum_{k=1}^n |f(t_k)|\Delta_k x - \int_a^b |f|\right) \\ &< \frac{\varepsilon}{2} + 0 + \frac{\varepsilon}{2} \\ &= \varepsilon. \end{aligned}$$

Since $\varepsilon > 0$ was arbitrary, this tells us that

$$\left|\int_a^b f\right| - \int_a^b |f| \leq 0,$$

as required. ☒

THE FUNDAMENTAL THEOREM OF CALCULUS

First, note that for any function F, defined on an interval containing $[a,b]$, we write

$$[F(x)]_a^b = F(b) - F(a). \qquad (\text{NT}1.30)$$

Let f be integrable on $[a,b]$.
Define $F : [a,b] \to \mathbb{R}$ by

$$F(x) = \int_a^x f(t)\,dt.$$

Then F is continuous on $[a,b]$.
Moreover, if f is continuous at a point $x \in [a,b]$, then F is differentiable at x and

$$F'(x) = f(x). \qquad (\text{T}1.31)$$

Proof. Let m be an upper bound for $|f|$ on $[a,b]$.
Then, for any $a \leq x, y \leq b$, we have

$$\begin{aligned}
|F(y) - F(x)| &= \left|\int_a^y f - \int_a^x f\right| \\
&= \left|\int_x^y f\right| \quad (\text{by additivity / linearity}) \\
&\leq \left|\int_x^y |f|\right| \quad (\text{by estimation}) \\
&\leq \left|\int_x^y m\right| \\
&= m|y-x|,
\end{aligned}$$

so that given an $\varepsilon > 0$, we can choose a $\delta = \frac{\varepsilon}{m}$ to get that $|y-x| < \delta$ implies that $|F(y) - F(x)| \leq m|y-x| < m\delta = \varepsilon$, showing F is continuous (indeed, uniformly continuous) on $[a,b]$.

Subsequently, suppose f is continuous at some $x \in [a,b]$.
Then, for any $a \leq x, y \leq b$ with $x \neq y$, we have that

$$\begin{aligned}
\left|\frac{F(y) - F(x)}{y-x} - f(x)\right| &= \left|\frac{\int_a^y f - \int_a^x f}{y-x} - f(x)\right| \\
&= \left|\frac{\int_x^y f}{y-x} - \frac{\int_x^y f(x)\,dt}{y-x}\right| \\
&= \frac{1}{|y-x|}\left|\int_x^y (f(t) - f(x))\,dt\right| \\
&\leq \frac{1}{|y-x|}\left|\int_x^y |f(t) - f(x)|\,dt\right|.
\end{aligned}$$

Let $\varepsilon > 0$ be arbitrary. Since f is continuous at x, it follows that we can choose a $\delta > 0$ so that if $|y-x| < \delta$, then $|f(y) - f(x)| < \varepsilon$.
So, if $0 < |y-x| < \delta$, then

$$\begin{aligned}
\left|\frac{F(y) - F(x)}{y-x} - f(x)\right| &\leq \frac{1}{|y-x|}\left|\int_x^y |f(t) - f(x)|\,dt\right| \\
&\leq \frac{1}{|y-x|}\left|\int_x^y \varepsilon\,dt\right| \\
&= \frac{1}{|y-x|}\,\varepsilon\,|y-x| \\
&= \varepsilon,
\end{aligned}$$

showing that $F'(x)$ exists and $F'(x) = f(x)$ (as $\varepsilon > 0$ was arbitrary). ☒

Let f be integrable on $[a,b]$, and F be differentiable on $[a,b]$ with $F' = f$.
Then

$$\int_a^b f = [F(x)]_a^b = F(b) - F(a). \qquad (\text{T}1.31)$$

Proof. Let $\varepsilon > 0$ be arbitrary.
Choose a $\delta > 0$ so that for every partition X of $[a,b]$ with $|X| < \delta$, we have that

$$\left|\int_a^b f - \sum_{k=1}^n f(t_k)\Delta_k x\right| < \varepsilon$$

for every choice of sample points $t_k \in [x_{k-1}, x_k]$.

Then, choose sample points $t_k \in [x_{k-1}, x_k]$ so that

$$F'(t_k) = \frac{F(x_k) - F(x_{k-1})}{x_k - x_{k-1}},$$

which we can do by the Mean Value Theorem.
This implies that $f(t_k)\Delta_k x = F(x_k) - F(x_{k-1})$.
Hence

$$\begin{aligned}
\sum_{k=1}^n f(t_k)\Delta_k x &= \sum_{k=1}^n (F(x_k) - F(x_{k-1})) \\
&= (\cancel{F(x_1)} - F(x_0)) + (\cancel{F(x_2)} - \cancel{F(x_1)}) + \cdots + (F(x_n) - \cancel{F(x_{n-1})}) \\
&= F(x_n) - F(x_0) \\
&= F(b) - F(a),
\end{aligned}$$

and consequently

$$\left|\int_a^b f - (F(b) - F(a))\right| < \varepsilon.$$

But since $\varepsilon > 0$ was arbitrary, it follows that

$$\int_a^b f = F(b) - F(a),$$

as needed. ☒

ANTIDERIVATIVE (D1.32)

We say F is an "antiderivative" for f on some interval $[a,b]$ if $F' = f$ on $[a,b]$.

In this case, we write

① $\int f = F + c,\ c \in \mathbb{R}$; or

② $\int f(x)\,dx = F(x) + c,\ c \in \mathbb{R}$. $\qquad$ (N1.34)

Note that if $G' = F' = f$ on $[a,b]$, then necessarily $(G - F)' = 0$, so that $G - F$ is constant on the interval;
ie $G = F + c$ for some $c \in \mathbb{R}$. $\qquad$ (N1.33)

EXAMPLE: $\int_0^{\sqrt{3}} \frac{dx}{1+x^2}$ (E1.35)

We can use the Fundamental Theorem of Calculus to calculate integrals of specific functions;
eg $\int_0^{\sqrt{3}} \frac{dx}{1+x^2}$.

Since $\frac{d}{dx}(\tan^{-1}(x)) = \frac{1}{1+x^2}$, it follows that

$$\begin{aligned}
\int_0^{\sqrt{3}} \frac{dx}{1+x^2} &= [\tan^{-1}(x)]_0^{\sqrt{3}} \\
&= \tan^{-1}(\sqrt{3}) - \tan^{-1}(0)
\end{aligned}$$

$$\therefore \int_0^{\sqrt{3}} \frac{dx}{1+x^2} = \frac{\pi}{3}.$$

Chapter 2: Methods of Integration

BASIC INTEGRALS (N2.1)

Here is a list of basic integrals:

① $\int x^p\,dx = \frac{x^{p+1}}{p+1} + c, \quad p \neq -1$

② $\int \frac{1}{x}\,dx = \ln(x) + c$

③ $\int e^x\,dx = e^x + c$

④ $\int a^x\,dx = \frac{a^x}{\ln(a)} + c$

⑤ $\int \ln(x)\,dx = x\ln(x) - x + c$

⑥ $\int \sin(x)\,dx = -\cos(x) + c$

⑦ $\int \cos(x)\,dx = \sin(x) + c$

⑧ $\int \sec^2(x)\,dx = \tan(x) + c$

⑨ $\int \sec(x)\tan(x)\,dx = \sec(x) + c$

⑩ $\int \tan(x)\,dx = \ln|\cos(x)| + c$

⑪ $\int \sec(x)\,dx = \ln|\sec(x) + \tan(x)| + c$

⑫ $\int \frac{1}{1+x^2}\,dx = \tan^{-1}(x) + c$

⑬ $\int \frac{1}{\sqrt{1-x^2}}\,dx = \sin^{-1}(x) + c$

⑭ $\int \frac{1}{x\sqrt{x^2-1}}\,dx = \sec^{-1}(x) + c.$

Proof. Each of these could be verified by taking the derivative of the RHS, and confirming it matches with the function in the integral. The proof then follows from the Fundamental Theorem of Calculus. ☒

EXAMPLE 1: $\int_1^4 \frac{x^2-5}{\sqrt{x}}\,dx$ (E2.2)

We can solve the integral $\int_1^4 \frac{x^2-5}{\sqrt{x}}\,dx$ using the Fundamental Theorem of Calculus.

$$\int_1^4 \frac{x^2-5}{\sqrt{x}}\,dx = \int_1^4 x^{\frac{3}{2}} - 5x^{-\frac{1}{2}}\,dx$$
$$= \left[\frac{2}{5}x^{\frac{5}{2}} - 10x^{\frac{1}{2}}\right]_1^4$$
$$= \left(\frac{64}{5} - 20\right) - \left(\frac{2}{5} - 10\right)$$

$$\therefore \int_1^4 \frac{x^2-5}{\sqrt{x}}\,dx = \frac{12}{5}.$$

EXAMPLE 2: $\int_{\pi/6}^{\pi/3} \sin(2x) + \cos(3x)\,dx$ (E2.3)

We can also solve the integral $\int_{\pi/6}^{\pi/3} \sin(2x) + \cos(3x)\,dx$ using the Fundamental Theorem of Calculus.

First, note $\frac{d}{dx}(\cos(2x)) = -2\sin(2x)$ and $\frac{d}{dx}(\sin(3x)) = 3\cos(3x)$, it follows that $\frac{d}{dx}(-\frac{1}{2}\cos(2x)) = \sin(2x)$ and $\frac{d}{dx}(\frac{1}{3}\sin(3x)) = \cos(3x)$.

Hence

$$\int_{\pi/6}^{\pi/3} \sin(2x) + \cos(3x)\,dx = \left[-\frac{1}{2}\cos(2x) + \frac{1}{3}\sin(3x)\right]_{\pi/6}^{\pi/3}$$
$$= \left(\frac{1}{4} + 0\right) - \left(-\frac{1}{4} + \frac{1}{3}\right)$$
$$= \frac{1}{6}.$$

SUBSTITUTION / CHANGE OF VARIABLES (T2.4)

Let $u = g(x)$ be differentiable on an interval, and let $f(u)$ be continuous on the range of $g(x)$.

Then

$$\int f(g(x))\, g'(x)\, dx = \int f(u)\, du$$

and

$$\int_{x=a}^{x=b} f(g(x))\, g'(x)\, dx = \int_{u=g(a)}^{u=g(b)} f(u)\, du.$$

Proof. Let $F(u)$ be an antiderivative of $f(u)$, so that $F'(u) = f(u)$ and $\int f(u)\, du = F(u) + c$.

Then, by the Chain Rule, we know that

$$\frac{d}{dx} F(g(x)) = F'(g(x))\, g'(x) = f(g(x))\, g'(x),$$

and so, by the Fundamental Theorem of Calculus,

$$\int f(g(x))\, g'(x)\, dx = F(g(x)) + c = F(u) + c = \int f(u)\, du$$

and

$$\int_{x=a}^{x=b} f(g(x))\, g'(x)\, dx = \left[F(g(x))\right]_{x=a}^{x=b}$$
$$= F(g(b)) - F(g(a))$$
$$= \left[F(u)\right]_{u=g(a)}^{u=g(b)}$$

$$\therefore \int_{x=a}^{x=b} f(g(x))\, g'(x)\, dx = \int_{u=g(a)}^{u=g(b)} f(u)\, du. \quad \blacksquare$$

Note that if $f(u) = g(x)$, we often write $f'(u)\, du = g'(x)\, dx$. (NT2.5)

EXAMPLE 1: $\int \sqrt{2x+3}\, dx$ (E2.6)

Substitution can be used to compute integrals such as $\int \sqrt{2x+3}\, dx$.

Let $u = 2x+3$, so that $du = 2\, dx$ (using the notation from above).

Then

$$\int \sqrt{2x+3}\, dx = \int u^{\frac{1}{2}} \left(\frac{du}{2}\right)$$
$$= \frac{1}{3} u^{\frac{3}{2}} + c$$

$$\therefore \int \sqrt{2x+3}\, dx = \frac{1}{3}(2x+3)^{\frac{3}{2}} + c.$$

EXAMPLE 2: $\int x e^{x^2}\, dx$ (E2.7)

The integral $\int x e^{x^2}\, dx$ can also be solved using substitution.

Let $u = x^2$ so that $du = 2x\, dx$.

Then

$$\int x e^{x^2}\, dx = \frac{1}{2} \int e^u\, du$$
$$= \frac{1}{2} e^u + c$$

$$\therefore \int x e^{x^2}\, dx = \frac{1}{2} e^{x^2} + c.$$

EXAMPLE 3: $\int \frac{\ln(x)}{x}\, dx$ (E2.8)

Substitution can also be used to solve integrals like $\int \frac{\ln(x)}{x}\, dx$.

Let $u = \ln(x)$ so that $du = \frac{1}{x}\, dx$.

Then

$$\int \frac{\ln(x)}{x}\, dx = \int u\, du$$
$$= \frac{u^2}{2} + c$$

$$\therefore \int \frac{\ln(x)}{x}\, dx = \frac{(\ln(x))^2}{2} + c.$$

EXAMPLE 4: $\int \tan(x)\, dx$ (E2.9)

We can use substitution to solve more complicated integrals like $\int \tan(x)\, dx$.

First, note $\tan(x) = \frac{\sin(x)}{\cos(x)}$.

Then, let $u = \cos(x)$, so that $du = -\sin(x)\, dx$.

It follows that

$$\int \tan(x)\, dx = \int \frac{\sin(x)\, dx}{\cos(x)}$$
$$= \int \frac{-du}{u}$$
$$= -\ln|u| + c$$

$$\therefore \int \tan(x)\, dx = -\ln|\cos(x)| + c.$$

EXAMPLE 5: $\int \frac{dx}{x+\sqrt{x}}$ (E2.10)

Sometimes, we might have to do two substitutions to calculate some integrals;

eg $\int \frac{dx}{x+\sqrt{x}}$

First, let $u = \sqrt{x}$, so that $x = u^2$ and $2u\, du = dx$.

Then

$$\int \frac{dx}{x+\sqrt{x}} = \int \frac{2u\, du}{u^2+u} = \int \frac{2\, du}{u+1}.$$

Next, let $v = u+1$, so that $dv = du$. It follows that

$$\int \frac{dx}{x+\sqrt{x}} = \int \frac{2\, du}{u+1}$$
$$= \int \frac{2\, dv}{v}$$
$$= 2\ln|v| + c$$
$$= 2\ln|u+1| + c$$

$$\therefore \int \frac{dx}{x+\sqrt{x}} = 2\ln|\sqrt{x}+1| + c$$

EXAMPLE 6: $\int_0^2 \frac{x\, dx}{\sqrt{2x^2+1}}$ (E2.11)

When doing substitution, we need to change the values of the "endpoints" accordingly;

eg $\int_0^2 \frac{x}{\sqrt{2x^2+1}}\, dx$.

Let $u = 2x^2+1$, so that $du = 4x\, dx$.

Note that $u=1$ and $u=9$ when $x=0$ and $x=2$ respectively.

Then

$$\int_{x=0}^{x=2} \frac{x}{\sqrt{2x^2+1}}\, dx = \int_{u=1}^{u=9} \frac{\frac{1}{4} du}{\sqrt{u}}$$
$$= \int_{u=1}^{u=9} \frac{1}{4} u^{-\frac{1}{2}}\, du$$
$$= \left[\frac{1}{2} u^{\frac{1}{2}}\right]_1^9$$
$$= \frac{3}{2} - \frac{1}{2}$$

$$\therefore \int_{x=0}^{x=2} \frac{x}{\sqrt{2x^2+1}}\, dx = 1.$$

EXAMPLE 7: $\int_0^1 \frac{dx}{1+3x^2}$ (E2.12)

Sometimes, we might have to make a weird substitution to solve an integral;

eg $\int_0^1 \frac{dx}{1+3x^2}$.

Let $u = \sqrt{3}x$, so that $du = \sqrt{3}\, dx$.

Note that $x=0 \Rightarrow u=0$, and $x=1 \Rightarrow u=\sqrt{3}$.

Then

$$\int_{x=0}^{x=1} \frac{dx}{1+3x^2} = \int_{u=0}^{u=\sqrt{3}} \frac{1}{\sqrt{3}} \cdot \frac{du}{1+u^2}$$
$$= \left[\frac{1}{\sqrt{3}} \tan^{-1}(u)\right]_0^{\sqrt{3}}$$
$$= \frac{1}{\sqrt{3}}\left(\frac{\pi}{3} - 0\right)$$

$$\therefore \int_{x=0}^{x=1} \frac{dx}{1+3x^2} = \frac{\pi}{3\sqrt{3}}.$$

INTEGRATION BY PARTS (T2.13)

Let $f(x)$ and $g(x)$ be differentiable in an interval.

Then
$$\int f(x)\,g'(x)\,dx = f(x)g(x) - \int g(x)f'(x)\,dx,$$
so that
$$\int_{x=a}^{x=b} f(x)\,g'(x)\,dx = \Big[f(x)g(x) - \int g(x)f'(x)\,dx\Big]_{x=a}^{x=b}.$$

Proof. By the Product Rule,
$$\frac{d}{dx}(f(x)g(x)) = f'(x)g(x) + f(x)g'(x).$$
Hence, by the Fundamental Theorem of Calculus,
$$\int (f'(x)g(x) + f(x)g'(x))\,dx = f(x)g(x) + c,$$
which can be rewritten as
$$\int f(x)\,g'(x)\,dx = f(x)g(x) - \int g(x)f'(x)\,dx.$$
(the arbitrary constant c is not needed since there is an integral on both sides of the equation.) ☒

If we let $u = f(x)$, $du = f'(x)\,dx$, $v = g(x)$ and $dv = g'(x)\,dx$, the above formula becomes
$$\int u\,dv = uv - \int v\,du. \quad \text{(NT2.14)}$$

POLYNOMIAL × TRIGONOMETRIC OR EXPONENTIAL FUNCTION

If the integral involves a polynomial multiplied by an exponential function, or a trigonometric function, try integrating by parts with u equal to the polynomial. (N2.15)

* note: multiple applications of integration by parts may be required if the degree of the polynomial is high.

EXAMPLE 1: $\int x\sin(x)\,dx$ (E2.16)

We employ the above strategy to evaluate the integral $\int x\sin(x)\,dx$.

Integrate by parts using $\begin{pmatrix} u = x & v = -\cos(x) \\ du = 1\,dx & dv = \sin(x)\,dx \end{pmatrix}$

$$\int x\sin(x)\,dx = -x\cos(x) - \int -\cos(x)\,dx$$
$$= -x\cos(x) + \int \cos(x)\,dx$$
$$\therefore \int x\sin(x)\,dx = -x\cos(x) + \sin(x) + c.$$

EXAMPLE 2: $\int (x^2+1)e^{2x}\,dx$ (E2.17)

Similarly, we can use the above strategy to evaluate the integral $\int (x^2+1)e^{2x}\,dx$.

First, integrate by parts using $\begin{pmatrix} u = x^2+1 & v = \frac{1}{2}e^{2x} \\ du = 2x\,dx & dv = e^{2x}\,dx \end{pmatrix}$ to get
$$\int (x^2+1)e^{2x}\,dx = \frac{1}{2}(x^2+1)e^{2x} - \int \frac{1}{2}e^{2x}(2x\,dx)$$
$$= \frac{1}{2}(x^2+1)e^{2x} - \int xe^{2x}\,dx.$$

To find $\int xe^{2x}$, we integrate by parts again, this time using $\begin{pmatrix} u_2 = x & v_2 = \frac{1}{2}e^{2x}\,dx \\ du_2 = 1\,dx & dv_2 = e^{2x}\,dx \end{pmatrix}$:
$$\int (x^2+1)e^{2x}\,dx = \frac{1}{2}(x^2+1)e^{2x} - \int xe^{2x}\,dx$$
$$= \frac{1}{2}(x^2+1)e^{2x} - \left(\frac{1}{2}xe^{2x} - \int \frac{1}{2}e^{2x}\,dx\right)$$
$$= \frac{1}{2}(x^2+1)e^{2x} - \frac{1}{2}xe^{2x} + \frac{1}{4}e^{2x} + c$$
$$\therefore \int (x^2+1)e^{2x}\,dx = \frac{1}{4}(2x^2-2x+3)e^{2x} + c.$$

POLYNOMIAL × LOGARITHMIC OR INVERSE TRIGONOMETRIC FUNCTION

If the integral involves a polynomial multiplied by a logarithmic or inverse trigonometric function, try integrating by parts with u equal to the logarithmic / inverse trigonometric function. (N2.15)

EXAMPLE 1: $\int \ln(x)\,dx$ (E2.18)

We can use the above strategy to evaluate the integral $\int \ln(x)\,dx$.

Integrate by parts using $\begin{pmatrix} u = \ln(x) & v = x \\ du = \frac{1}{x}dx & dv = 1\,dx \end{pmatrix}$ to get
$$\int \ln(x)\,dx = x\ln(x) - \int x \cdot \frac{1}{x}\,dx$$
$$= x\ln(x) - \int 1\,dx$$
$$\therefore \int \ln(x)\,dx = x\ln(x) - x + c.$$

EXAMPLE 2: $\int_1^4 \sqrt{x}\ln(x)\,dx$ (E2.19)

The above strategy can even be used when the polynomial contains terms with non-integer powers, eg $\int_1^4 \sqrt{x}\ln(x)\,dx$.

Integrate by parts using $\begin{pmatrix} u = \ln(x) & v = \frac{2}{3}x^{\frac{3}{2}} \\ du = \frac{1}{x}dx & dv = x^{\frac{1}{2}}\,dx \end{pmatrix}$ to get
$$\int_1^4 \sqrt{x}\ln(x)\,dx = \left[\frac{2}{3}x^{\frac{3}{2}}\ln(x) - \int \frac{2}{3}x^{\frac{1}{2}}\,dx\right]_1^4$$
$$= \left[\frac{2}{3}x^{\frac{3}{2}}\ln(x) - \frac{4}{9}x^{\frac{3}{2}}\right]_1^4$$
$$= \left(\frac{16}{3}\ln(4) - \frac{32}{9}\right) - \left(\frac{2}{3}\ln(1) - \frac{4}{9}\right)$$
$$\therefore \int_1^4 \sqrt{x}\ln(x)\,dx = \frac{16}{3}\ln(4) - \frac{28}{9}.$$

EXPONENTIAL × SINE/COSINE FUNCTION

If the integral involves an exponential function times a sine/cosine function, try integrating by parts twice, letting u be the exponential function both times. (N2.15)

EXAMPLE: $\int e^x\sin(x)\,dx$ (E2.20)

We can use the above strategy to evaluate the integral $\int e^x\sin(x)\,dx$.

Proof. Let $I = \int e^x\sin(x)\,dx$.

Integrate by parts twice, first using $\begin{pmatrix} u_1 = e^x & v_1 = -\cos(x) \\ du_1 = e^x dx & dv_1 = \sin(x)\,dx \end{pmatrix}$, and then with $\begin{pmatrix} u_2 = e^x & v_2 = \sin(x) \\ du_2 = e^x dx & dv_2 = \cos(x)\,dx \end{pmatrix}$ to get
$$I = \int e^x\sin(x)\,dx = -e^x\cos(x) + \int e^x\cos(x)\,dx$$
$$= -e^x\cos(x) + \left(e^x\sin(x) - \int e^x\sin(x)\,dx\right)$$
$$\therefore I = -e^x\cos(x) + e^x\sin(x) - I.$$

Hence $2I = -e^x\cos(x) + e^x\sin(x) + c$,

so that $I = \frac{1}{2}(\sin(x) - \cos(x))e^x + d$.

OTHER SORTS OF PROBLEMS

EXAMPLE 1: $\int \sin^n(x)\,dx$ (E2.21)

We can use integration by parts to get a general formula for $\int \sin^n(x)\,dx$ in terms of $\int \sin^{n-2}(x)\,dx$.

Let $I = \int \sin^n(x)\,dx = \int \sin^{n-1}(x)\sin(x)\,dx$

Integrate by parts using $\begin{pmatrix} u = \sin^{n-1}(x) & v = -\cos(x) \\ du = (n-1)(\sin^{n-2}(x))(\cos(x))\,dx & dv = \sin(x)\,dx \end{pmatrix}$

to get

$$\begin{aligned} I = \int \sin^n(x)\,dx &= -\sin^{n-1}(x)\cos(x) - \int -\cos(x)(n-1)(\sin^{n-2}(x))(\cos(x))\,dx \\ &= -\sin^{n-1}(x)\cos(x) + \int (n-1)(\cos^2(x))(\sin^{n-2}(x))\,dx \\ &= -\sin^{n-1}(x)\cos(x) + \int (n-1)(1-\sin^2(x))(\sin^{n-2}(x))\,dx \\ \therefore\ I &= -\sin^{n-1}(x)\cos(x) + (n-1)\int \sin^{n-2}(x)\,dx - (n-1)I. \end{aligned}$$

Hence

$$(n-1)I + I = nI = -\sin^{n-1}(x)\cos(x) + (n-1)\int \sin^{n-2}(x)\,dx,$$

so that

$$I = -\frac{1}{n}\sin^{n-1}(x)\cos(x) + \frac{n-1}{n}\int \sin^{n-2}(x)\,dx.$$

In particular, we can use the attained above formula to evaluate $\int \sin^2(x)\,dx$ and $\int \sin^4(x)\,dx$.

In particular, when $n=2$, we get

$$\begin{aligned} \int \sin^2(x)\,dx &= -\frac{1}{2}\sin(x)\cos(x) + \frac{1}{2}\int 1\,dx \\ \therefore \int \sin^2(x)\,dx &= -\frac{1}{2}\sin(x)\cos(x) + \frac{1}{2}x + c. \end{aligned}$$

When $n=4$, we get

$$\begin{aligned} \int \sin^4(x)\,dx &= -\frac{1}{4}\sin^3(x)\cos(x) + \frac{3}{4}\int \sin^2(x)\,dx \\ &= -\frac{1}{4}\sin^3(x)\cos(x) + \frac{3}{4}\left(-\frac{1}{2}\sin(x)\cos(x) + \frac{1}{2}x\right) + c \\ \therefore \int \sin^4(x)\,dx &= -\frac{1}{4}\sin^3(x)\cos(x) - \frac{3}{8}\sin(x)\cos(x) + \frac{3}{8}x + c. \end{aligned}$$

EXAMPLE 2: $\int \sec^n(x)\,dx$ (E2.22)

In a similar manner to the above, we can use integration by parts to attain a general formula for $\int \sec^n(x)\,dx$ in terms of $\int \sec^{n-2}(x)\,dx$.

Let $I = \int \sec^n(x)\,dx = \int \sec^{n-2}(x)\sec^2(x)\,dx$.

Integrate by parts using $\begin{pmatrix} u = \sec^{n-2}(x) & v = \tan(x) \\ du = (n-2)(\sec^{n-3}(x))(\sec(x)\tan(x)) & dv = \sec^2(x)\,dx \\ \quad = (n-2)(\sec^{n-2}(x))(\tan(x)) & \end{pmatrix}$

to get

$$\begin{aligned} I = \int \sec^n(x)\,dx &= \sec^{n-2}(x)\tan(x) - \int (n-2)(\sec^{n-2}(x))(\tan^2(x))\,dx \\ &= \sec^{n-2}(x)\tan(x) - \int (n-2)(\sec^{n-2}(x))(\sec^2 x - 1)\,dx \\ \therefore\ I &= \sec^{n-2}(x)\tan(x) - (n-2)I + (n-2)\int \sec^{n-2}(x)\,dx. \end{aligned}$$

Hence

$$(n-1)I = \sec^{n-2}(x)\tan(x) + (n-2)\int \sec^{n-2}(x)\,dx,$$

so that

$$I = \frac{1}{n-1}\sec^{n-2}(x)\tan(x) + \frac{n-2}{n-1}\int \sec^{n-2}(x)\,dx.$$

We can use the above formula to evaluate the integral $\int \sec^3(x)\,dx$.

In particular, when $n=3$, we have that

$$\begin{aligned} \int \sec^3(x)\,dx &= \frac{1}{2}\sec(x)\tan(x) + \frac{1}{2}\int \sec(x)\,dx \\ \therefore \int \sec^3(x)\,dx &= \frac{1}{2}\sec(x)\tan(x) + \frac{1}{2}\ln|\sec(x) + \tan(x)| + c. \end{aligned}$$

TRIGONOMETRIC INTEGRALS

$\int f(\sin(x))\cos^{2n+1}(x)\,dx$ OR $\int f(\cos(x))\sin^{2n+1}(x)\,dx$

1. To find $\int f(\sin(x))\cos^{2n+1}(x)\,dx$, write $\cos^{2n+1}(x) = (1-\sin^2(x))^n\cos(x)$ and then try the substitution $u=\sin(x)$, $du=\cos(x)\,dx$. (N2.23 (1))

2. Similarly, to find $\int f(\cos(x))\sin^{2n+1}(x)\,dx$, write $\sin^{2n+1}(x) = (1-\cos^2(x))^n\sin(x)$ and then try the substitution $u=\cos(x)$, $du=-\sin(x)\,dx$. (N2.23 (2))

EXAMPLE: $\int_0^{\pi/3}\frac{\sin^3(x)}{\cos^2(x)}\,dx$ (E2.24)

We can use the above strategy to solve the integral $\int_0^{\pi/3}\frac{\sin^3(x)}{\cos^2(x)}\,dx$.

Make the substitution $u=\cos(x)$, so that $du=-\sin(x)\,dx$.
Then

$$\int_0^{\pi/3}\frac{\sin^3(x)}{\cos^2(x)}\,dx = \int_{x=0}^{x=\frac{\pi}{3}}\frac{(1-\cos^2 x)\sin(x)\,dx}{\cos^2(x)}$$
$$= \int_{u=1}^{u=\frac12} -\frac{(1-u^2)\,du}{u^2}$$
$$= \int_{u=1}^{u=\frac12} -\frac{1}{u^2}+1\,du$$
$$= \left[\frac{1}{u}+u\right]_1^{\frac12}$$
$$= (2+\tfrac12)-(1+1)$$

$\therefore \int_0^{\pi/3}\frac{\sin^3(x)}{\cos^2(x)}\,dx = \frac12.$

$\int \sin^{2m}(x)\cos^{2n}(x)\,dx$

1. To find $\int \sin^{2m}(x)\cos^{2n}(x)$, try using the trigonometric identities $\sin^2(\theta)=\frac12-\frac12\cos(2\theta)$ and $\cos^2(\theta)=\frac12+\frac12\cos(2\theta)$.

2. Alternatively, write $\cos^{2n}(x) = (1-\sin^2(x))^n$ and use the formula from E2.21. (N2.23 (3))

EXAMPLE: $\int_0^{\pi/4}\sin^6(x)\,dx$ (E2.25)

We can use either strategy 1 or 2 to evaluate the integral $\int_0^{\pi/4}\sin^6(x)\,dx$.

We use strategy 1.
Note that

$$\int_0^{\pi/4}\sin^6(x)\,dx = \int_0^{\pi/4}\left(\tfrac12-\tfrac12\cos(2x)\right)^3dx \quad \text{(using the half-angle formula)}$$
$$= \int_0^{\pi/4}\tfrac18-\tfrac38\cos(2x)+\tfrac38\cos^2(2x)-\tfrac18\cos^3(2x)\,dx$$
$$= \int_0^{\pi/4}\tfrac18-\tfrac38\cos(2x)+\tfrac38\left(\tfrac12+\tfrac12\cos(4x)\right)-\tfrac18(1-\sin^2(2x))\cos(2x)\,dx$$
$$= \int_0^{\pi/4}\tfrac{5}{16}-\tfrac12\cos(2x)+\tfrac{3}{16}\cos(4x)+\tfrac18\sin^2(2x)\cos(2x)\,dx$$
$$= \left[\tfrac{5}{16}x-\tfrac14\sin(2x)+\tfrac{3}{64}\sin(4x)+\tfrac{1}{48}\sin^3(2x)\right]_0^{\pi/4}$$
$$= \tfrac{5\pi}{64}-\tfrac14+\tfrac{1}{48}$$

$\therefore \int_0^{\pi/4}\sin^6(x)\,dx = \frac{5\pi}{64}-\frac{11}{48}.$

$\int f(\tan(x))\sec^{2n+2}(x)\,dx$

To find $\int f(\tan(x))\sec^{2n+2}(x)\,dx$, write $\sec^{2n+2}(x) = (1+\tan^2(x))^n\sec^2(x)$ and try the substitution $u=\tan(x)$, $du=\sec^2(x)$. (N2.23 (4))

EXAMPLE 1: $\int_0^{\pi/4}\tan^4(x)\,dx$ (E2.26)

The above strategy can be used to solve the integral $\int_0^{\pi/4}\tan^4(x)\,dx$.

Note first that

$$\int_0^{\pi/4}\tan^4(x)\,dx = \int_0^{\pi/4}\tan^2(x)\sec^2(x)-\tan^2x\,dx$$
$$= \int_0^{\pi/4}\tan^2(x)\sec^2(x)-\sec^2(x)+1\,dx.$$

To find $\int\tan^2(x)\sec^2(x)\,dx$, make the substitution $u=\tan(x)$, $du=\sec^2(x)\,dx$ to get that

$$\int\tan^2(x)\sec^2(x)\,dx = \int u^2\,du$$
$$= \frac{u^3}{3}+c$$

$\therefore \int\tan^2(x)\sec^2(x)\,dx = \frac{\tan^3(x)}{3}+c.$

It follows that

$$\int_0^{\pi/4}\tan^4(x) = \left[\frac{\tan^3(x)}{3}-\tan(x)+x\right]_0^{\pi/4}$$
$$= \frac13-1+\frac\pi4$$

$\therefore \int_0^{\pi/4}\tan^4(x) = -\frac23+\frac\pi4.$

EXAMPLE 2: $\int_0^{\pi/4}\frac{\sec^4(x)}{\sqrt{\tan(x)+1}}\,dx$ (E2.27)

We can again use the above strategy to evaluate the integral $\int_0^{\pi/4}\frac{\sec^4(x)}{\sqrt{\tan(x)+1}}\,dx$.

Make the substitution $u=\tan(x)$, so that $du=\sec^2(x)\,dx$.
Then

$$\int_0^{\pi/4}\frac{\sec^4(x)}{\sqrt{\tan(x)+1}}\,dx = \int_{x=0}^{x=\frac\pi4}\frac{(\tan^2(x)+1)\sec^2(x)\,dx}{\sqrt{\tan(x)+1}}$$
$$= \int_{u=0}^{u=1}\frac{(u^2+1)}{\sqrt{u+1}}\,du.$$

Next, make the substitution $v=u+1$, so that $du=dv$.
Then

$$\int_0^{\pi/4}\frac{\sec^4(x)}{\sqrt{\tan(x)+1}}\,dx = \int_{u=0}^{u=1}\frac{(u^2+1)}{\sqrt{u+1}}\,du$$
$$= \int_{v=1}^{v=2}\frac{((v-1)^2+1)}{\sqrt v}\,dv$$
$$= \int_1^2 v^{\frac32}-2v^{\frac12}+2v^{-\frac12}\,dv$$
$$= \left[\tfrac25 v^{\frac52}-\tfrac43 v^{\frac32}+4v^{\frac12}\right]_1^2$$
$$= \left(\tfrac25(4\sqrt2)-\tfrac43(2\sqrt2)+4(\sqrt2)\right)-\left(\tfrac25-\tfrac43+4\right)$$

$\therefore \int_0^{\pi/4}\frac{\sec^4(x)}{\sqrt{\tan(x)+1}}\,dx = \frac{44\sqrt2-46}{15}.$

$\int f(\sec(x))\tan^{2n+1}(x)\,dx$

To solve $\int f(\sec(x))\tan^{2n+1}(x)\,dx$, write $\tan^{2n+1}(x) = \frac{(\sec^2(x)-1)^n}{\sec(x)}\sec(x)\tan(x)$ and try the substitution $u=\sec(x)$, $du=\sec(x)\tan(x)\,dx$. (N2.23 (5))

$\int \sec^{2n+1}(x)\tan^{2n}(x)\,dx$

To solve $\int\sec^{2n+1}(x)\tan^{2n}(x)\,dx$, write $\tan^{2n}(x) = (\sec^2(x)-1)^n$ and use the formula from E2.22. (N2.23 (6))

$\int \sin(ax)\sin(bx)\,dx$, $\int \cos(ax)\cos(bx)\,dx$ OR $\int \sin(ax)\cos(bx)\,dx$ (N2.28)

To evaluate $\int \sin(ax)\sin(bx)\,dx$, $\int \cos(ax)\cos(bx)\,dx$ or $\int \sin(ax)\cos(bx)\,dx$, use the identities

1. $\cos(A-B) - \cos(A+B) = 2\sin(A)\sin(B)$;
2. $\cos(A-B) + \cos(A+B) = 2\cos(A)\cos(B)$; or
3. $\sin(A-B) + \sin(A+B) = 2\sin(A)\cos(B)$.

EXAMPLE : $\int_0^{\pi/6} \cos(3x)\cos(2x)\,dx$ (E2.29)

We can employ the above strategy to evaluate the integral $\int_0^{\pi/6} \cos(3x)\cos(2x)\,dx$.

By ② in the above, we have that

$$\begin{aligned}2\cos(3x)\cos(2x) &= \cos(3x-2x) + \cos(3x+2x)\\ &= \cos(x) + \cos(5x).\end{aligned}$$

Hence

$$\begin{aligned}\int_0^{\pi/6} \cos(2x)\cos(3x)\,dx &= \int_0^{\pi/6} \tfrac{1}{2}(\cos(x)+\cos(5x))\,dx\\ &= \left[\tfrac{1}{2}\sin(x) + \tfrac{1}{10}\sin(5x)\right]_0^{\pi/6}\\ &= \tfrac{1}{4} + \tfrac{1}{20}\end{aligned}$$

$$\therefore \int_0^{\pi/6} \cos(2x)\cos(3x)\,dx = \frac{3}{10}.$$

WEIERSTRASS SUBSTITUTION (N2.30)

1. The Weierstrass substitution is letting $u = \tan(\frac{x}{2})$, so that $x = 2\tan^{-1}(u)$, $dx = \frac{2}{1+u^2}\,du$.

2. Additionally, it implies $\sin(\frac{x}{2}) = \frac{u}{\sqrt{1+u^2}}$ & $\cos(\frac{x}{2}) = \frac{1}{\sqrt{1+u^2}}$, so that

① $$\begin{aligned}\sin(x) &= 2\sin(\tfrac{x}{2})\cos(\tfrac{x}{2})\\ &= 2\left(\frac{u}{\sqrt{1+u^2}}\right)\left(\frac{1}{\sqrt{1+u^2}}\right)\end{aligned}$$

$$\therefore \sin(x) = \frac{2u}{1+u^2};$$ and

② $$\begin{aligned}\cos(x) &= \cos^2(\tfrac{x}{2}) - \sin^2(\tfrac{x}{2})\\ &= \left(\frac{1}{\sqrt{1+u^2}}\right)^2 - \left(\frac{u}{\sqrt{1+u^2}}\right)^2\end{aligned}$$

$$\therefore \cos(x) = \frac{1-u^2}{1+u^2}.$$

EXAMPLE: $\int \frac{dx}{1-\cos(x)}$ (E2.31)

The Weierstrass substitution can be used to solve some integrals; eg $\int \frac{dx}{1-\cos(x)}$.

Let $u = \tan(\frac{x}{2})$, so that $dx = \frac{2}{1+u^2}\,du$, and $\cos(x) = \frac{1-u^2}{1+u^2}$.

Then

$$\begin{aligned}\int \frac{dx}{1-\cos(x)} &= \int \frac{1}{1-\left(\frac{1-u^2}{1+u^2}\right)}\left(\frac{2}{1+u^2}\,du\right)\\ &= \int \frac{2}{1+u^2-(1-u^2)}\,du\\ &= \int \frac{du}{u^2}\\ &= -\frac{1}{u} + C\end{aligned}$$

$$\therefore \int \frac{dx}{1-\cos(x)} = -\cot(\tfrac{x}{2}) + C.$$

INVERSE TRIGONOMETRIC SUBSTITUTION

$\int f\left(\sqrt{a^2 - b^2(x+c)^2}\right) dx$

For an integral involving $\sqrt{a^2 - b^2(x+c)^2}$, try the substitution $\theta = \sin^{-1}\left(\frac{b(x+c)}{a}\right)$, so that

① $a\sin\theta = b(x+c)$;

② $a\cos\theta = \sqrt{a^2 - b^2(x+c)^2}$; and

③ $a\cos\theta \, d\theta = b\, dx$. (N2.32 (2))

EXAMPLE 1: $\int_0^1 \frac{dx}{(4-3x^2)^{3/2}}$ (E2.33)

The above method can be used to evaluate the integral $\int_0^1 \frac{dx}{(4-3x^2)^{3/2}}$.

Let $2\sin\theta = \sqrt{3}x$, so that $2\cos\theta = \sqrt{4-3x^2}$ and $2\cos\theta d\theta = \sqrt{3}\, dx$.

Then

$$\int_{x=0}^{x=1} \frac{dx}{(4-3x^2)^{\frac{3}{2}}} = \int_{\theta=0}^{\theta=\frac{\pi}{3}} \frac{\frac{2}{\sqrt{3}}\cos\theta \, d\theta}{(2\cos\theta)^3}$$

$$= \int_0^{\pi/3} \frac{1}{4\sqrt{3}} \sec^2\theta \, d\theta$$

$$= \left[\frac{1}{4\sqrt{3}} \tan\theta\right]_0^{\pi/3}$$

$$\therefore \int_{x=0}^{x=1} \frac{dx}{(4-3x^2)^{\frac{3}{2}}} = \frac{1}{4}.$$

EXAMPLE 2: $\int_2^3 (4x-x^2)^{\frac{3}{2}} dx$ (E2.36)

The above strategy can also be applied to more complex integrals, like $\int_2^3 (4x-x^2)^{\frac{3}{2}} dx$.

Let $2\sin\theta = x-2$, so that $2\cos\theta = \sqrt{4x-x^2}$ and $2\cos\theta \, d\theta = dx$. Then

$$\int_{x=2}^{x=3} (4x-x^2)^{\frac{3}{2}} dx = \int_{\theta=0}^{\theta=\pi/6} (2\cos\theta)^3 (2\cos\theta \, d\theta)$$

$$= \int_0^{\pi/6} 16\cos^4\theta \, d\theta$$

$$= \int_0^{\pi/6} 4(1+\cos 2\theta)^2 \, d\theta$$

$$= \int_0^{\pi/6} 4 + 8\cos 2\theta + 4\cos^2 2\theta \, d\theta$$

$$= \int_0^{\pi/6} 4 + 8\cos 2\theta + 2 + 2\cos 4\theta \, d\theta$$

$$= \left[6\theta + 4\sin 2\theta + \tfrac{1}{2}\sin 4\theta\right]_0^{\pi/6}$$

$$\therefore \int_{x=2}^{x=3} (4x-x^2)^{\frac{3}{2}} dx = \pi + \frac{9\sqrt{3}}{4}.$$

$\int f\left(\sqrt{a^2 + b^2(x+c)^2}\right) dx$

For an integral involving $\sqrt{a^2 + b^2(x+c)^2}$ (or $\frac{1}{\sqrt{a^2+b^2(x+c)^2}}$), try the substitution $\theta = \tan^{-1}\left(\frac{b(x+c)}{a}\right)$, so that

① $a\tan\theta = b(x+c)$;

② $a\sec\theta = \sqrt{a^2 + b^2(x+c)^2}$; and

③ $a\sec^2\theta \, d\theta = b\, dx$. (N2.32(1))

EXAMPLE: $\int_1^{\sqrt{3}} \frac{dx}{x^2\sqrt{x^2+3}}$ (E2.34)

We can use the above strategy to evaluate the integral $\int_1^{\sqrt{3}} \frac{dx}{x^2\sqrt{x^2+3}}$.

Let $\sqrt{3}\tan\theta = x$, so that $\sqrt{3}\sec\theta = \sqrt{x^2+3}$ and $\sqrt{3}\sec^2\theta \, d\theta = dx$. Then

$$\int_{x=1}^{x=\sqrt{3}} \frac{dx}{x^2\sqrt{x^2+3}} = \int_{\theta=\frac{\pi}{4}}^{\theta=\frac{\pi}{6}} \frac{\sqrt{3}\sec^2\theta \, d\theta}{3\tan^2\theta(\sqrt{3}\sec\theta)}$$

$$= \int_{\frac{\pi}{4}}^{\frac{\pi}{6}} \frac{1}{3} \frac{\sec\theta}{\tan^2\theta} d\theta$$

$$\therefore \int_{x=1}^{x=\sqrt{3}} \frac{dx}{x^2\sqrt{x^2+3}} = \int_{\frac{\pi}{4}}^{\frac{\pi}{6}} \frac{1}{3} \frac{\cos\theta}{\sin^2\theta} d\theta$$

Then, let $u = \sin\theta$, so that $du = \cos\theta \, d\theta$. It follows that

$$\int_{x=1}^{x=\sqrt{3}} \frac{dx}{x^2\sqrt{x^2+3}} = \int_{\theta=\frac{\pi}{4}}^{\theta=\frac{\pi}{6}} \frac{1}{3} \frac{\cos\theta}{\sin^2\theta} d\theta$$

$$= \int_{u=\frac{1}{2}}^{u=\frac{1}{\sqrt{2}}} \frac{1}{3} \frac{1}{u^2} du$$

$$= \left[-\frac{1}{3u}\right]_{1/2}^{1/\sqrt{2}}$$

$$\therefore \int_{x=1}^{x=\sqrt{3}} \frac{dx}{x^2\sqrt{x^2+3}} = \frac{2-\sqrt{2}}{3}.$$

$\int f\left(\sqrt{b^2(x+c)^2 - a^2}\right) dx$

For an integral involving $\sqrt{b^2(x+c)^2 - a^2}$, try the substitution $\theta = \sec^{-1}\left(\frac{b(x+c)}{a}\right)$, so that

① $a\sec\theta = b(x+c)$;

② $a\tan\theta = \sqrt{b^2(x+c)^2 - a^2}$; and

③ $a\sec\theta\tan\theta = b\,dx$. (N2.32 (3))

EXAMPLE: $\int_2^4 \frac{\sqrt{x^2-4}}{x^2} dx$ (E2.35)

The above strategy can be used to evaluate the integral $\int_2^4 \frac{\sqrt{x^2-4}}{x^2} dx$.

Let $2\sec\theta = x$, so that $2\tan\theta = \sqrt{x^2-4}$ and $2\sec\theta\tan\theta d\theta = dx$.

Then

$$\int_{x=2}^{x=4} \frac{\sqrt{x^2-4}}{x^2} = \int_{\theta=0}^{\theta=\frac{\pi}{3}} \frac{\tan^2\theta \sec\theta \, d\theta}{\sec^2\theta}$$

$$= \int_0^{\pi/3} \frac{\tan^2\theta}{\sec\theta} d\theta$$

$$= \int_0^{\pi/3} \frac{\sec^2\theta - 1}{\sec\theta} d\theta$$

$$= \int_0^{\pi/3} (\sec\theta - \cos\theta) \, d\theta$$

$$= \left[\ln|\sec\theta + \tan\theta| - \sin\theta\right]_0^{\pi/3}$$

$$\therefore \int_{x=2}^{x=4} \frac{\sqrt{x^2-4}}{x^2} = \ln(2+\sqrt{3}) - \frac{\sqrt{3}}{2}.$$

PARTIAL FRACTIONS (N2.37)

We can find the integral of a rational function $\frac{f(x)}{g(x)}$ (where f & g are polynomials) as follows:

① Use long division to find polynomials $q(x)$ and $r(x)$ such that

$$f(x) \equiv g(x)q(x) + r(x),$$

where $\deg(r) < \deg(g)$.

* if $\deg(f) < \deg(g)$, then $q(x)=0$ and $r(x)=f(x)$.

② Then, note that $\frac{f(x)}{g(x)} = q(x) + \frac{r(x)}{g(x)}$,

and it follows that

$$\int \frac{f(x)}{g(x)}\,dx = \int q(x) + \frac{r(x)}{g(x)}\,dx.$$

③ Next, factor $g(x)$ into linear and irreducible quadratic factors.

*we can always do this! (MATH 145 R34)

④ Finally, split $\frac{r(x)}{g(x)}$ into its "partial fraction decomposition";

ie write $\frac{r(x)}{g(x)}$ as a sum of terms

so that

i) for each linear factor $(ax+b)^k$, we have the k terms

$$\frac{A_1}{(ax+b)} + \frac{A_2}{(ax+b)^2} + \cdots + \frac{A_k}{(ax+b)^k}\ ; \quad \text{and}$$

ii) for each irreducible quadratic factor $(ax^2+bx+c)^k$, we have the k terms

$$\frac{B_1x + C_1}{(ax^2+bx+c)} + \frac{B_2x + C_2}{(ax^2+bx+c)^2} + \cdots + \frac{B_kx + C_k}{(ax^2+bx+c)^k}.$$

eg if $g(x) = x(x-1)^3(x^2+2x+3)^2$, then we would write

$$\frac{r(x)}{g(x)} = \left(\frac{A}{x}\right) + \left(\frac{B}{x-1} + \frac{C}{(x-1)^2} + \frac{D}{(x-1)^3}\right) + \left(\frac{E}{x^2+2x+3} + \frac{F}{(x^2+2x+3)^2}\right),$$

and then solve for the various constants. (E2.38)

⑤ From here, we can solve the integral.

EXAMPLE 1: $\int_2^3 \frac{x-7}{(x-1)^2(x+2)}\,dx$ (E2.39)

The above strategy can be used to solve the integral $\int_2^3 \frac{x-7}{(x-1)^2(x+2)}\,dx$.

First, we need to find A,B,C such that

$$\frac{x-7}{(x-1)^2(x+2)} = \frac{A}{x-1} + \frac{B}{(x-1)^2} + \frac{C}{x+2},$$

or

$$x-7 = A(x-1)(x+2) + B(x+2) + C(x-1)^2.$$

Equating coefficients, we get that

$$\begin{cases} A+C=0 \\ A+B-2C=1 \\ -2A+2B+C=-7. \end{cases}$$

Solving this system gives us that $A=1$, $B=-2$ & $C=-1$.

Hence,

$$\int_2^3 \frac{x-7}{(x-1)^2(x+2)}\,dx = \int_2^3 \frac{1}{x-1} - \frac{2}{(x-1)^2} - \frac{1}{x+2}\,dx$$

$$= \left[\ln(x-1) - \frac{2}{x-1} - \ln(x+2)\right]_2^3$$

$$\therefore \int_2^3 \frac{x-7}{(x-1)^2(x+2)}\,dx = \ln\left(\frac{8}{5}\right) - 1.$$

EXAMPLE 2: $\int_1^{\sqrt{3}} \frac{x^4-x^3+1}{x^3+x}\,dx$ (E2.40)

As mentioned in step ① of the method, sometimes long division is needed before partial fraction decomposition can be carried out;

eg $\int_1^{\sqrt{3}} \frac{x^4-x^3+1}{x^3+x}\,dx$.

First, use polynomial long division to get that

$$\frac{x^4-x^3+1}{x^3+x} = (x-1) + \frac{-x^2+x+1}{x^3+x}.$$

Then, note that to get

$$\frac{-x^2+x+1}{x^3+x} = \frac{A}{x} + \frac{Bx+C}{x^2+1}$$

we need

$$-x^2+x+1 = A(x^2+1) + (Bx+C)(x).$$

Equating coefficients gives $A+B=-1$, $C=1$ and $A=1$.

Solving these equations gives $A=1$, $B=-2$ and $C=1$.

Thus

$$\int_1^{\sqrt{3}} \frac{x^4-x^3+1}{x^3+x}\,dx = \int_1^{\sqrt{3}} x-1 + \frac{1}{x} - \frac{2x}{x^2+1} + \frac{1}{x^2+1}\,dx$$

$$= \left[\frac{1}{2}x^2 - x + \ln(x) - \ln(x^2+1) + \tan^{-1}(x)\right]_1^{\sqrt{3}}$$

$$\therefore \int_1^{\sqrt{3}} \frac{x^4-x^3+1}{x^3+x}\,dx = 2-\sqrt{3} + \ln\left(\frac{\sqrt{3}}{2}\right) + \frac{\pi}{12}.$$

EXAMPLE 3: $\int_1^2 \frac{x^5+x^4-2x^3-2x^2-5x-25}{x^2(x^2-2x+5)^2}\,dx$ (E2.41)

Partial fraction decomposition can also be applied in tandem with substitution to solve integrals;

eg $\int_1^2 \frac{x^5+x^4-2x^3-2x^2-5x-25}{x^2(x^2-2x+5)^2}\,dx$.

Let I be the above integral.

To get

$$\frac{x^5+x^4-2x^3-2x^2-5x-25}{x^2(x^2+2x+5)^2} = \frac{A}{x} + \frac{B}{x^2} + \frac{Cx+D}{x^2+2x+5} + \frac{Ex+F}{(x^2+2x+5)^2}$$

we need

$$x^5+x^4-2x^3-2x^2-5x-25 = Ax(x^2-2x+5)^2 + B(x^2-2x+5)^2 + (Cx+D)(x^2+2x+5)(x^2) + (Ex+F)(x^2).$$

Comparing coefficients, we get that $A+C=1$; $-4A+B-2C+D=1$; $14A-4B+5C-2D+E=-2$; $-20A+14B+5D+F=-2$; $25A-20B=-5$; and $25B=-25$.

Solving these equations gives $A=-1$, $B=-1$, $C=2$, $D=2$, $E=2$ and $F=-18$.

Hence

$$I = \int_1^2 -\frac{1}{x} - \frac{1}{x^2} + \frac{2x+2}{x^2-2x+5} + \frac{2x-18}{(x^2-2x+5)^2}\,dx$$

$$= \int_1^2 -\frac{1}{x} - \frac{1}{x^2} + \frac{(2x-2)+4}{x^2-2x+5} + \frac{(2x-2)-16}{(x^2-2x+5)^2}\,dx$$

$$I = \int_1^2 -\frac{1}{x} - \frac{1}{x^2} + \frac{2x-2}{x^2-2x+5} + \frac{4}{x^2-2x+5} + \frac{2x-2}{(x^2-2x+5)^2} - \frac{16}{(x^2-2x+5)^2}\,dx.$$

To compute $\int \frac{2x-2}{x^2-2x+5}\,dx$ and $\int \frac{2x-2}{(x^2-2x+5)^2}\,dx$, make the substitution $u = x^2-2x+5$, so that $du = (2x-2)\,dx$.

Then

$$\int \frac{2x-2}{x^2-2x+5}\,dx = \int \frac{du}{u} = \ln|u|+c = \ln|x^2-2x+5|+c\ ;$$

and

$$\int \frac{2x-2}{(x^2-2x+5)^2}\,dx = \int \frac{du}{u^2} = \frac{-1}{u}+c = \frac{-1}{x^2-2x+5}+c.$$

To compute $\int \frac{4\,dx}{x^2-2x+5}$ and $\int \frac{16\,dx}{(x^2-2x+5)^2}$, make the substitution $2\tan\theta = x-1$, so that $2\sec\theta = \sqrt{x^2-2x+5}$ and $2\sec^2\theta\,d\theta = dx$.

Then

$$\int \frac{4\,dx}{x^2-2x+5} = \int \frac{4\cdot 2\sec^2\theta\,d\theta}{(2\sec\theta)^2} = \int 2\,d\theta = 2\theta + c = 2\tan^{-1}\left(\frac{x-1}{2}\right) + c$$

and

$$\int \frac{16\,dx}{(x^2-2x+5)^2} = \int \frac{16\cdot 2\sec^2\theta\,d\theta}{(2\sec\theta)^4} = \int \frac{2\,d\theta}{\sec^2\theta} = \int 2\cos^2\theta\,d\theta = \int 1+\cos(2\theta)\,d\theta$$

$$= \theta + \frac{1}{2}\sin(2\theta) + c = \theta + \sin\theta\cos\theta + c = \tan^{-1}\left(\frac{x-1}{2}\right) + \frac{2(x-1)}{x^2-2x+5} + c.$$

Thus

$$I = \left[-\ln(x) + \frac{1}{x} + \ln(x^2-2x+5) + 2\tan^{-1}\left(\frac{x-1}{2}\right) - \frac{1}{x^2-2x+5} - \tan^{-1}\left(\frac{x-1}{2}\right) - \frac{2(x-1)}{x^2-2x+5}\right]_1^2$$

$$\therefore I = \ln\left(\frac{5}{8}\right) - \frac{17}{20} + \tan^{-1}\left(\frac{1}{2}\right).$$

EXAMPLE 4: $\int \frac{\sec^3(x)}{\sec(x)-1}\,dx$ (E2.42)

Partial fraction decomposition can also be applied even if the function is not rational (at first);

eg $\int \frac{\sec^3(x)}{\sec(x)-1}\,dx$.

First, note that

$$\int \frac{\sec^3(x)}{\sec(x)-1}\,dx = \int \frac{\sec^3(x)}{\sec(x)-1} \cdot \frac{\sec(x)+1}{\sec(x)+1}\,dx$$
$$= \int \frac{\sec^4(x)+\sec^3(x)}{\sec^2(x)-1}\,dx$$
$$= \int \frac{\sec^4(x)+\sec^3(x)}{\tan^2(x)}\,dx$$

$$\therefore \int \frac{\sec^3(x)}{\sec(x)-1}\,dx = \int \frac{\sec^4(x)}{\tan^2(x)}\,dx + \int \frac{\sec^3(x)}{\tan^2(x)}.$$

To find $\int \frac{\sec^4(x)}{\tan^2(x)}\,dx$, make the substitution $u=\tan(x)$, so that $du=\sec^2(x)\,dx$. Then

$$\int \frac{\sec^4(x)}{\tan^2(x)}\,dx = \int \frac{(\tan^2(x)+1)\sec^2(x)}{\tan^2(x)}\,dx$$
$$= \int \frac{(u^2+1)}{u^2}\,du$$
$$= \int 1+\frac{1}{u^2}\,du$$
$$= u-\frac{1}{u}+c$$

$$\therefore \int \frac{\sec^4(x)}{\tan^2(x)}\,dx = \tan(x)-\cot(x)+c;$$

To find $\int \frac{\sec^3(x)}{\tan^2(x)}\,dx$, make the substitution $v=\sin(x)$, so that $dv=\cos(x)\,dx$. Then

$$\int \frac{\sec^3(x)}{\tan^2(x)}\,dx = \int \frac{dx}{\cos(x)\sin^2(x)}$$
$$= \int \frac{\cos(x)\,dx}{(1-\sin^2(x))\sin^2(x)}$$
$$= \int \frac{dv}{(1-v^2)v^2}.$$

Then, note that $\frac{1}{(1-v^2)v^2} = \frac{1/2}{1-v} + \frac{1/2}{1+v} + \frac{0}{v} + \frac{1}{v^2}$ (by partial fraction decomposition), so that

$$\int \frac{\sec^3(x)}{\tan^2(x)}\,dx = \int \frac{1/2}{1-v} + \frac{1/2}{1+v} + \frac{1}{v^2}\,dv$$
$$= -\frac{1}{2}\ln|1-v| + \frac{1}{2}\ln|1+v| - \frac{1}{v} + c$$

$$\therefore \int \frac{\sec^3(x)}{\tan^2(x)}\,dx = -\frac{1}{2}\ln|1-\sin(x)| + \frac{1}{2}\ln|1+\sin(x)| - \csc(x) + c.$$

Finally, it follows that

$$\int \frac{\sec^3(x)}{\sec(x)-1}\,dx = \int \frac{\sec^4(x)}{\tan^2(x)}\,dx + \int \frac{\sec^3(x)}{\tan^2(x)}\,dx$$
$$= \tan(x)-\cot(x)-\frac{1}{2}\ln|1-\sin(x)|+\frac{1}{2}\ln|1+\sin(x)|-\csc(x)+c$$

$$\therefore \int \frac{\sec^3(x)}{\sec(x)-1}\,dx = \tan(x)-\cot(x)+\ln|\sec(x)+\tan(x)|-\csc(x)+c.$$

APPROXIMATE INTEGRATION (D2.43 (1))

💡 Let f be integrable on $[a,b]$.
Then, we can approximate the integral of f on $[a,b]$ by any Riemann sum

$$I = \int_a^b f(x)\,dx \cong \sum_{k=1}^{n} f(c_k)\,\Delta_k x,$$

where $a = x_0 < x_1 < \cdots < x_n = b$, $\Delta_k x = x_{k-1} - x_k$
and $c_k \in [x_{k-1}, x_k]$ $\forall k \in \{1, 2, \ldots, n\}$.

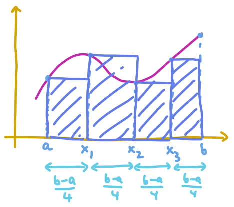
LEFT ENDPOINT APPROXIMATION (D2.43 (2))

💡 Let f be integrable on $[a,b]$.
Then, the "n^{th} left endpoint approximation" for $I = \int_a^b f$, denoted by L_n, is defined to be

$$L_n = \sum_{k=1}^{n} f(x_{k-1})\,\Delta_k x;$$

ie

$$L_n = \frac{b-a}{n} \sum_{k=1}^{n} f\left(a + \tfrac{b-a}{n}(k-1)\right).$$



*the subintervals are equally sized!

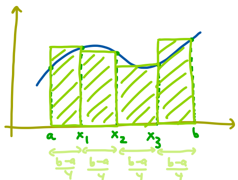
RIGHT ENDPOINT APPROXIMATION (D2.43 (3))

💡 Let f be integrable on $[a,b]$.
Then, the "n^{th} right endpoint approximation" for $I = \int_a^b f$, denoted by R_n, is defined to be

$$R_n = \sum_{k=1}^{n} f(x_k)\,\Delta_k x;$$

ie

$$R_n = \frac{b-a}{n} \sum_{k=1}^{n} f\left(a + \tfrac{b-a}{n}k\right).$$



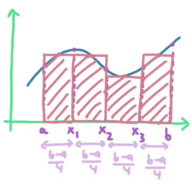
MIDPOINT APPROXIMATION (D2.43 (4))

💡 Let f be integrable on $[a,b]$.
Then, the "n^{th} midpoint approximation" for $I = \int_a^b f$, denoted by M_n, is defined to be

$$M_n = \sum_{k=1}^{n} f\left(\frac{x_{k-1}+x_k}{2}\right)\Delta_k x;$$

ie

$$M_n = \frac{b-a}{n} \sum_{k=1}^{n} f\left(a + \tfrac{b-a}{n}\left(\tfrac{2k-1}{2}\right)\right).$$



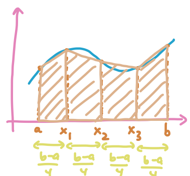
TRAPEZOIDAL APPROXIMATION (D2.44)

💡$_1$ Let f be integrable on $[a,b]$.
Then, the "n^{th} trapezoidal approximation" for $I = \int_a^b f$, denoted by T_n, is defined by

$$T_n = \sum_{k=1}^{n} \frac{f(x_{k-1}) + f(x_k)}{2}\,\Delta_k x;$$

ie

$$T_n = \frac{b-a}{n} \sum_{k=1}^{n} \frac{f\left(a+\frac{b-a}{n}(k-1)\right) + f\left(a + \frac{b-a}{n}k\right)}{2}.$$



*note each "area" is a trapezoid!

💡$_2$ Note that $T_n = \dfrac{L_n + R_n}{2}$.

SIMPSON APPROXIMATION (D2.45)

💡$_1$ Let f be integrable on $[a,b]$.
Then, for some $n \in 2\mathbb{Z}^+$, the "nth Simpson approximation" for $I = \int_a^b f$, denoted by S_n, is defined to be

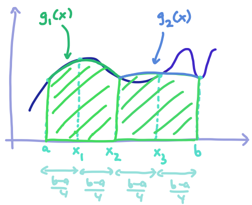
$$S_n = \sum_{k=1}^{n/2} \int_{x_{2k-2}}^{x_{2k}} g(x)\,dx,$$

where $x_m = a + \frac{b-a}{n}m$ $\forall m \in \{0, 1, \ldots, n\}$, and

$g(x) = g_k(x)$ $\forall k \in \{1, 2, \ldots, \frac{n}{2}\}$,

where for each k, g_k is a quadratic polynomial such that

$$\begin{cases} g_k(x_{2k-2}) = f(x_{2k-2}); \\ g_k(x_{2k-1}) = f(x_{2k-1}); \quad \text{and} \\ g_k(x_{2k}) = f(x_{2k}). \end{cases}$$



💡$_2$ We can prove that

$$S_n = \sum_{k=1}^{n/2} \frac{f(x_{2k-2}) + 4f(x_{2k-1}) + f(x_{2k})}{3}\,\Delta_k x;$$

ie

$$S_n = \frac{b-a}{n} \sum_{k=1}^{n/2} \frac{f\left(a+\frac{b-a}{n}(2k-2)\right) + 4f\left(a+\frac{b-a}{n}(2k-1)\right) + f\left(a+\frac{b-a}{n}(2k)\right)}{3}.$$

Proof. First, note if $h(x) = Ax^2 + Bx + C$ satisfies $h(-1) = u$, $h(0) = v$ and $h(1) = w$, then necessarily

$$\begin{cases} A - B + C = u; \\ C = v; \quad \text{and} \\ A + B + C = w. \end{cases}$$

Solving these equations yields that $A = \frac{u-2v+w}{2}$, $B = \frac{w-u}{2}$ and $C = w$, so that

$$\int_{-1}^{1} h(x)\,dx = \int_{-1}^{1} \frac{u-2v+w}{2}x^2 + \frac{w-u}{2}x + v\,dx$$

$$= \left[\frac{u-2v+w}{6}x^3 + \frac{w-u}{4}x^2 + vx\right]_{-1}^{1}$$

$$= \frac{u-2v+w}{3} + 2v$$

$$\therefore \int_{-1}^{1} h(x)\,dx = \frac{u+4v+w}{3}.$$

Then, by shifting and scaling, it follows that

$$\int_{x_{2k-2}}^{x_{2k}} g_k(x)\,dx = \frac{f(x_{2k-2}) + 4f(x_{2k-1}) + f(x_{2k})}{3}\left(\frac{b-a}{n}\right). \quad \blacksquare$$

ERROR BOUNDS FOR APPROXIMATE INTEGRATION (T2.46)

Let f be integrable on $[a,b]$, and suppose the higher order derivatives of f exist.

Denote $I = \int_a^b f(x)\,dx$. Then

① $|L_n - I| \leq \frac{(b-a)^2}{2n} \max_{a \leq x \leq b} |f'(x)|$;

② $|R_n - I| \leq \frac{(b-a)^2}{2n} \max_{a \leq x \leq b} |f'(x)|$;

③ $|T_n - I| \leq \frac{(b-a)^3}{12n^2} \max_{a \leq x \leq b} |f''(x)|$;

④ $|M_n - I| \leq \frac{(b-a)^3}{24n^2} \max_{a \leq x \leq b} |f''(x)|$; and

⑤ $|S_n - I| \leq \frac{(b-a)^5}{180n^4} \max_{a \leq x \leq b} |f''''(x)|$.

EXAMPLE : ERROR BOUNDS OF APPROXIMATIONS OF $\int_0^{4\pi/3} \sin^2(x)\,dx$ (E2.47)

We can use the above theorem to find the bounds on the errors for L_8, R_8, M_8, T_8 & S_8 on $I = \int_0^{4\pi/3} \sin^2(x)\,dx$.

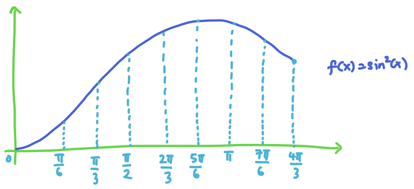
First, note that

$$I = \int_0^{4\pi/3} \sin^2(x)\,dx = \int_0^{4\pi/3} \frac{1}{2} - \frac{1}{2}\cos(2x)\,dx$$

$$= \left[\frac{1}{2}x - \frac{1}{4}\sin(2x)\right]_0^{4\pi/3}$$

$$\therefore I = \frac{4\pi}{3} - \frac{\sqrt{3}}{8}.$$

Next, when we divide the interval $[0, \frac{4\pi}{3}]$ into 8 equal sub-intervals, the size of each subinterval is $\frac{\pi}{6}$ and the endpoints of the sub-intervals are $0, \frac{\pi}{6}, \frac{\pi}{3}, \frac{\pi}{2}, \frac{2\pi}{3}, \frac{5\pi}{6}, \pi, \frac{7\pi}{6}$ and $\frac{4\pi}{3}$.



For convenience, let $f(x) = \sin^2(x)$.

Thus, the approximations are

① $L_8 = \frac{b-a}{8} \sum_{k=1}^{8} f(x_{k-1})$

$= \frac{1}{8}\left(\frac{4\pi}{3} - 0\right)\left(f(0) + f(\frac{\pi}{6}) + f(\frac{\pi}{3}) + f(\frac{\pi}{2}) + f(\frac{2\pi}{3}) + f(\frac{5\pi}{6}) + f(\pi) + f(\frac{7\pi}{6})\right)$

$= \frac{1}{8}\left(\frac{4\pi}{3}\right)\left(0 + \frac{1}{4} + \frac{3}{4} + 1 + \frac{3}{4} + \frac{1}{4} + 0 + \frac{1}{4}\right)$

$\therefore L_8 = \frac{13\pi}{24}$;

② $R_8 = \frac{b-a}{8} \sum_{k=1}^{8} f(x_{k-1})$

$= \frac{1}{8}\left(\frac{4\pi}{3}\right)\left(f(\frac{\pi}{6}) + f(\frac{\pi}{3}) + f(\frac{\pi}{2}) + f(\frac{2\pi}{3}) + f(\frac{5\pi}{6}) + f(\pi) + f(\frac{7\pi}{6}) + f(\frac{4\pi}{3})\right)$

$= \frac{1}{8}\left(\frac{4\pi}{3}\right)\left(\frac{1}{4} + \frac{3}{4} + 1 + \frac{3}{4} + \frac{1}{4} + 0 + \frac{1}{4} + \frac{3}{4}\right)$

$\therefore R_8 = \frac{2\pi}{3}$;

③ $T_8 = \frac{1}{2}(L_8 + R_8)$

$= \frac{1}{2}\left(\frac{13\pi}{24} + \frac{2\pi}{3}\right)$

$\therefore T_8 = \frac{29\pi}{48}$;

④ $M_8 = \frac{b-a}{8} \sum_{k=1}^{8} f\left(\frac{x_{k-1}+x_k}{2}\right)$

$= \frac{1}{8}\left(\frac{4\pi}{3}\right)\left(f(\frac{\pi}{12}) + f(\frac{\pi}{4}) + f(\frac{5\pi}{12}) + f(\frac{7\pi}{12}) + f(\frac{3\pi}{4}) + f(\frac{11\pi}{12}) + f(\frac{13\pi}{12}) + f(\frac{15\pi}{12})\right)$

$= \frac{\pi}{6}\left(\frac{2-\sqrt{3}}{4} + \frac{1}{2} + \frac{2+\sqrt{3}}{4} + \frac{2+\sqrt{3}}{4} + \frac{1}{2} + \frac{2-\sqrt{3}}{4} + \frac{2-\sqrt{3}}{4} + \frac{1}{2}\right)$

*using the identity $\sin^2(x) = \frac{1}{2}(1-\cos(2x))$ to figure out the values of $f(\frac{4\pi}{12})$.

$= \frac{\pi}{6}\left(4 - \frac{\sqrt{3}}{4}\right)$; and

⑤ $S_8 = \frac{b-a}{8} \sum_{k=1}^{8} \frac{f(x_{2k-2}) + 4f(x_{2k-1}) + f(x_{2k})}{3}$

$= \frac{1}{24}\left(\frac{4\pi}{3}\right)\left(f(0) + 4f(\frac{\pi}{6}) + 2f(\frac{\pi}{3}) + 4f(\frac{\pi}{2}) + 2f(\frac{2\pi}{3}) + 4f(\frac{5\pi}{6}) + 2f(\pi) + 4f(\frac{7\pi}{6}) + f(\frac{4\pi}{3})\right)$

$= \frac{\pi}{18}\left(0 + 1 + \frac{3}{2} + 4 + \frac{3}{2} + 1 + 0 + 1 + \frac{3}{4}\right)$

$\therefore S_8 = \frac{43\pi}{72}$.

Then, for $f(x) = \sin^2(x)$; note that

① $f'(x) = 2\sin(x)\cos(x) = \sin(2x)$;

② $f''(x) = 2\cos(2x)$;

③ $f'''(x) = -4\sin(2x)$; and

④ $f''''(x) = -8\cos(2x)$.

It follows that

① $\max_{0 \leq x \leq \frac{4\pi}{3}} |f'(x)| = \max_{0 \leq x \leq \frac{4\pi}{3}} |\sin(2x)| = 1$;

② $\max_{0 \leq x \leq \frac{4\pi}{3}} |f''(x)| = \max_{0 \leq x \leq \frac{4\pi}{3}} |2\cos(2x)| = 2$; and

③ $\max_{0 \leq x \leq \frac{4\pi}{3}} |f''''(x)| = \max_{0 \leq x \leq \frac{4\pi}{3}} |-8\cos(2x)| = 8$.

Finally, by the above theorem, we get that

① $|L_8 - I| \leq \frac{1}{16}\left(\frac{4\pi}{3}\right)^2(1) = \frac{\pi^2}{9}$;

② $|R_8 - I| \leq \frac{1}{16}\left(\frac{4\pi}{3}\right)^2(1) = \frac{\pi^2}{9}$;

③ $|T_8 - I| \leq \frac{1}{12 \cdot 6^2}\left(\frac{4\pi}{3}\right)^3(2) = \frac{8\pi^3}{729}$;

④ $|M_8 - I| \leq \frac{1}{24 \cdot 6^2}\left(\frac{4\pi}{3}\right)^3(2) = \frac{4\pi^3}{729}$; and

⑤ $|S_8 - I| \leq \frac{1}{180 \cdot 6^4}\left(\frac{4\pi}{3}\right)^5(8) = \frac{2^7\pi^5}{5 \cdot 3^{11}}$.

IMPROPER INTEGRATION

IMPROPER INTEGRATION ON $[a,b)$ (D2.48 (1))

1. Suppose that $f: [a,b) \to \mathbb{R}$ is integrable on every closed interval contained in $[a,b)$.
Then the "improper integral of f" on $[a,b)$ is defined to be

$$\int_a^b f = \lim_{t \to b^-} \int_a^t f.$$

2. We say f is "improperly integrable" on $[a,b)$, or that the improper integral of f on $[a,b)$ "converges", if $\int_a^b f$ exists and is finite.

3. We also allow the case where $b = \infty$, and in this case we have

$$\int_a^\infty f = \lim_{t \to \infty} \int_a^t f.$$

IMPROPER INTEGRATION ON $(a,b]$ (D2.48 (2))

1. Suppose that $f: (a,b] \to \mathbb{R}$ is integrable on every closed interval contained in $(a,b]$.
Then, the "improper integral of f" on $(a,b]$ is defined to be

$$\int_a^b f = \lim_{t \to a^+} \int_t^b f.$$

2. Similarly, we say f is "improperly integrable" on $(a,b]$, or that the improper integral of f on $(a,b]$ "converges", if $\int_a^b f$ exists and is finite.

3. We also allow the case where $a = -\infty$, and in this case we have

$$\int_{-\infty}^b f = \lim_{t \to -\infty} \int_t^b f.$$

IMPROPER INTEGRATION ON (a,b) (D2.48 (3))

1. Suppose that $f: (a,b) \to \mathbb{R}$ is integrable on every closed interval in (a,b).
Suppose further that for any point $c \in (a,b)$, the integrals $\int_a^c f$ and $\int_c^b f$ both exist and can be added.
Then the "improper integral of f" on (a,b) is defined to be

$$\int_a^b f = \int_a^c f + \int_c^b f,$$

*the choice of c does not matter!

where $c \in (a,b)$ is arbitrary.

2. We say f is "improperly integrable" on (a,b) when both $\int_a^c f$ and $\int_c^b f$ are finite.

EVALUATING IMPROPER INTEGRALS

We write

① $[F(x)]_{a^+}^{b^-} = \lim_{x\to b^-} F(x) - \lim_{x\to a^+} F(x)$;

② $[F(x)]_{a^+}^{b} = F(b) - \lim_{x\to a^+} F(x)$; and

③ $[F(x)]_{a}^{b^-} = \lim_{x\to b^-} F(x) - F(a)$. (NT2.49)

Suppose that $f: (a,b)\to\mathbb{R}$ is integrable on every closed interval contained in (a,b), and assume that F is differentiable with $F'=f$ on (a,b).

Then

$$\int_a^b f = [F(x)]_{a^+}^{b^-}. \qquad (\text{N}2.50)$$

(A similar result holds for functions defined on $[a,b)$ and $(a,b]$).

Proof. Choose some $c\in(a,b)$. Then, by the Fundamental Theorem of Calculus,

$$\int_a^b f = \int_a^c f + \int_c^b f$$

$$= \lim_{s\to a^+}\int_s^c f + \lim_{t\to b^-}\int_c^t f$$

$$= \lim_{s\to a^+}(F(c)-F(s)) + \lim_{t\to b}(F(t)-F(c))$$

$$= \lim_{t\to b^-} F(t) - \lim_{s\to a^+} F(s)$$

$$\therefore \int_a^b f = [F(x)]_{a^+}^{b^-}. \quad \blacksquare$$

EXAMPLE 1: $\int_0^1 \frac{dx}{x}$ (E2.51 (1))

The above strategy can help us evaluate the integral $\int_0^1 \frac{dx}{x}$.

$$\int_0^1 \frac{dx}{x} = [\ln(x)]_{0^+}^1$$

$$= 0-(-\infty)$$

$$\therefore \int_0^1 \frac{dx}{x} = \infty.$$

EXAMPLE 2: $\int_0^1 \frac{dx}{\sqrt{x}}$ (E2.51 (2))

Similarly, we can evaluate $\int_0^1 \frac{dx}{\sqrt{x}}$ by the above method.

$$\int_0^1 \frac{dx}{\sqrt{x}} = [2\sqrt{x}]_{0^+}^1$$

$$= 2-0$$

$$\therefore \int_0^1 \frac{dx}{\sqrt{x}} = 2.$$

EXAMPLE 3: $\int_0^1 \frac{dx}{x^p}$ CONVERGES $\Leftrightarrow p<1$ (E2.52)

By extension of the previous two examples, we can in fact show $\int_0^1 \frac{dx}{x^p}$ converges if and only if $p<1$.

Proof. The case with $p=1$ was dealt in E2.50.

If $p>1$, then $p-1>0$, so that

$$\int_0^1 \frac{dx}{x^p} = \left[\frac{-1}{(p-1)x^{p-1}}\right]_{0^+}^1 = \left(-\frac{1}{p-1}\right)-(-\infty) = \infty,$$

and if $p<1$, then $1-p>0$, so that

$$\int_0^1 \frac{dx}{x^p} = \left[\frac{x^{1-p}}{1-p}\right]_{0^+}^1 = \left(\frac{1}{1-p}\right)-(0) = \frac{1}{1-p},$$

and these deductions are sufficient to prove the claim. $\blacksquare$

EXAMPLE 4: $\int_1^\infty \frac{dx}{x^p}$ CONVERGES $\Leftrightarrow p>1$ (E2.53)

Similarly, we can prove $\int_1^\infty \frac{dx}{x^p}$ converges if and only if $p>1$.

Proof. When $p=1$, then

$$\int_1^\infty \frac{dx}{x^p} = \int_1^\infty \frac{dx}{x} = [\ln(x)]_1^\infty = \infty - 0 = \infty.$$

When $p>1$, then $p-1>0$, so that

$$\int_1^\infty \frac{dx}{x^p} = \left[\frac{-1}{(p-1)x^{p-1}}\right]_1^\infty = (0)-\left(-\frac{1}{p-1}\right) = \frac{1}{p-1}.$$

When $p<1$, then $1-p>0$, so that

$$\int_1^\infty \frac{dx}{x^p} = \left[\frac{x^{1-p}}{1-p}\right]_1^\infty = (\infty)-\left(\frac{1}{1-p}\right) = \infty,$$

and these deductions are sufficient to prove the claim. $\blacksquare$

EXAMPLE 5: $\int_0^\infty e^{-x}\,dx$ (E2.54)

In a similar manner, we can evaluate the integral $\int_0^\infty e^{-x}\,dx$.

$$\int_0^\infty e^{-x}\,dx = [-e^{-x}]_0^\infty$$

$$= 0-(-1)$$

$$\therefore \int_0^\infty e^{-x}\,dx = 1.$$

EXAMPLE 6: $\int_0^1 \ln(x)\,dx$ (E2.55)

Similarly, we can evaluate the integral $\int_0^1 \ln(x)\,dx$ using the strategy in N2.50.

$$\int_0^1 \ln(x)\,dx = [x\ln(x)-x]_{0^+}^1$$

$$= (-1) - \lim_{x\to 0^+}(x\ln(x))$$

$$= -1 - \lim_{x\to 0^+}\frac{\ln(x)}{\left(\frac{1}{x}\right)}$$

$$= -1 - \lim_{x\to 0^+}\frac{\left(\frac{1}{x}\right)}{\left(-\frac{1}{x^2}\right)} \quad \left(\text{by L'Hôpital's Rule, since } \frac{\ln(0)}{\left(\frac{1}{0}\right)} = \frac{\infty}{\infty}\right)$$

$$= -1 - \lim_{x\to 0^+}(-x)$$

$$= -1-(0)$$

$$\therefore \int_0^1 \ln(x)\,dx = -1.$$

COMPARISON FOR IMPROPER INTEGRALS (T2.56)

Let f and g be integrable on any closed intervals contained in (a,b), and suppose further that
$0 \leq f(x) \leq g(x) \quad \forall x \in (a,b)$.

Suppose g is improperly integrable on (a,b). Then so is f, and

$$\int_a^b f \leq \int_a^b g.$$

On the other hand, if $\int_a^b f$ diverges, then $\int_a^b g$ diverges as well.

(Similar results hold for functions f & g defined on half-open intervals)

EXAMPLE 1: $\int_0^{\pi/2} \sqrt{\sec(x)}\,dx$ CONVERGES (E2.57)

Using comparison, we can show that $\int_0^{\pi/2} \sqrt{\sec(x)}\,dx$ converges.

Proof. First, note $\forall x \in [0, \frac{\pi}{2})$, we have that $\cos(x) \geq 1 - \frac{2}{\pi}x$, so that $\sec(x) \leq \frac{1}{1-\frac{2}{\pi}x}$, and hence $\sqrt{\sec(x)} \leq \sqrt{\frac{1}{1-\frac{2}{\pi}x}}$.

Let $u = 1 - \frac{2}{\pi}x$, so that $du = -\frac{2}{\pi}dx$. Then

$$\int_{x=0}^{x=\frac{\pi}{2}} \frac{1}{\sqrt{1-\frac{2}{\pi}x}}\,dx = \int_{u=1}^{u=0} -\frac{\pi}{2}u^{-\frac{1}{2}}\,du = \left[-\pi u^{\frac{1}{2}}\right]_1^0$$

$$\therefore \int_{x=0}^{x=\frac{\pi}{2}} \frac{1}{\sqrt{1-\frac{2}{\pi}x}}\,dx = \pi,$$

which is clearly finite.

It follows by comparison that $\int_0^{\pi/2} \sqrt{\sec(x)}\,dx$ converges. ▨

EXAMPLE 2: $\int_0^\infty e^{-x^2}\,dx$ CONVERGES (E2.58)

Similarly, we can show $\int_0^\infty e^{-x^2}\,dx$ converges using comparison.

Proof. First, note for $\forall u \in [0,\infty)$, $e^u \geq 1+u$; hence $e^{x^2} \geq 1+x^2 > 0$, so that $e^{-x^2} \leq \frac{1}{1+x^2}$.

Then, since

$$\int_0^\infty \frac{dx}{1+x^2} = \left(\tan^{-1}(x)\right]_0^\infty \leq \frac{\pi}{2},$$

which is finite, we see that $\int_0^\infty e^{-x^2}\,dx$ converges by comparison. ▨

ESTIMATION FOR IMPROPER INTEGRALS (T2.59)

Let $f:(a,b) \to \mathbb{R}$ be integrable on any closed interval contained within (a,b).

Suppose $|f|$ is improperly integrable on (a,b).

Then so is f, and in this case

$$\left| \int_a^b f \right| \leq \int_a^b |f|$$

(Similar results hold for functions defined on half-open intervals).

EXAMPLE: $\int_0^\infty \frac{\sin(x)}{x}\,dx$ CONVERGES (E2.60)

Using estimation, we can show that $\int_0^\infty \frac{\sin(x)}{x}\,dx$ converges.

Proof. We show $\int_0^1 \frac{\sin(x)}{x}\,dx$ and $\int_1^\infty \frac{\sin(x)}{x}\,dx$ converge.

First, since $\lim_{x\to0^+} \frac{\sin(x)}{x} = 1$ by the Fundamental Trigonometric Limit, the function $f(x) = \begin{cases} 1, & x=0 \\ \frac{\sin(x)}{x}, & x>0 \end{cases}$ is continuous on $[0,1]$,

and so by T1.17 $f(x)$ is also integrable on $[0,1]$.

Then, by the Fundamental Theorem of Calculus, $\int_r^1 f(x)\,dx$ is continuous for $r \in [0,1]$, and so

$$\int_0^1 \frac{\sin(x)}{x}\,dx = \lim_{r\to0^+} \int_r^1 \frac{\sin(x)}{x}\,dx = \lim_{r\to0^+} \int_r^1 f(x)\,dx = \int_0^1 f(x)\,dx,$$

which is finite, so $\int_0^1 \frac{\sin(x)}{x}\,dx$ converges as well.

Next, integrate by parts using $\begin{pmatrix} u = \frac{1}{x} & v = \frac{1}{x} \\ du = -\frac{1}{x^2}dx & dv = \frac{-1}{x^2}dx \end{pmatrix}$ to get

$$\int_1^\infty \frac{\sin(x)}{x}\,dx = \left[\frac{-\cos(x)}{x}\right]_1^\infty - \int_1^\infty \frac{\cos(x)}{x^2}\,dx$$

$$\therefore \int_1^\infty \frac{\sin(x)}{x}\,dx = \cos(1) - \int_1^\infty \frac{\cos(x)}{x^2}\,dx.$$

Then, since $\left|\frac{\cos(x)}{x^2}\right| \leq \frac{1}{x^2}$ and $\int_1^\infty \frac{dx}{x^2}$ converges, necessarily $\int_1^\infty \left|\frac{\cos(x)}{x^2}\right|dx$ converges too by comparison.

Hence, by estimation, $\int_1^\infty \frac{\cos(x)}{x^2}\,dx$ also converges.

Finally, since

$$\int_0^\infty \frac{\sin(x)}{x}\,dx = \int_0^1 \frac{\sin(x)}{x}\,dx + \int_1^\infty \frac{\sin(x)}{x}\,dx,$$

and both $\int_0^1 \frac{\sin(x)}{x}\,dx$ and $\int_1^\infty \frac{\sin(x)}{x}$ are finite, it follows that $\int_0^\infty \frac{\sin(x)}{x}\,dx$ converges, and we are done. ▨

Chapter 3: Applications of the Definite Integral

AREA BETWEEN CURVES (D3.2)

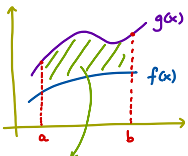
Let $f, g : [a,b] \to \mathbb{R}$ be integrable with $f(x) \le g(x)$ $\forall x \in [a,b]$.

Then, we define the "area" of the region R given by

$$a \le x \le b, \quad f(x) \le y \le g(x)$$

to be

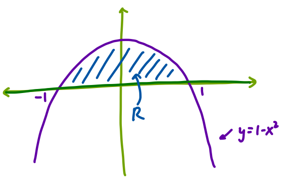
$$A = \int_a^b (g(x) - f(x))\,dx.$$



$R = \int_a^b (g(x) - f(x))\,dx.$

EXAMPLE 1: AREA OF REGION BETWEEN X-AXIS AND $y = 1 - x^2$ (E3.3)

We can use the above formula to calculate the area of the region between the x-axis and the parabola $y = 1 - x^2$.



Note that the region R is given by

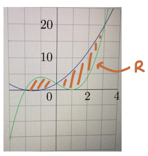
$$-1 \le x \le 1, \quad 0 \le y \le 1 - x^2,$$

so the area is

$$A = \int_{-1}^{1} (1 - x^2)\,dx = \left[x - \frac{x^3}{3}\right]_{-1}^{1} = \frac{4}{3}.$$

EXAMPLE 2: AREA OF REGION BETWEEN $y = x^2 + 3x + 2$ & $y = x^3 - 3x + 2$ (E3.4)

Similarly, we can use the method above to calculate the area of the region between the curves $y = x^2 + 3x + 2$ and $y = x^3 - 3x + 2$.



Let $f(x) = x^2 + 3x + 2$ and $g(x) = x^3 - 3x + 2$.
Then, note that

$$\begin{aligned} f(x) - g(x) &= (x^2 + 3x + 2) - (x^3 - 3x + 2) \\ &= -(x^3 - x^2 - 6x) \\ &= -x(x-3)(x+2) \end{aligned}$$

and so $f(x) = g(x)$ when $x = 0$, $x = 3$ and $x = -2$.
Moreover, $f(x) \ge g(x)$ $\forall x \in (-\infty, -2] \cup [0, 3]$ and $f(x) \le g(x)$ $\forall x \in [-2, 0] \cup [3, \infty)$.

Then, from the diagram, observe that

$$\begin{aligned} A &= \int_{-2}^{0} (g(x) - f(x))\,dx + \int_0^3 (f(x) - g(x))\,dx \\ &= \int_{-2}^{0} (x^3 - x^2 - 6x)\,dx + \int_0^3 (-x^3 + x^2 + 6x)\,dx \\ &= \left[\tfrac{1}{4}x^4 - \tfrac{1}{3}x^3 - 3x^2\right]_{-2}^{0} + \left[-\tfrac{1}{4}x^4 + \tfrac{x^3}{3} + 3x^2\right]_0^3 \end{aligned}$$

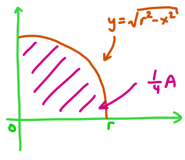
$$A = \frac{253}{12}.$$

EXAMPLE 3: AREA OF A CIRCLE OF RADIUS r (E3.5)

We can use a similar method to calculate the area of a circle with radius r.

Observe that the area of a circle with radius r is simply 4 times the area of the region given by

$$0 \le x \le r, \quad 0 \le y \le \sqrt{r^2 - x^2},$$



so that the area of the circle is

$$A = 4\int_0^r \sqrt{r^2 - x^2}\,dx.$$

Make the substitution $r\sin\theta = x$, so that $r\cos\theta = \sqrt{r^2 - x^2}$ and $r\cos\theta\,d\theta = dx$ to get that

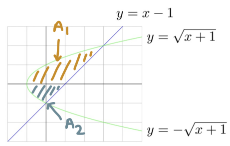
$$\begin{aligned} A &= 4\int_{\theta=0}^{\theta=\pi/2} r\cos\theta\,(r\cos\theta\,d\theta) \\ &= 4r^2 \int_0^{\pi/2} \left(\tfrac{1}{2} - \tfrac{1}{2}\cos 2\theta\right) d\theta \quad \text{(by the double angle formula)} \\ &= r^2 \int_0^{\pi/2} (2 - 2\cos 2\theta)\,d\theta \\ &= r^2 \left[2\theta - 2\sin 2\theta\right]_0^{\pi/2} \end{aligned}$$

$$\therefore A = \pi r^2.$$

EXAMPLE 4: AREA OF REGION BETWEEN $y = x - 1$ & $y^2 = x + 1$ (E3.6)

We can find the area of the region between the curves $y = x - 1$ and $y^2 = x + 1$ using a similar method.

To find the area of the region, we consider two parts separately:



(orange) This is the region defined by

$$-1 \le x \le 0, \quad -\sqrt{x+1} \le y \le \sqrt{x+1}.$$

It follows that

$$\begin{aligned} A_1 &= \int_{-1}^{0} \sqrt{x+1} - (-\sqrt{x+1})\,dx \\ &= \int_{-1}^{0} 2(x+1)^{\frac{1}{2}}\,dx \\ &= \left[\tfrac{4}{3}(x+1)^{3/2}\right]_{-1}^{0} \end{aligned}$$

$$\therefore A_1 = \frac{4}{3}.$$

(blue) This is the region defined by

$$0 \le x \le 3, \quad x - 1 \le y \le \sqrt{x+1}.$$

It follows that

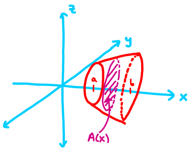
$$\begin{aligned} A_2 &= \int_0^3 \sqrt{x+1} - (x-1)\,dx \\ &= \left[\tfrac{2}{3}(x+1)^{\frac{3}{2}} - \tfrac{1}{2}x^2 + x\right]_0^3 \\ &= \frac{19}{6}. \end{aligned}$$

Hence, the total area of the region is given by

$$\begin{aligned} A &= A_1 + A_2 \\ &= \frac{4}{3} + \frac{19}{6} \end{aligned}$$

$$\therefore A = \frac{9}{2}.$$

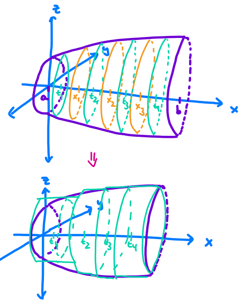
VOLUME BY CROSS-SECTION



(1) Suppose that a solid S lies in space between $x=a$ and $x=b$, and its cross-sectional area at "position" x is equal to $A(x)$, and assume that $A(x)$ is integrable on $[a,b]$.

(2) We can approximate the volume of S as follows:

① Choose a partition of $[a,b]$
$$a<x_0<x_1<\cdots<x_n=b,$$
with corresponding sample points
$$c_k\in[x_{k-1},x_k]\quad\forall k\in\{1,2,\dots,n\}.$$



② Divide the solid into strips, where the k^{th} strip lies between $x=x_{k-1}$ and $x=x_k$ and has thickness
$$\Delta_k x = x_k - x_{k-1}.$$

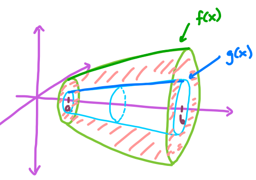
③ Then, note that the total volume of S is equal to
$$V=\sum_{k=1}^{n}\Delta_k V \approx \sum_{k=1}^{n}A(c_k)\Delta_k x. \quad \text{(N3.7)}$$

VOLUME (BY CROSS-SECTION) (D3.8)

Suppose that a solid S lies in space between $x=a$ and $x=b$, and that its cross-sectional area at x is equal to $A(x)$, where $A(x)$ is integrable on $[a,b]$. Then, we define the "volume" of S to be
$$V=\lim_{n\to\infty}\sum_{k=1}^{n}A(c_k)\Delta_k x,$$
or alternatively,
$$V=\int_a^b A(x)\,dx.$$

VOLUME OF AN "ANNULUS SOLID" (E3.9)

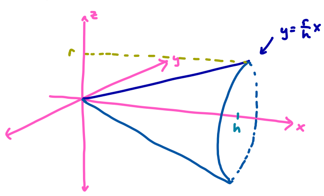
(1) Let f and g be integrable on $[a,b]$ with $0\le f(x)\le g(x)$ $\forall x\in[a,b]$.
Let R be the region in the xy-plane given by
$$a\le x\le b,\quad f(x)\le y\le g(x)$$
and let S be the solid obtained by revolving R about the x-axis.



(2) Then, the area function at position x is given by
$$A(x)=\pi([g(x)]^2-[f(x)]^2),$$
so that the volume of S is
$$V=\int_a^b A(x)\,dx = \int_a^b \pi([g(x)]^2-[f(x)]^2)\,dx.$$

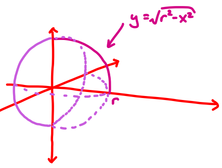
VOLUME OF A CONE (E3.10)

(1) Note that a cone with base radius r and height h can be obtained by revolving the triangular region R given by
$$0\le x\le h,\quad 0\le y\le \frac{r}{h}x$$
about the x-axis.



(2) It follows that the volume of the cone is
$$V=\int_0^r \pi\left(\frac{r}{h}x\right)^2 dx$$
$$=\frac{\pi r^2}{h^2}\left[\frac{x^3}{3}\right]_0^h$$
$$\therefore\ V=\frac{1}{3}\pi r^2 h.$$

VOLUME OF A SPHERE (E3.11)

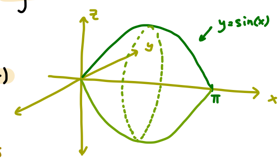


(1) Similarly, we can find the volume of one half of a sphere by revolving the region R given by
$$0\le x\le r,\quad 0\le y\le\sqrt{r^2-x^2}$$
around the x-axis.

(2) Hence, the volume of the sphere is
$$V=2\int_0^r \pi\left(\sqrt{r^2-x^2}\right)^2 dx$$
$$=2\pi\left[r^2x-\frac{1}{3}x^3\right]_0^r$$
$$\therefore\ V=\frac{4}{3}\pi r^3.$$

VOLUME OF A "FOOTBALL" (E3.12)

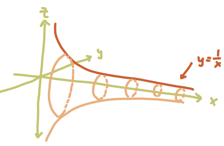
We can use the same formula to calculate the volume of the "football-shaped" solid S which is obtained by revolving the region R given by
$$0\le x\le\pi,\quad 0\le y\le \sin(x)$$
around the x-axis.



The volume of the solid is hence given by
$$V=\int_0^\pi \pi\sin^2(x) = \int_0^\pi \pi\left(\frac{1}{2}-\frac{1}{2}\cos(2x)\right)dx$$
$$=\pi\left[\frac{1}{2}x-\frac{1}{4}\sin(2x)\right]_0^\pi$$
$$\therefore\ V=\frac{\pi^2}{2}.$$

VOLUME OF GABRIEL'S HORN (E3.13)

(1) "Gabriel's horn" refers to the solid S obtained by revolving the region R given by
$$1\le x<\infty,\quad 0\le y\le\frac{1}{x}$$
around the x-axis.



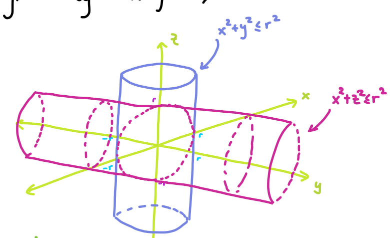
Then, the volume of Gabriel's horn is just
$$V=\int_1^\infty \pi\left(\frac{1}{x^2}\right) = \pi\left[-\frac{1}{x}\right]_1^\infty = \pi.$$

(2) Note that Gabriel's horn has infinite surface area, but only finite volume. (R3.14)

VOLUME OF INTERSECTION OF TWO CYLINDERS (E3.15)

Finally, we can use our formula to calculate the volume of the solid given by $x^2+y^2\le r^2$, $x^2+z^2\le r^2$.

To find the cross-section at $x\in[-r,r]$, note that the region of said cross-section is given by $y^2\le r^2-x^2$ & $z^2\le r^2-x^2$, or equivalently $|y|\le\sqrt{r^2-x^2}$ and $|z|\le\sqrt{r^2-x^2}$.



Hence, the cross-section at x is the square given by $|y|\le\sqrt{r^2-x^2}$ and $|z|\le\sqrt{r^2-x^2}$, so that
$$A(x)=\left[2\sqrt{r^2-x^2}\right]^2 = 4(r^2-x^2).$$
(since the square has side length $2\sqrt{r^2-x^2}$.)

It follows that the volume of S is
$$V=\int_{-r}^{r}A(x)\,dx = \int_{-r}^{r}4(r^2-x^2)\,dx = 4\left[r^2x-\frac{1}{3}x^3\right]_{-r}^{r} = \frac{16}{3}r^3.$$

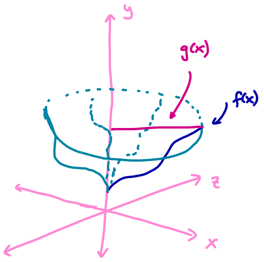
VOLUME BY CYLINDRICAL SHELLS (N3.16)

Suppose that f and g are integrable on $[a,b]$, with $f(x) \leq g(x)$ $\forall x \in [a,b]$.

Let R be the region in the xy-plane given by

$$a \leq x \leq b, \quad f(x) \leq y \leq g(x)$$

and let S be the solid obtained by revolving R about the y-axis.



We can find the volume of S by the following:

1. Choose a partition of $[a,b]$
$$a = x_0 < x_1 < \cdots < x_n = b,$$
with corresponding sample points
$$c_k \in [x_{k-1}, x_k] \quad \forall k \in \{1, 2, \ldots, n\}.$$

2. Divide the region into "strips", where the kth strip R_k is given by
$$x_{k-1} \leq x \leq x_k, \quad f(x) \leq y \leq g(x).$$

3. Revolve each strip R_k around the y-axis to create cylindrical "shells".

4. Note that these shells approximate the total volume of the solid.

5. Then, the volume of the kth shell is
$$\Delta_k V \simeq 2\pi c_k \left(g(c_k) - f(c_k)\right) \Delta_k x,$$
so the total volume of the solid is
$$V \simeq \sum_{k=1}^{n} \Delta_k V = \sum_{k=1}^{n} 2\pi c_k \left(g(c_k) - f(c_k)\right) \Delta_k x.$$

VOLUME (BY CYLINDRICAL SHELLS) (D3.17)

Suppose that f and g are integrable on $[a,b]$ with $f(x) \leq g(x)$ $\forall x \in [a,b]$, and let R be the region in the xy-plane given by

$$a \leq x \leq b, \quad f(x) \leq y \leq g(x).$$

Let S be the solid by revolving R about the y-axis.

Then, the "volume" of S is defined to be

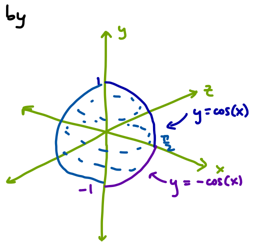
$$V = \int_a^b 2\pi x \left(f(x) - g(x)\right) dx.$$

VOLUME OF A "DISCUS" (E3.19)

Let S be the solid obtained by revolving the region R given by

$$0 \leq x \leq \frac{\pi}{2}, \quad -\cos(x) \leq y \leq \cos(x)$$

about the y-axis.



The volume of S is

$$V = \int_0^{\pi/2} 2\pi x \left[\cos(x) - (-\cos(x))\right] dx$$

$$\therefore V = \int_0^{\pi/2} 4\pi x \cos(x)\, dx.$$

To solve this integral, integrate by parts using

$$\begin{pmatrix} u = x & v = \sin(x) \\ \downarrow & \uparrow \\ du = 1\,dx & dv = \cos(x)\,dx \end{pmatrix}$$

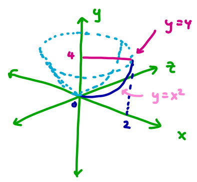
to get that

$$V = 4\pi\left[x\sin(x) - \int \sin(x)\,dx\right]_0^{\pi/2}$$
$$= 4\pi\left[x\sin(x) - \cos(x)\right]_0^{\pi/2}$$

$$\therefore V = 2\pi^2 - 4\pi.$$

VOLUME OF A "BOWL" (E3.20)

We can use similar reasoning to find the capacity of a "bowl", which is the solid obtained by revolving the parabola $y = x^2$ with $0 \leq x \leq 2$ about the y-axis.



The volume of the bowl is given by

$$V = \int_0^2 2\pi x\,(4 - x^2)\,dx$$
$$= \pi\left[4x^2 - \tfrac{1}{2}x^4\right]_0^2$$

$$\therefore V = 8\pi.$$

VOLUME OF A TORUS (E3.21)

A "torus" can be obtained by revolving the disc D given by

$$(x-R)^2 + y^2 \leq r^2$$

about the y-axis.

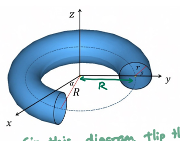


Image source: The coordinate system & parametrisation of a circular torus of radii R and r (ResearchGate)

(in this diagram, flip the y & z axes)

The disc D is given by

$$R - r \leq x \leq R + r, \quad -\sqrt{r^2 - (x-R)^2} \leq y \leq \sqrt{r^2 - (x-R)^2},$$

so that the volume of the torus is

$$V = \int_{R-r}^{R+r} 2\pi x \cdot 2\sqrt{r^2 - (x-R)^2}\,dx.$$

Let $r\sin\theta = x - R$, so that $r\cos\theta = \sqrt{r^2 - (x-R)^2}$ and $r\cos\theta\,d\theta = dx$, to get

$$V = \int_{x=R-r}^{x=R+r} 4\pi x\sqrt{r^2 - (x-R)^2}\,dx = \int_{\theta=-\frac{\pi}{2}}^{\theta=\frac{\pi}{2}} 4\pi(R + r\sin\theta)(r\cos\theta)(r\cos\theta\,d\theta)$$

$$= 4\pi r^2 \int_{\theta=-\frac{\pi}{2}}^{\theta=\frac{\pi}{2}} R\cos^2\theta + r\sin\theta\cos^2\theta\,d\theta$$

$$= 4\pi r^2\left[R\left(\tfrac{1}{2}\theta + \tfrac{1}{4}\sin 2\theta\right) + \tfrac{r}{3}\cos^3\theta\right]_{-\pi/2}^{\pi/2}$$

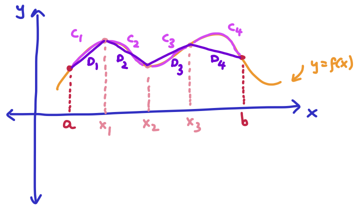
$$\therefore V = 2\pi^2 r^2 R.$$

ARCLENGTH (N3.22)

Let f be differentiable on $[a,b]$, or let f be differentiable on (a,b) and continuous on $[a,b]$.

Let C be the curve $y=f(x)$ with $a \leq x \leq b$.

We can approximate the length of C as follows.



① Choose a partition of $[a,b]$
$a = x_0 < x_1 < \cdots < x_n = b$.

② By the Mean Value Theorem, there exist sample points $c_k \in [x_{k-1}, x_k]$ such that

$$f'(c_k) = \frac{f(x_k) - f(x_{k-1})}{x_k - x_{k-1}} = \frac{\Delta_k y}{\Delta_k x}.$$

③ Let C_k be the part of the curve C from $x_{k-1} \leq x \leq x_k$, and let D_k be the line segment from $(x_{k-1}, f(x_{k-1}))$ to $(x_k, f(x_k))$.

④ Then, note that the total length of C_k is approximately equal to the length of D_k; ie

$$\Delta_k L \cong |D_k| = \sqrt{(\Delta_k x)^2 + (\Delta_k y)^2}$$

$$= \sqrt{1 + \left(\frac{\Delta_k y}{\Delta_k x}\right)^2} \cdot \Delta_k x$$

$$\therefore \Delta_k L = \sqrt{1 + [f'(c_k)]^2} \cdot \Delta_k x,$$

so that the total length of C is approximately

$$L \cong \sum_{k=1}^{n} \Delta_k L = \sum_{k=1}^{n} \sqrt{1 + [f'(c_k)]^2} \cdot \Delta_k x.$$

LENGTH (D3.23)

Let f be differentiable on $[a,b]$, or let f be differentiable on (a,b) and continuous on $[a,b]$.

Then, we define the "length", or "arclength", of the curve $y=f(x)$ from $x=a$ to $x=b$ to be

$$L = \int_a^b \sqrt{1 + f'(x)^2}\, dx.$$

RECTIFIABLE (D3.23)

We say that f (from $x=a$ to $x=b$) is "rectifiable" if its arclength from $x=a$ to $x=b$ is finite.

EXAMPLE 1: LENGTH OF $y=x^2$ (E3.24)

Using the above formula, we can find the length of the curve $y=x^2$ with $0 \leq x \leq 2$.

Let $f(x) = x^2$, so that $f'(x) = 2x$. Then, the length of the curve is

$$L = \int_0^2 \sqrt{1 + f'(x)^2}\, dx = \int_0^2 \sqrt{1+4x^2}\, dx.$$

To solve this integral, let $2x = \tan\theta$ so $\sec\theta = \sqrt{1+4x^2}$ & $\sec^2\theta\, d\theta = 2dx$ to get

$$\int \sqrt{1+4x^2}\, dx = \int \tfrac{1}{2}\sec^3\theta\, d\theta$$

$$= \tfrac{1}{4}\sec\theta\tan\theta + \tfrac{1}{4}\ln|\sec\theta + \tan\theta| + c$$

$$= \tfrac{1}{2}x\sqrt{1+4x^2} + \tfrac{1}{4}\ln|2x + \sqrt{1+4x^2}| + c.$$

It follows that

$$L = \int_0^2 \sqrt{1+4x^2}\, dx = \left[\tfrac{1}{2}x\sqrt{1+4x^2} + \tfrac{1}{4}\ln(2x + \sqrt{1+4x^2})\right]_0^2$$

$$= \sqrt{17} + \tfrac{1}{4}\ln(4+\sqrt{17}).$$

SURFACE AREA (N3.26)

Let f be differentiable on $[a,b]$, or let f be differentiable on (a,b) and continuous on $[a,b]$.

Let C be the curve in the xy-plane given by $y=f(x)$ with $a \le x \le b$, and let S be the surface obtained by revolving C about the x-axis.

We can approximate the area of S by the following:

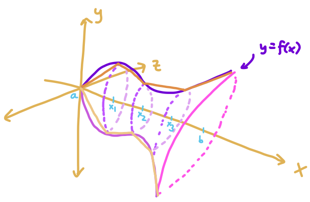
① Choose a partition of $[a,b]$
$a = x_0 < x_1 < \cdots < x_n < b$.

② By the Mean Value Theorem, there exists sample points $c_k \in [x_{k-1}, x_k]$ such that

$$f'(c_k) = \frac{f(x_k) - f(x_{k-1})}{x_k - x_{k-1}} = \frac{\Delta_k y}{\Delta_k x}.$$

③ Let C_k be the part of C with $x_{k-1} \le x \le x_k$, and let S_k denote the "slice" of S which is obtained by revolving C_k around the x-axis.

④ Let D_k be the line segment from $(x_{k-1}, f(x_{k-1}))$ to $(x_k, f(x_k))$, and let T_k be the slice of a cone obtained by revolving D_k about the x-axis.



⑤ Then, the area of the slice S_k is approximately equal to the area of T_k, so that

$$\Delta_k A \cong \pi(f(x_{k-1}) + f(x_k))\Delta_k L$$

(using the formula for the area of a cone (N2.35))

$$= \pi(f(x_{k-1}) + f(x_k))\sqrt{1+f'(c_k)^2}\,\Delta_k x$$

$\therefore \Delta_k A \cong 2\pi f(c_k)\sqrt{1+f'(c_k)^2}\,\Delta_k x$,

and it follows the area of S is approximately equal to

$$A \cong \sum_{k=1}^{n} \Delta_k A \cong \sum_{k=1}^{n} 2\pi f(c_k)\sqrt{1+f'(c_k)^2}\,\Delta_k x.$$

AREA (D3.27)

Let f be differentiable on $[a,b]$, or let f be differentiable on (a,b) and continuous on $[a,b]$.

Let C be the curve given by $y=f(x)$ with $a \le x \le b$, and S the surface obtained by revolving C about the x-axis.

Then, we define the "area" of S to be

$$A = \int_a^b 2\pi f(x)\sqrt{1+f'(x)^2}\,dx.$$

* if the curve is revolved about the y-axis instead, replace the $f(x)$ with x.
(D3.29)

AREA OF A SPHERE (E3.30)

Note that a sphere can be obtained by revolving the curve $y=\sqrt{r^2-x^2}$ with $-r \le x \le r$ about the x-axis.

Let $f(x) = \sqrt{r^2-x^2}$, so that $f'(x) = -\frac{x}{\sqrt{r^2-x^2}}$,

so that

$$\sqrt{1+f'(x)^2} = \sqrt{1+\frac{x^2}{r^2-x^2}} = \sqrt{\frac{r^2}{r^2-x^2}} = \frac{r}{\sqrt{r^2-x^2}}.$$

So, the area of the sphere is

$$A = \int_{-r}^{r} 2\pi f(x)\sqrt{1+f'(x)^2}\,dx$$

$$= \int_{-r}^{r} 2\pi\sqrt{r^2-x^2}\cdot\frac{r}{\sqrt{r^2-x^2}}\,dx$$

$$= \int_{-r}^{r} 2\pi r\,dx$$

$\therefore A = 4\pi r^2$.

AREA OF A TORUS (E3.31)

Similarly, half of a "torus" can be obtained by revolving the curve $y=\sqrt{r^2-(R-x)^2}$ with $R-r \le x \le R+r$ about the y-axis.

Let $f(x) = \sqrt{r^2-(x-R)^2}$, so that $f'(x) = \frac{-(x-R)}{\sqrt{r^2-(x-R)^2}}$,

so that

$$\sqrt{1+f'(x)^2} = \sqrt{1+\frac{(x-R)^2}{r^2-(x-R)^2}} = \sqrt{\frac{r^2}{r^2-(x-R)^2}} = \frac{r}{\sqrt{r^2-(x-R)^2}}.$$

It follows that the surface area of the torus is

$$A = 2\int_{R-r}^{R+r} 2\pi x\cdot\frac{r}{\sqrt{r^2-(x-R)^2}}\,dx = 4\pi r\int_{R-r}^{R+r}\frac{x\,dx}{\sqrt{r^2-(x-R)^2}}.$$

Make the substitution $r\sin\theta = x-R$, so that $r\cos\theta = \sqrt{r^2-(x-R)^2}$ and $r\cos\theta\,d\theta = dx$. Then

$$A = 4\pi r\int_{x=R-r}^{x=R+r}\frac{x\,dx}{\sqrt{r^2-(x-R)^2}} = 4\pi r\int_{\theta=-\frac{\pi}{2}}^{\theta=\frac{\pi}{2}}\frac{(R+r\sin\theta)\cdot r\cos\theta}{r\cos\theta}\,d\theta$$

$$= 4\pi r\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}(R+r\sin\theta)\,d\theta$$

$$= 4\pi r\left[R\theta - r\cos\theta\right]_{-\pi/2}^{\pi/2}$$

$\therefore A = 4\pi^2 rR$.

MASS & DENSITY (E3.32)

1. Suppose a rod lies along the x-axis from $x=a$ to $x=b$, with linear density (mass per unit length) equal to $\rho(x)$, where $\rho(x)$ is integrable on $[a,b]$.

2. We can approximate the mass of the rod as follows:

   ① Choose a partition of $[a,b]$
   $$a = x_0 < x_1 < \dots < x_n = b$$
   with corresponding sample points $c_k \in [x_{k-1}, x_k]$.

   ② Then, the mass of the part of the rod between $x=x_{k-1}$ and $x=x_k$ is approximately
   $$\Delta_k M \cong \rho(c_k)\Delta_k x,$$
   so that the total mass of the rod is
   $$M = \sum_{k=1}^{n} \Delta_k M = \sum_{k=1}^{n} \rho(c_k)\Delta_k x.$$

MASS (E3.32)

Suppose a rod lies along the x-axis from $x=a$ to $x=b$, with linear density $\rho(x)$, where $\rho(x)$ is integrable on $[a,b]$.

Then the "mass" of the rod is
$$M = \int_a^b \rho(x)\,dx.$$

MASS OF A SPHERE WITH VARYING DENSITY (E3.33)

Suppose that a ball of radius R has varying density, such that the density of each point r units away from the origin is equal to $\rho(r)$, where $\rho(r)$ is integrable on $[0,r]$.

We approximate the mass of the ball as follows:

Choose a partition $0=r_0<r_1<\dots<r_n=R$ of $[0,R]$, with sample points $c_k \in [r_{k-1}, r_k]$.

Divide the sphere into spherical shells using concentric spheres of radius r_k.

Then, the volume of the k^{th} spherical shell is
$$\Delta_k V \cong 4\pi c_k^2 \Delta_k r,$$
so the shell's mass is
$$\Delta_k M \cong \rho(c_k)\Delta_k V \cong \rho(c_k)\,4\pi c_k^2 \Delta_k r.$$

Hence, the total mass of the sphere is
$$M = \int_a^b 4\pi r^2 \rho(r)\,dr.$$

FORCE

PRESSURE EXAMPLE (E3.34)

3.34 Example: A tank is in the shape of the parabolic sheet given by $y=x^2$, $-2\le x\le 2$, $-5\le z\le 5$ together with the two ends given by $-2\le x\le 2$, $x^2\le y\le 4$ with $z=\pm 5$ (where the y-axis is pointing upwards). The tank is filled with a liquid of density ρ. The pressure $P(h)$ (force per unit area) exerted by the liquid on each wall at all points which lie at a depth h is given by
$$P = \rho g h$$

Along one of the ends of the tank, consider a thin horizontal slice at position y of thickness Δy.

The slice is at a "depth" of $h=4-y$, so the pressure at all points on the slice is $P = \rho g h = \rho g(4-y)$.

The width of the slice is equal to $2\sqrt{y}$, so the area of the slice is
$$\Delta A = 2\sqrt{y}\,\Delta y,$$
so the force exerted by the water on the slice is
$$\Delta F = P\Delta A = \rho g(4-y)\cdot 2\sqrt{y}\,\Delta y.$$

Hence, the total force exerted on the end of the tank is
$$F = \int_0^4 \rho g(4-y)\cdot 2\sqrt{y} = \rho g \int_0^4 8y^{\frac{1}{2}} - 2y^{\frac{3}{2}}\,dy$$
$$= \rho g\left[\tfrac{16}{3}y^{\frac{3}{2}} - \tfrac{4}{5}y^{\frac{5}{2}}\right]_0^4$$
$$\therefore\ F = \frac{256}{15}\rho g.$$

CHARGED ROD (COULOMB'S LAW) (E3.35)

3.35 Example: A charged rod, of charge Q (with its charge evenly distributed along its length) lies along the x-axis from $x=0$ to $x=2$. A small object of charge q lies at position $(x,y)=(2,1)$. Find the force exerted by the rod on the object. Use the fact that the force exerted by one small object of charge q_1 at position p_1 on another of charge q_2 at position p_2 is equal to
$$F = \frac{k\,q_1 q_2}{|u|^2}\cdot\frac{u}{|u|}$$
where k is a constant and u is the direction vector from p_1 to p_2, that is $u = p_2 - p_1$.

First, consider a small slice of rod at position x, of thickness Δx.

Since the rod has length 2, the charge per unit length is $\frac{Q}{2}$, so that the charge on the slice of the rod is $\Delta Q = \frac{Q}{2}\Delta x$.

Then, the distance from the slice, which is at position $(x,0)$, to the small object (at $(2,1)$) is
$$r = |u| = \sqrt{(2-x)^2+1},$$
so that
$$|\Delta F| = \frac{kq\cdot\frac{Q}{2}\Delta x}{(2-x)^2+1}.$$

Next, by similar triangles, the x & y-components of the force exerted by the slice on the object are given by
$$\Delta F_x = \frac{2-x}{\sqrt{(2-x)^2+1}}\Delta F = \frac{kqQ(2-x)\Delta x}{2((2-x)^2+1)^{3/2}}$$
&
$$\Delta F_y = \frac{1}{\sqrt{(2-x)^2+1}}\Delta F = \frac{kqQ\Delta x}{2((2-x)^2+1)^{3/2}}.$$

It follows that the x and y-components of the total force are
$$F_x = \int_0^2 \frac{kqQ(2-x)}{2((2-x)^2+1)^{3/2}}\,dx$$
and
$$F_y = \int_0^2 \frac{kqQ}{2((2-x)^2+1)^{3/2}}\,dx.$$

Solving, we get that the total force exerted by the rod on the object, expressed as a vector, is
$$F = (F_x, F_y) = \tfrac{1}{2}kqQ\left(1-\tfrac{1}{\sqrt{5}},\ \tfrac{2}{\sqrt{5}}\right).$$

WORK

TANK EXAMPLE (E3.36)

3.36 Example: A tank is in the shape of the parabolic sheet given by $y = x^2$, $-2 \le x \le 2$, $-5 \le z \le 5$ together with the two ends given by $-2 \le x \le 2$, $x^2 \le y \le 4$ with $z = \pm 5$ (where the y-axis is pointing vertically). The tank is filled with a liquid of density ρ. Find the work required to pump all the liquid out of the tank, bringing it all to the level of the top of the tank. Use the fact that the work required to raise a small object of mass m from height h_1 to height h_2 is equal to

$$W = mgh$$

where $h = h_2 - h_1$.

Consider a thin slice of liquid at position y of thickness Δy.

The slice is in the shape of a thin rectangle of length $\ell = 10$, width $w = 2\sqrt{y}$ and thickness Δy, so it follows that its volume is

$$\Delta V = 20\sqrt{y}\,\Delta y.$$

Hence, its mass is

$$\Delta M = \rho \Delta V = 20\rho\sqrt{y}\,\Delta y.$$

Then, all of the water in this slice must be raised from height $h_1 = 4 - y$ to height $h_2 = 4$, so that the work done "raising" the water in this slice is

$$\Delta W = gh\,\Delta M = 20\rho g(4-y)\sqrt{y}\,\Delta y.$$

It follows that the total work required to pump all of the water in the tank is

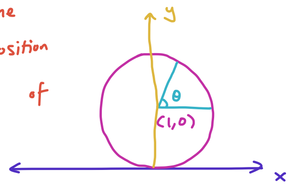
$$W = \int_0^4 20\rho g(4-y)\sqrt{y}\,dy = 20\rho g\left[\frac{8}{3}y^{\frac{3}{2}} - \frac{2}{5}y^{\frac{5}{2}}\right]_0^4$$

$$\therefore\ W = \frac{2560}{3}\rho g.$$

CHAIN EXAMPLE (E3.37)

3.37 Example: A chain, of length π and mass M, lies along the x-axis. Find the work required to lift the chain and lie it along the top half of the circle $x^2 + (y-1)^2 = 1$ (where the y-axis points upwards).

Note that for a thin slice of the chain (when it is lying on the top half of the circle) at position θ of thickness $\Delta\theta$, the mass of the slice is



$$\Delta M = \frac{M}{\pi}\Delta\theta,$$

and the height above the x-axis is

$$y = 1 + \sin\theta,$$

so that the work done is lifting the slice from the x-axis is

$$\Delta W = gy\,\Delta M = \frac{gM}{\pi}(1+\sin\theta)\,\Delta\theta.$$

It follows that the total work done is

$$W = \int_0^{\pi} \frac{gM}{\pi}(1+\sin\theta)\,d\theta = \frac{gM}{\pi}\left[\theta - \cos\theta\right]_0^{\pi} = \frac{gM}{\pi}(\pi+2).$$

Chapter 4: Parametric and Polar Curves

PARAMETRIC CURVES

GRAPH (D4.1)

Let $f: I \to \mathbb{R}^2$, where I is an interval of $\mathbb{R}$.
Then, we define the "graph" of f, denoted as "graph(f)", to be the set
$$\text{graph}(f) = \{(x, f(x)) : x \in I\}.$$

CURVE IN $\mathbb{R}^2$ / EXPLICITLY DEFINED (D4.1)

1. Let $f: I \to \mathbb{R}^2$, where I is an interval of $\mathbb{R}$.
Then, we say f is a "curve in $\mathbb{R}^2$" if f is continuous.

2. In this case, we say the curve is defined "explicitly" by the equation
$y = f(x)$.

NULL SET / IMPLICITLY DEFINED (D4.1)

1. Let $f: U \to \mathbb{R}$, where U is a connected set in $\mathbb{R}^2$.
Then, we define the "null space" of f, denoted by Null(f), to be the set
$$\text{Null}(f) = \{(x,y) \in U \mid f(x,y) = 0\}.$$

2. Note that when f is continuous, Null(f) is typically a curve in $\mathbb{R}^2$; in this case, we say the curve is defined "implicitly" by the equation
$f(x,y) = 0$.

RANGE / IMAGE / PARAMETRICALLY DEFINED (D4.1)

1. Let $f: I \to \mathbb{R}^2$, where I is an interval in $\mathbb{R}$, be defined by $f(t) = (x(t), y(t))$.
Then we define the "range" of f, denoted by Range(f), to be the set
$$\text{Range}(f) = \{f(t) \mid t \in I\} = \{(x(t), y(t)) \mid t \in I\}.$$

2. When $x(t)$ and $y(t)$ are continuous, Range(f) is typically a curve in $\mathbb{R}^2$; in this case, we say the curve is defined "parametrically" by the equation
$p = (x,y) = f(t)$;
or by the set of equations
$p = (x,y)$; $x = x(t)$, $y = y(t)$,
where t is called the "parameter" of this equation.

EXAMPLE 1: 2D CIRCLE (E4.2)

We can assign three "definitions" to the equation describing a circle of radius 1 centred at $(0,0)$:

① The circle is defined implicitly by the equation
$$x^2 + y^2 = 1;$$

② The circle is defined explicitly by the set of equations
$$\begin{cases} y = \sqrt{1-x^2}, & |x| \le 1 \\ y = -\sqrt{1-x^2}, & |x| \le 1 \end{cases}; \text{ and}$$

③ The circle is defined parametrically by the equation
$$(x,y) = (\cos(t), \sin(t)), \quad 0 \le t \le 2\pi.$$

EXAMPLE 2: 3D SPHERE (R4.3)

Similarly, we can assign three "definitions" to the equation describing the sphere of radius 1 centred at $(0,0,0)$:

① The sphere can be defined implicitly by the equation
$$x^2 + y^2 + z^2 = 1;$$

② The sphere can be defined explicitly by the set of equations
$$\begin{cases} z = \sqrt{1-x^2-y^2}, & |x| \le 1, \ |y| \le 1 \\ z = -\sqrt{1-x^2-y^2}, & |x| \le 1, \ |y| \le 1 \end{cases}; \text{ and}$$

③ The sphere can be defined parametrically by the equation
$$(x,y,z) = (\cos\theta \sin\phi, \ \sin\theta \sin\phi, \ \cos\phi),$$
where θ denotes the "latitude" and ϕ the "longitude" of the sphere.

- we let $\theta \in [0, 2\pi]$ and $\phi \in [0, \pi]$, such that $\phi = 0$ at the "North pole" and $\phi = \pi$ at the "South pole"

EXAMPLE 3: LINE SEGMENT (E4.4)

Given two points $a, b \in \mathbb{R}^2$, the line segment from a to b can be defined parametrically by
$$p = f(t) = a + t(b-a), \quad 0 \le t \le 1.$$

EXAMPLE 4: CIRCULAR ARC (E4.5)

The arc of the circle of radius r centred at (a,b) can be defined parametrically by
$$(x,y) = (a + r\cos(t), \ b + r\sin(t)), \quad \alpha \le t \le \beta,$$
where $\alpha, \beta \in \mathbb{R}$ are arbitrary.

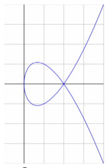
SKETCHING PARAMETRIC CURVES (N4.6)

To sketch a parametric curve, we can simply make a table of values and plot points.

EXAMPLE 1: "ALPHA" CURVE (E4.7)

The "alpha curve" is given by the parametric curve $(x,y) = (t^2, t^3 - 2t)$. (D4.9)

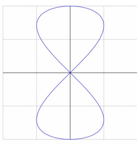
t	x	y
-2	4	-4
$-\sqrt{2}$	2	0
-1	1	1
0	0	0
1	1	-1
$\sqrt{2}$	2	0
2	4	4



EXAMPLE 2: "FIGURE EIGHT" CURVE (E4.8)

The "figure eight" curve is given by the parametric curve $(x,y) = (\sin(2t), 2\sin(t))$. (D4.9)

t	x	y
0	0	0
$\pi/6$	$\sqrt{3}/2$	1
$\pi/4$	1	$\sqrt{2}$
$\pi/3$	$\sqrt{3}/2$	$\sqrt{3}$
$\pi/2$	0	2
$2\pi/3$	$-\sqrt{3}/2$	$\sqrt{3}$
$3\pi/4$	-1	$\sqrt{2}$
$5\pi/6$	$-\sqrt{3}/2$	1
π	0	0



PARAMETRIC → IMPLICIT (N4.10)

Sometimes, we can "eliminate" the parameter in a parametric equation to obtain an implicit/explicit equation for the curve.

EXAMPLE 1: $(x,y) = (t^2+1, t^3)$ (E4.11)

For instance, we can eliminate the parameter in the equation $(x,y) = (t^2+1, t^3)$ to find an implicit equation for the curve.

Note that $x = t^2+1$, $y = t^3 \Rightarrow (x-1)^3 = (t^2)^3 = t^6 = y^2$,

so the curve is given implicitly by

$$y^2 = (x-1)^3$$

EXAMPLE 2: $(x,y) = (\sin(t), \sec(t))$ (E4.11)

We can use a similar method to extract the implicit equation for the curve defined parametrically as $(x,y) = (\sin(t), \sec(t))$.

Note that

$$y^2 = \sec^2(t) = \frac{1}{\cos^2(t)} = \frac{1}{1-\sin^2(t)} = \frac{1}{1-x^2},$$

so that $y^2 = \frac{1}{1-x^2}$ is the implicit definition for this curve.

EXAMPLE 3: ALPHA CURVE (E4.12)

We can do similarly to find the implicit equation of the alpha curve, defined by $(x,y) = (t^2, t^3 - 2t)$.

Note that

$$y^2 = (t^3-2t)^2 = t^6 - 4t^4 + 4t^2 = x^3 - 4x^2 + 4x$$

$$\Rightarrow y^2 = x(x-2)^2.$$

EXAMPLE 4: FIGURE EIGHT CURVE (E4.12)

We can do similarly to find the implicit equation of the figure eight curve, given by $(x,y) = (\sin(2t), 2\sin(t))$.

Note that

$$x^2 = \sin^2(2t) = 4\sin^2(t)\cos^2(t)$$
$$= 4\sin^2(t)(1-\sin^2(t))$$
$$= y^2(1-(\tfrac{y}{2})^2)$$
$$= \tfrac{1}{4}y^2(4-y^2),$$

so that the implicit definition for the curve is

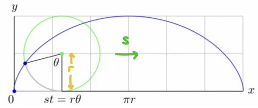
$$x^2 = \tfrac{1}{4}y^2(2-y)(2+y).$$

f(t) REPRESENTS POSITION OF A MOVING POINT (N4.13)

If we take t as time, then the parametric curve $(x,y) = f(t) = (x(t), y(t))$ represents the position of a moving point.

EXAMPLE: CYCLOID (E4.14)

1. A "cycloid" is the curve generated by a point on a circle in the xy-plane which rolls (without slipping) about the x-axis.



2. We can use this to find the parametric equation for a cycloid.

Let the circle be described as in the diagram, and suppose it rolls with speed s.
Let the point in question be the origin $(0,0)$ on the circle.

Then, at time t, the centre will be at position (st, r).
Let $\theta = \theta(t)$ be the angle through which the circle has revolved about its centre at time t.
Since the circle revolves at a constant rate, necessarily $\theta(t) = ct$ for some constant c.
Moreover, since the circle rolls without slipping, it makes one full revolution about its centre when $x(t) = 2\pi r$;
hence $st = 2\pi r$ when $\theta(t) = 2\pi$;
ie $ct = 2\pi$ when $t = \frac{2\pi r}{s}$, so that $c = \frac{s}{r}$.

Next, since the centre of the circle is at $(st, 0) = (r\theta(t), r)$ at time t, and the circle has rotated (clockwise) by the angle $\theta(t) = \frac{s}{r}t$,
it follows that the point on the circle originally at $(0,0)$ will have moved to the position

$$(x,y) = (r\theta(t), r) - (r\sin(\theta(t)), r\cos(\theta(t))).$$

We use θ as our parameter, and write this as

$$(x,y) = (x(\theta), y(\theta)) = (r\theta, r) - (r\sin\theta, r\cos\theta)$$
$$\Rightarrow (x,y) = r(\theta - \sin\theta, 1-\cos\theta),$$

or use t as our parameter instead and write

$$(x,y) = (x(t), y(t)) = (st, r) - (r\sin(\tfrac{s}{r}t), r\cos(\tfrac{s}{r}t)).$$

TANGENT

TANGENT VECTOR (D4.15)

Let $(x,y) = f(t) = (x(t), y(t))$ be a parametric curve.
Then, the "tangent vector" to the curve at the point where $t = t_0$ is the vector

$$f'(t_0) = (x'(t_0), y'(t_0)).$$

LINEARISATION (D4.15)

Let $(x,y) = f(t) = (x(t), y(t))$ be a parametric curve.
Then, we define the "linearisation" of f at $t = t_0$, denoted by $L(t)$, to be the function

$$L(t) = f(t_0) + f'(t_0)(t - t_0).$$

VELOCITY / SPEED / ACCELERATION (D4.15)

Let $(x,y) = f(t) = (x(t), y(t))$ be a parametric curve.
Suppose t represents time, and $f(t)$ represents the position of a moving point.
Then:

1. The "velocity" of the point at time t is equal to
$$v = f'(t) = (x'(t), y'(t));$$
2. The "speed" of the point at time t is equal to
$$|v| = |f'(t)|;$$ and
3. The "acceleration" of the point at time t is equal to
$$a = f''(t) = (x''(t), y''(t)).$$

EXAMPLE 1: TANGENT TO ALPHA CURVE (E4.16)

We can use the formulas above to find the tangent to the alpha curve where $t = 1$.

Recall the equation for the alpha curve is $(x,y) = (t^2, t^3 - 2t)$.

First, note that $(x(1), y(1)) = (1, -1)$.
Next,

$$\frac{dx}{dt} = 2t \quad \& \quad \frac{dy}{dt} = 3t^2 - 2.$$

It follows that

$$(x'(t), y'(t)) = (2t, 3t^2 - 2),$$

so that

$$(x'(1), y'(1)) = (2, 1).$$

Hence, the tangent line is the line through $(1,-1)$ in the direction of the vector $(2,1)$.
This line has slope $\frac{1}{2}$, so its equation is

$$y + 1 = \frac{1}{2}(x - 1),$$

or

$$y = \frac{1}{2}x - \frac{3}{2}.$$

EXAMPLE 2: STONE PROBLEM (E4.17)

4.17 Example: A small stone is stuck in the tread of the tire of a car. The tire has radius $r = 0.25$ (in meters) and the car moves at speed $s = 10$ (in meters per second). The stone moves along a cycloid with its position (in meters) at time t (in seconds) given by

$$(x,y) = \big(x(t), y(t)\big) = (st, r) - \left(r\sin\left(\tfrac{s}{r}t\right), r\cos\left(\tfrac{s}{r}t\right)\right).$$

Find the position, the velocity, and the speed of the stone at time $t = \pi/120$.

Put $r = \frac{1}{4}$ & $s = 10$ into the parametric equations to get

$$(x,y) = (10t, \tfrac{1}{4}) - (\tfrac{1}{4}\sin(40t), \tfrac{1}{4}\cos(40t))$$

and

$$(x', y') = (10, 0) - (10\cos(40t), -10\sin(40t))$$

Then, when $t = \frac{\pi}{120}$, the position, velocity and speed are

$$p = (x,y) = (\tfrac{\pi}{12}, \tfrac{1}{4}) - (\tfrac{\sqrt{3}}{8}, \tfrac{1}{8}) = (\tfrac{\pi}{12} - \tfrac{\sqrt{3}}{8}, \tfrac{1}{8});$$

$$v = (x', y') = (10, 0) - (5, -5\sqrt{3}) = (5, 5\sqrt{3}); \quad \&$$

$$|v| = \sqrt{(x')^2 + (y')^2} = \sqrt{25 + 75} = 10.$$

"DERIVATIVE" OF X-COMPONENT OF PARAMETRIC EQUATION (N4.18)

Consider the parametric curve $(x,y) = f(t) = (x(t), y(t))$ with $r \leq t \leq s$.
Suppose we can eliminate the parameter, and express the curve in the form $y = g(x)$.
Then

1. $y(t) = g(x(t))$;
2. $g'(x(t)) = \dfrac{y'(t)}{x'(t)}$; and
3. $g''(x(t)) = \dfrac{\frac{d}{dt}\left(\frac{y'(t)}{x'(t)}\right)}{x'(t)}$.

__Proof.__ ① follows by construction.
Take the derivative of both sides of ① wrt t to get

$$y'(t) = [g(x(t))]' = g'(x(t))\,x'(t),$$

So that

$$g'(x(t)) = \frac{y'(t)}{x'(t)} \quad \forall x'(t) \neq 0,$$

proving ②.
Then, take the derivative of both sides of ② wrt t to get

$$y''(t) = [g'(x(t))]' = g''(x(t))\,x'(t) = \frac{d}{dt}\left(\frac{y'(t)}{x'(t)}\right),$$

or

$$g''(x(t)) = \frac{\frac{d}{dt}\left(\frac{y'(t)}{x'(t)}\right)}{x'(t)},$$

proving ③. ∎

EXAMPLE: "DERIVATIVE" OF THE FIGURE EIGHT CURVE (E4.19)

Consider the curve $(x,y) = (\sin(2t), 2\sin(t))$.
Suppose the portion of the curve with $-\frac{\pi}{4} \leq t \leq \frac{\pi}{4}$ is given explicitly by $y = g(x)$ with $-1 \leq x \leq 1$.
Then, we can use the formulas above to find $g'(\frac{\sqrt{3}}{2})$ and $g''(\frac{\sqrt{3}}{2})$.

First, note that for $-\frac{\pi}{4} \leq t \leq \frac{\pi}{4}$, we have

$$x(t) = \frac{\sqrt{3}}{2} \Leftrightarrow \sin(2t) = \frac{\sqrt{3}}{2} \Leftrightarrow t = \frac{\pi}{6}.$$

Moreover,

$$(x(t), y(t)) = (\sin(2t), 2\sin(t))$$

$$\Rightarrow (x'(t), y'(t)) = (2\cos(2t), 2\cos(t)),$$

so

$$g'(x(t)) = \frac{y'(t)}{x'(t)} = \frac{\cos(t)}{\cos(2t)}$$

and

$$g''(x(t)) = \frac{\frac{d}{dx}\left(\frac{y'(t)}{x'(t)}\right)}{x'(t)} = \frac{\frac{d}{dt}\left(\frac{\cos(t)}{\cos(2t)}\right)}{2\cos(2t)} = \frac{-\sin(t)\cos(2t) + 2\cos(t)\sin(2t)}{2\cos^3(2t)}.$$

Put in $t = \frac{\pi}{6}$ to get

$$g'(\tfrac{\sqrt{3}}{2}) = \frac{\cos \pi/6}{\cos \pi/3} = \sqrt{3}; \quad \text{and}$$

$$g''(\tfrac{\sqrt{3}}{2}) = \frac{-\sin \pi/6 \cos \pi/3 + 2\cos\frac{\pi}{6}\sin\frac{\pi}{3}}{2\cos^3\frac{\pi}{3}} = 5.$$

"INTEGRAL" OF THE X-COMPONENT OF PARAMETRIC EQUATION (N4.20)

Consider the curve given parametrically by $(x,y) = f(t) = (x(t), y(t))$ with $r \leq t \leq s$, and Suppose that $y(t) \geqslant 0$ and $x'(t) \geqslant 0$ $\forall t \in [r,s]$.
Let $a = x(r)$ and $b = x(s)$.
(Note that $a \geqslant b$, since $x'(t) \geqslant 0$ $\forall t \in [r,s]$.)

Next, suppose that we can eliminate t to express the curve explicitly by $y = g(x)$ $\forall x \in [a,b]$.
Then $y(t) = g(x(t))$ $\forall t \in [r,s]$.
Subsequently, using the Substitution Rule, we obtain the following formulas:

1. The area of the region R given by $a \leq x \leq b$, $0 \leq y \leq g(x)$ is
$$A = \int_{x=a}^{x=b} g(x)\,dx = \int_{t=r}^{t=s} g(x(t))\,x'(t)\,dt = \int_r^s y(t)x'(t)\,dt\,;$$

2. The volume of the solid obtained by revolving R about the x-axis is
$$V = \int_{x=a}^{x=b} \pi g(x)^2\,dx = \int_{t=r}^{t=s} \pi y(t)^2 x'(t)\,dt\,;$$

3. If $a \geqslant 0$, the volume of the solid obtained by revolving R about the y-axis is
$$V = \int_{x=a}^{x=b} 2\pi x\,g(x)\,dx = \int_{t=r}^{t=s} 2\pi x(t)y(t)\,x'(t)\,dt\,;$$

4. The length of the curve C with $a \leq x \leq b$ is given by
$$L = \int_{x=a}^{x=b} \sqrt{1+g'(x)^2}\,dx = \int_{t=r}^{t=s} \sqrt{1+\frac{y'(t)^2}{x'(t)^2}}\,x'(t)\,dt$$
$$\therefore\ L = \int_r^s \sqrt{x'(t)^2 + y'(t)^2}\,dt\,;$$

5. The surface area of the surface obtained by revolving C about the x-axis is
$$A = \int_{x=a}^{x=b} 2\pi g(x)\sqrt{1+g'(x)^2}\,dx = \int_{t=r}^{t=s} 2\pi y(t)\sqrt{x'(t)^2+y'(t)^2}\,dt\,;$$ and

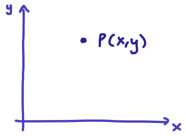
6. When $a \geqslant 0$, the surface area of the surface obtained by revolving C about the y-axis is
$$A = \int_{x=a}^{x=b} 2\pi x\sqrt{1+g'(x)^2}\,dx = \int_{t=r}^{t=s} 2\pi x(t)\sqrt{x'(t)^2+y'(t)^2}\,dt.$$

POLAR COORDINATES

CARTESIAN GRID / COORDINATES (D4.26)

1. Let P be a point in the $\mathbb{R}^2$ plane.
Then, we say the "Cartesian coordinates" of P is the ordered pair (x, y), where
   ① "x" denotes the horizontal position of P; and
   ② "y" denotes the vertical position of P.

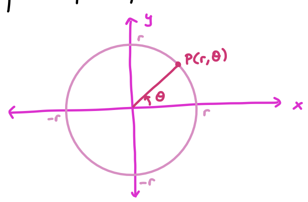
2. To plot "Cartesian points", we use a Cartesian grid.



POLAR GRID / COORDINATES (D4.26)

1. Let P be a point in the $\mathbb{R}^2$ plane.
Then, we say the "polar coordinates" of P is the ordered pair (r, θ); where
   ① "r" (≥ 0) represents the distance from the origin; and
   ② θ ($\in \mathbb{R}$) represents the angle between the positive x-axis to P in the counter-clockwise direction.

2. Similarly, to plot "polar points", we use a polar grid.



3. Note that for any $r>0$,
$(-r, \theta) = (r, \pi+\theta)$. (N4.29)

CONVERSION BETWEEN CARTESIAN AND POLAR COORDINATES (N4.27)

1. Let the point P be represented in both Cartesian coordinates (x,y) and polar coordinates (r,θ).
Then the following must hold:
   ① $x = r\cos\theta$;
   ② $y = r\sin\theta$;
   ③ $x^2+y^2 = r^2$; and
   ④ $\tan\theta = \frac{y}{x}$.

2. For every $(x,y) \neq (0,0)$, there exists a unique $r = \sqrt{x^2+y^2}$ and a unique θ (up to a multiple of 2π) such that $(x,y) \cong (r,\theta)$.
In particular,

$$\theta = \begin{cases} \tan^{-1}\left(\frac{y}{x}\right) + 2\pi k \text{ for some } k \in \mathbb{Z}, & x>0 \\ \cos^{-1}\left(\frac{x}{\sqrt{x^2+y^2}}\right) + 2\pi k \text{ for some } k \in \mathbb{Z}, & y>0 \\ \pi + \tan^{-1}\left(\frac{y}{x}\right) + 2\pi k \text{ for some } k \in \mathbb{Z}, & x<0 \\ -\cos^{-1}\left(\frac{x}{\sqrt{x^2+y^2}}\right) + 2\pi k \text{ for some } k \in \mathbb{Z}, & y<0. \end{cases} \quad \text{(N4.28)}$$

SKETCHING POLAR CURVES (N4.32)

1. We can sketch curves in the $\mathbb{R}^2$-plane described by polar coordinates
   ① explicitly (ie in the form $r = f(\theta)$);
   ② implicitly (ie in the form $f(r,\theta) = 0$); and
   ③ parametrically (ie in the form $(r,\theta) = f(t) = (r(t), \theta(t))$).

LIMAÇON (D4.34)

1. A "limaçon" is a polar curve of the form
$r = a + b\cos\theta$
or
$r = a + b\sin\theta$
for some $a, b \in \mathbb{R}$.

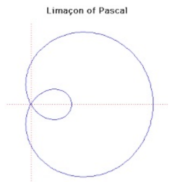
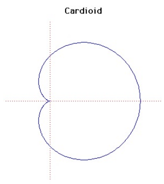


Image source: Limacon of Pascal - MacTutor History of Mathematics

CARDIOID (D4.34)

1. A "cardioid" is a limaçon of the form
$r = a + a\cos\theta$
or
$r = a + a\sin\theta$
for some $a \in \mathbb{R}^+$.



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ROSE (D4.34)

1. A "rose" is a polar curve of the form
$r = a\cos(n\theta)$
or
$r = a\sin(n\theta)$
for some $a \in \mathbb{R}^+$ and $n \in \mathbb{Z}^+$.
   - when n is odd, the number of petals is equal to n;
   - when n is even, the number of petals is equal to $2n$.

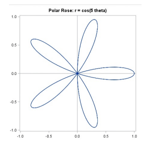


Image source: Lo, how a polar rose e'er blooming - The DO Loop (Blogs - SAS)

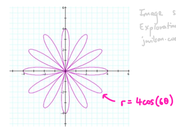


Image source: Explorations of Polar Equations - jwilson.coe.uga.edu

$y=f(x) \iff r\sin\theta = f(r\cos\theta)$ (N4.35)

1. If a curve is described explicitly in Cartesian coordinates by $y = f(x)$, then the same curve can be described implicitly in polar coordinates by
$r\sin\theta = f(r\cos\theta)$.

$f(x,y) = 0 \iff f(r\cos\theta, r\sin\theta) = 0$ (N4.35)

2. Similarly, if a curve is described implicitly in Cartesian coordinates by $f(x,y) = 0$, then the same curve can be described implicitly in polar coordinates by
$f(r\cos\theta, r\sin\theta) = 0$.

$(x,y) = (x(t), y(t)) \iff$ POLAR COORDINATES (N4.35)

3. Lastly, if a curve is described parametrically in Cartesian coordinates by $(x,y) = (x(t), y(t))$, we can (sometimes but not always) use algebraic manipulation to express the curve in polar form.
* using the formulas in N4.27

$(r,\theta) = (r(t), \theta(t)) \Rightarrow (x,y) = (r(t)\cos\theta(t),\ r(t)\sin\theta(t))$ (N4.36)

Note that if a curve is given parametrically in polar coordinates by $(r,\theta) = (r(t), \theta(t))$, then it is given parametrically in Cartesian coordinates by

$$(x,y) = (r(t)\cos\theta(t),\ r(t)\sin\theta(t)).$$

$r = r(\theta) \Rightarrow (r,\theta) = (r(t), t)$ & $(x,y) = (r(t)\cos(t),\ r(t)\sin(t))$ (N4.36)

Similarly, if a curve is described explicitly in polar coordinates by $r = r(\theta)$, then

① it is given parametrically in polar coordinates by $(r,\theta) = (r(t), t)$; and

② it is given parametrically in Cartesian coordinates by

$$(x,y) = (r(t)\cos(t),\ r(t)\sin(t)).$$

TRANSFORM POLAR INTO PARAMETRIC CARTESIAN COORDINATES TO PERFORM CALCULATIONS (N4.39)

EXAMPLE 1: SLOPE OF $r = r(\theta)$ AT $\theta = t$ (E4.40)

To find a formula for the polar curve $r = r(\theta)$ at the point where $\theta = t$, we can convert the curve into its "parametric Cartesian form" (used in N4.36).

First, write $r = r(\theta)$ as

$$(x,y) = (r(t)\cos(t),\ r(t)\sin(t)).$$

Using the product rule, the slope at the point where $\theta = t$ is equal to

$$\frac{dy}{dx} = \frac{\left(\frac{dy}{dt}\right)}{\left(\frac{dx}{dt}\right)} = \frac{r'(t)\sin(t) + r(t)\cos(t)}{r'(t)\cos(t) - r(t)\sin(t)}.$$

EXAMPLE 2: FIND CARTESIAN COORDINATES OF ALL HORIZONTAL & VERTICAL POINTS ON $r = 1+\cos\theta$ (E4.41)

We can employ a similar method to find all the horizontal and vertical points on the cardioid $r = 1+\cos\theta$.

First, express the curve $r = 1+\cos\theta$ parametrically in Cartesian coordinates:

$$(x,y) = ((1+\cos(t))\cos(t),\ ((1+\cos(t))\sin(t)).$$

Then

$$x'(t) = \frac{d}{dt}(\cos(t) + \cos^2(t)) = -\sin(t) - 2\sin(t)\cos(t)$$
$$= -\sin(t)(1+2\cos(t)).$$

Hence $x'(t) = 0$ when $\sin(t) = 0$ or $\cos(t) = -\frac{1}{2}$.
This occurs when $t = 0, \pi$ and $t = \pm\frac{2\pi}{3}$ respectively, plus integer multiples of 2π.

Similarly,

$$y'(t) = \frac{d}{dt}(\sin(t) + \sin(t)\cos(t)) = (2\cos(t)-1)(\cos(t)+1),$$

and so $y'(t) = 0$ when $\cos(t) = \frac{1}{2}$ or $\cos(t) = -1$.
This occurs when $t = \pm\frac{\pi}{3}$ and $t = \pm\pi$ respectively, plus multiples of 2π.

Lastly, plug in the values of t into the parametric Cartesian equation to get that

$$t = 0 \Rightarrow (x,y) = (2,0)$$
$$t = \pm\frac{2\pi}{3} \Rightarrow (x,y) = \left(-\frac{1}{4}, \pm\frac{\sqrt{3}}{4}\right),$$

and since $x'(t) = 0$ & $y'(t) \neq 0$ at these points, the curve is vertical at these points.

At

$$t = \pm\frac{\pi}{3} \Rightarrow (x,y) = \left(\frac{3}{4}, \frac{\pm 3\sqrt{3}}{4}\right),$$

and since $y'(t) = 0$ & $x'(t) \neq 0$ at these points, the curve is horizontal at these points.

When $t = \pi$, this is at $(x,y) = (0,0)$. We cannot determine whether it is vertical or horizontal yet, since $x'(t) = 0 = y'(t)$.
Apply L'Hôpital's rule to get

$$\lim_{t\to\pi}\frac{y'(t)}{x'(t)} = \lim_{t\to\pi}\frac{(2\cos(t)-1)(\cos(t)+1)}{-\sin(t)(1+2\cos(t))}$$
$$= \lim_{t\to\pi}\frac{2\cos(t)-1}{1+2\cos(t)} \cdot \lim_{t\to\pi}\frac{\cos(t)+1}{-\sin(t)}$$
$$= \frac{-2-1}{1-2} \cdot \lim_{t\to\pi}\frac{\cos(t)+1}{-\sin(t)}$$
$$= 3 \cdot 0$$
$$= 0,$$

So that $(x,y) = (0,0)$ is a horizontal point.

EXAMPLE 3: FORMULA FOR LENGTH OF $r = r(\theta)$ WITH $\alpha \le \theta \le \beta$ (E4.43)

We can use a similar strategy to find a formula for the length of the polar curve $r = r(\theta)$ with $\alpha \le \theta \le \beta$.

Write the curve in its Cartesian parametric form:

$$(x,y) = (r(t)\cos(t),\ r(t)\sin(t)),$$

so that

$$x'(t) = r'(t)\cos(t) - r(t)\sin(t)$$
$$y'(t) = r'(t)\sin(t) + r(t)\sin(t).$$

It follows that

$$x'(t)^2 + y'(t)^2 = r'(t)^2 + r(t)^2;$$

thus

$$L = \int_{t=\alpha}^{t=\beta}\sqrt{x'(t)^2 + y'(t)^2}\,dt = \int_{t=\alpha}^{t=\beta}\sqrt{r'(t)^2 + r(t)^2}\,dt.$$

AREA UNDER A POLAR CURVE (N4.45)

Let the region R be the region given in polar coordinates by

$$\alpha \le \theta \le \beta, \quad f(\theta) \le r \le g(\theta).$$

We can approximate the area of R as follows:

① Choose a partition of $[\alpha, \beta]$

$$\alpha = \theta_0 < \theta_1 < \cdots < \theta_n = \beta,$$

with corresponding sample points $c_k \in [\theta_{k-1}, \theta_k]$.

② "Slice" R into thin wedges, with the k^{th} wedge given by

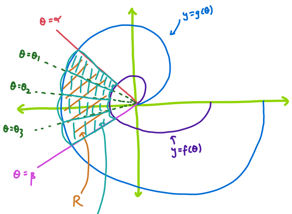
$$\theta_{k-1} \le \theta \le \theta_k, \quad f(\theta) \le r \le g(\theta).$$

③ Then, the area of the k^{th} wedge is approximately

$$\Delta_k A \cong \frac{1}{2}\left(g(c_k)^2 - f(c_k)^2\right)\Delta_k \theta.$$

④ The area of R is approximately the area of these sums, so that

$$A \cong \sum_{k=1}^{n} \Delta_k A \cong \sum_{k=1}^{n} \frac{1}{2}\left(g(c_k)^2 - f(c_k)^2\right)\Delta_k \theta.$$



each "slice" is the sector of a circle with angle $\Delta_k \theta$

$$\Rightarrow \Delta_k A = \frac{1}{2} g(\theta)^2 \Delta_k \theta - \frac{1}{2} f(\theta)^2 \Delta_k \theta = \frac{1}{2}\Delta_k \theta \left(g(\theta)^2 - f(\theta)^2\right).$$

The RHS is a Riemann sum, so that the exact area of R is the limit of these Riemann sums:

$$A = \int_{\theta=\alpha}^{\theta=\beta} \frac{1}{2}\left(g(\theta)^2 - f(\theta)^2\right) d\theta.$$

EXAMPLE 1: AREA INSIDE $r = 1 + \cos\theta$ (E4.46)

Using the above formula, we can calculate the area of the region inside the cardioid $r = 1 + \cos\theta$.

$$A = \int_{\theta=0}^{\theta=2\pi} \frac{1}{2}(1+\cos\theta)^2 \, d\theta = \int_0^{2\pi} \frac{1}{2} + \cos\theta + \frac{1}{2}\cos^2\theta \, d\theta$$

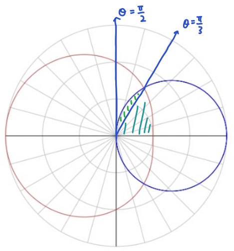
$$= \int_0^{2\pi} \frac{3}{4} + \cos\theta + \frac{1}{4}\cos 2\theta \, d\theta$$

$$= \left[\frac{3}{4}\theta + \sin\theta + \frac{1}{8}\sin 2\theta\right]_0^{2\pi}$$

$$\therefore A = \frac{3\pi}{2}.$$

EXAMPLE 2: AREA OF INTERSECTION OF $r = 3\cos\theta$ AND $r = 2 - \cos\theta$ (E4.48)

We can apply the formula to more complicated contexts as well, such as finding the area of the intersection of the circle $r = 3\cos\theta$ and the limaçon $r = 2 - \cos\theta$.



$$▨ = \int_0^{\pi/3} \frac{1}{2}(2-\cos\theta)^2 \, d\theta$$

$$▨ = \int_{\pi/3}^{\pi/2} \frac{1}{2}(3\cos\theta)^2 \, d\theta$$

So, the total area is

$$A = 2(▨ + ▨)$$

$$= 2\left(\int_0^{\pi/3} \frac{1}{2}(2-\cos\theta)^2 \, d\theta + \int_{\pi/3}^{\pi/2} \frac{1}{2}(3\cos\theta)^2 \, d\theta\right)$$

$$= \left[\frac{9}{2}\theta - 4\sin\theta + \frac{1}{4}\sin 2\theta\right]_0^{\pi/3} + \left[\frac{9}{2}\theta + \frac{9}{4}\sin 2\theta\right]_{\pi/3}^{\pi/2}$$

$$\therefore A = \frac{9\pi}{4} - 3\sqrt{3}.$$

Chapter 5: Differential Equations

An "(ordinary) differential equation", or "DE", is an equation which involves a function, say $y=y(x)$, of a single variable x, along with some of its derivatives (eg $y'(x)$, $y''(x)$ etc). (D5.1)

ORDER (D5.1)

The "order" of a DE is the highest of the orders of the derivatives which occur in the equation.

eg the equation "$y''(x) + 2y'(x)y(x)^3 = \sin(x)$" is a second order DE.

SOLUTION/GENERAL SOLUTION (D5.1)

A "solution" to a DE is a function $y=y(x)$ which makes the equation hold for all x in some interval.

* a DE can have many solutions.

The "general solution" to a DE is an expression which "contains" all the solutions for the said DE.

- the general solution will usually involve arbitrary constants;
- the number of arbitrary constants is equal to the order of the DE.

INITIAL CONDITIONS (D5.1)

"Initial conditions" are one or more additional conditions that we might require a solution to a DE to satisfy.

INITIAL VALUE PROBLEM / IVP (D5.1)

An "initial value problem", or "IVP", is a DE paired together with an initial condition / a set of initial conditions.

- often, the # of initial conditions = order of the DE
- and there is exactly one solution

EXAMPLE 1: SOLUTION TO $y''y'+x^2=y$ OF THE FORM $y=ax^2+bx+c$ (E5.2)

We can find a solution of the form $y=ax^2+bx+c$ to the DE $y''y'+x^2=y$.

Let $y=ax^2+bx+c \Rightarrow y'=2ax+b \Rightarrow y''=2a$.

So $y''y'+x^2=y = (2a)(2ax+b)+x^2 = ax^2+bx+c$.

$\Rightarrow 4a^2x+2ab+x^2 = ax^2+bx+c$

Equating coefficients, we get $a=1$, $4a^2=b$, $2ab=c$,

so that $a=1$, $b=4$ & $c=8$.

Hence the only solution is $y=x^2+4x+8$. ∎

EXAMPLE 2: DE W/ EXPONENTIAL FUNCTIONS (E5.3)

5.3 Example: Find two distinct constants r_1 and r_2 such that $y=e^{r_1x}$ and e^{r_2x} are both solutions to the DE $y''+3y'+2y=0$, show that $y=a\,e^{r_1x}+b\,e^{r_2x}$ is a solution for any constants a and b, and then find a solution to the DE with $y(0)=1$ and $y'(0)=0$.

Let $y=e^{rx}$, so that $y'=re^{rx}$ & $y''=r^2e^{rx}$, so that

$$y''+3y'+2y=0 \iff r^2e^{rx}+3re^{rx}+2e^{rx}=0$$
$$\iff (r^2+3r+2)e^{rx}=0$$
$$\iff (r+1)(r+2)e^{rx}=0$$
$$\iff r=-1 \text{ or } r=-2 \quad (\text{since } e^{rx}>0 \ \forall x\in\mathbb{R}).$$

Hence, we can take $r_1=-1$ & $r_2=-2$.

Now, let $y=ae^{r_1x}+be^{r_2x} = ae^{-x}+be^{-2x}$.

Then
$$y' = -ae^{-x}-2be^{-2x},$$
and
$$y'' = ae^{-x}+4be^{-2x}.$$

Hence
$$y''+3y'+2y = (ae^{-x}+4be^{-2x})+3(-ae^{-x}-2be^{-2x})+2(ae^{-x}+be^{-2x}) = 0,$$

showing $y=ae^{-x}+be^{-2x}$ is indeed a solution to the DE.

(So this is the general solution).

Then, since $y(0)=a+b$ and $y'(0)=-a-2b$, it follows that

$$\begin{cases} y(0)=1 \\ y'(0)=0 \end{cases} \Rightarrow \begin{cases} a+b=1 \\ -a-2b=0 \end{cases} \Rightarrow a=2,\ b=-1.$$

So, the required particular solution to the IVP is $y=2e^{-x}-e^{-2x}$.

EXAMPLE 3: "APPLIED" DE (E5.4)

5.4 Example: A rock is thrown downwards at 5 m/s from the top of a 100 m cliff and it falls to the ground. Assuming that the rock accelerates downwards at 10 m/s^2, find the speed of the rock when it lands.

Let $x(t)$ be the height of the rock (in meters) after t seconds.

We need to solve the IVP consisting of

- the 2nd order DE $x''(t)=-10$; and
- the two initial conditions $x'(t)=-5$ and $x(0)=100$.

Then, observe that
$$x''(t)=-10$$
$$\Rightarrow \int x''(t)\,dt = \int -10\,dt$$
$$\Rightarrow x'(t) = -10t+c_1,$$

where $c_1\in\mathbb{R}$ is some constant. Then, since $x'(0)=-5$, it follows that $c_1=-5$, so we have
$$x'(t) = -10t-5.$$

Hence
$$\int x'(t)\,dt = \int -10t-5\,dt$$
$$\Rightarrow x(t) = -5t^2-5t+c_2,$$

where c_2 is another constant. Then, since $x(0)=100$, we have $c_2=100$; hence, the solution to the IVP is $x(t)=-5t^2-5t+100$.

Then, to find out when the rock lands, we solve $x(t)=0$:
$$0 = -5t^2-5t+100$$
$$\Rightarrow 0 = -5(t+5)(t-4),$$

so (since $t\geq 0$) the rock lands when $t=4$.

Then, since $x'(4)=-45$, the rock lands at a speed of $45\,\text{ms}^{-1}$.

DIRECTION FIELDS

SOLUTION CURVE (DS.5)

1. A "solution curve" to a DE is the graph of a solution $y = y(x)$ of the said DE.

SLOPE/DIRECTION FIELD (NS.6)

1. The "slope field", or the "direction field", of a DE of the form
$$y'(x) = F(x, y(x))$$
is a sketch of the solution curves to the said DE.

2. We can sketch a solution curve to a DE of the above form as follows:
   1. Choose many points (x,y), and for each point compute $F(x,y)$.
   2. Suppose $y = y(x)$ is a solution to the DE, so that $y'(x) = F(x,y)$, which is the slope of the solution curve at the point (x,y).
   3. Then, at each point (x,y), draw a short line segment at the point (x,y) with slope $F(x,y)$.
   4. If we choose enough points (x,y), it should be possible to visualise the solution curves, since they follow the direction of the short line segments.

3. Then, to sketch the direction field of the DE $y'(x) = F(x,y)$:
   1. we first draw several isoclines, which are the curves $F(x,y) = m$, where $m \in \mathbb{R}$; then,
   2. we draw many short line segments of slope m along each isocline $F(x,y) = m$.

4. Finally, to sketch the graph of the solution to the IVP $y'(x) = F(x,y)$ with $y(x_0) = y_0$:
   1. we sketch the direction field for the DE $y'(x) = F(x,y)$; then,
   2. we draw the solution curve which passes through the point (x_0, y_0).

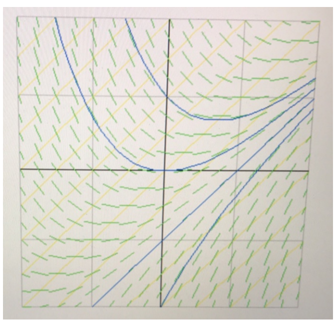
EXAMPLE: SKETCHING THE DF FOR $y' = x - y$ (ES.7)

Suppose we wanted to sketch the direction field for the DE $y' = x - y$, and the solution curves through each of the points $(x_0, y_0) = (0,-2), (0,-1), (0,0)$ and $(0,1)$.

The isoclines are the lines $x - y = m$, so to sketch the DF, we must lightly draw the lines $x - y = m$ for several values of m. (Shown in yellow).

Then, along each isocline, we draw many short line segments of slope m, where m is the value of the isocline $(x - y = m)$ passing through the point. (shown in green).

Lastly, we can sketch the solution curves (shown in blue).



EULER'S METHOD (NS.8)

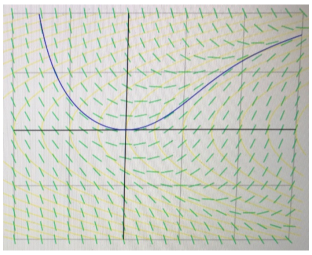
1. "Euler's Method" is a way to approximate the solution to the IVP $y'(x) = F(x, y(x))$ with $y(a) = b$.

2. Methodology:
   1. Pick a step size Δx (which is small).
   2. Let $x_0 = a$ and $y_0 = b$, and for each $n \geq 0$, let
   $$x_{n+1} = x_n + \Delta x; \text{ and}$$
   $$y_{n+1} = y_n + F(x_n, y_n)\Delta x.$$
   3. The solution curve $y = f(x)$ is then approximated for values $x \geq a$ by the piecewise linear curve whose graph has vertices at the points (x_n, y_n).
   4. Note that $\frac{y_{n+1} - y_n}{x_{n+1} - x_n}$ (ie the slope of the line segment joining (x_n, y_n) and (x_{n+1}, y_{n+1})) is equal to the slope of the direction field at the point (x_n, y_n).
   5. Lastly, if we wish to approximate solutions with values $x \leq a$, we can construct points with $n < 0$ by letting
   $$x_{n-1} = x_n - \Delta x$$
   $$y_{n-1} = y_n - F(x_n, y_n)\Delta x.$$

EXAMPLE: $y' = x - y^2$ (ES.9)

5.9 Example: Consider the IVP $y' = x - y^2$ with $y(0) = 0$. Sketch the direction field for the given DE along with the graph of the solution curve $y = f(x)$. With the help of a calculator, apply Euler's method with step size $\Delta x = \frac{1}{2}$ to approximate the value of $f(3)$.

The isocline $y' = m$ is the sideways parabola $m = x - y^2$, or $x = y^2 + m$. We draw the isoclines (yellow), the DF (green) and the solution curve (blue) below:



Next, let $x_0 = 0$ and $y_0 = 0$. For $k \geq 0$, set $x_{k+1} = x_k + \Delta x$ & $y_{k+1} = y_k + F(x_k, y_k)\Delta x$, where $F(x,y) = x - y^2$.

Then, observe

k	x_k	y_k	$F(x_k, y_k) = x_k - y_k^2$
0	0	0	0
1	0.5	0	0.5
2	1.0	0.25	0.9375
3	1.5	0.71875	0.98339
4	2.0	1.2104492	0.534812
5	2.5	1.477855	0.315942,
6	3.0	1.635827	

so that $f(3) \approx y_6 \approx 1.6$.

SEPARABLE FIRST ORDER EQUATIONS (DS.10)

A "separable first order DE" is any DE that can be written in the form

$$f(y(x))\,y'(x) = g(x),$$

where $f(x)$, $g(x)$ are continuous functions.

SOLVING SEPARABLE 1ST ORDER DES (NS.11)

1. Note $y = y(x)$ is a solution to the separable DE $f(y)\,y' = g(x)$ when

$$\int f(y(x))\,y'(x)\,dx = \int g(x)\,dx,$$

or

$$\int f(y(x))\,y'(x)\,dx = \int f(y)\,dy.$$

2. So to solve the DE, we rewrite it as $f(y)\,dy = g(x)dx$ and then integrate both sides.

LINEAR FIRST ORDER EQUATIONS (DS.14)

A "linear first order DE" is a DE which can be written in the form

$$y'(x) + p(x)\,y(x) = q(x)$$

for some continuous functions $p(x)$ and $q(x)$.

SOLVING LINEAR 1ST ORDER EQUATIONS (NS.15)

1. We can solve the linear DE $y' + py = q$ as follows:

① Find an "integrating factor" $\lambda = \lambda(x)$ such that $\lambda' = \lambda p$; this would imply

$$(\lambda y)' = \lambda' y + \lambda y' = \lambda y' + \lambda p y.$$

② Then $y' + py = q$ reduces to

$$\lambda y' + \lambda p y = \lambda q$$

$$\Rightarrow (\lambda y)' = \lambda q$$

$$\lambda y = \int \lambda q\,dx$$

$$\therefore\ y = \frac{1}{\lambda}\int \lambda q\,dx.$$

③ To find λ, we need to solve the DE

$$\lambda' = \lambda p,$$

which results in the solution

$$\lambda = e^{\int p(x)\,dx}.$$

2. In general, the solution to the DE

$$y'(x) + p(x)\,y(x) + q(x) = 0$$

is

$$y(x) = \frac{1}{e^{\int p(x)\,dx}}\int e^{\int p(x)\,dx}\,q(x)\,dx,$$

where $\lambda(x) = e^{\int p(x)\,dx}$ is the integrating factor. (TS.16)

APPLICATIONS

*mainly just "plug and chug" formulas.

ORTHOGONAL TRAJECTORY (DS.21)

For a given family of curves, an "orthogonal trajectory" is a curve that intersects each curve orthogonally; ie at a right angle to the curve.

EXAMPLE: ORTHOGONAL TRAJECTORY OF $x = ky^2$ (ES.22)

We can find the orthogonal trajectories of the family of parabolas $x = ky^2$ via the following:

Differentiating $x = ky^2$ wrt x, we get $1 = 2kyy'$, so the parabola $x = ky^2$ has slope $y' = \frac{1}{2ky}$ at each point.

Since $k = \frac{x}{y^2}$, it follows $y' = \frac{1}{2(\frac{x}{y^2})y} = \frac{y}{2x}$.

Then, as the orthogonal trajectories are perpendicular to the parabolas, necessarily their slope is $y' = \frac{-2x}{y}$.

So to find the orthogonal trajectories, we solve the DE $y' = -\frac{2x}{y}$.

This is a separable DE; so

$$y\,dy = -2x\,dx$$

$$\Rightarrow \int y\,dy = \int -2x\,dx$$

$$\frac{y^2}{2} = -x^2 + c$$

$$\Rightarrow x^2 + \frac{y^2}{2} = c,$$

implying the orthogonal trajectories are the ellipses $x^2 + \frac{y^2}{2} = c$, where $c \in \mathbb{R}^+$.

EXPONENTIAL GROWTH/DECAY (DS.23)

1. A quantity $y = y(t)$ is said to "grow/decay exponentially" if it satisfies the DE $y'(t) = ky(t)$ for some $k \in \mathbb{R}$.

This gives the solution

$$y = ce^{kt},$$

and note that $c = y(0)$, so that

$$y(t) = y(0)\,e^{kt}.$$

2. Then,

① when $y(0) > 0$ & $k > 0$, we say y grows exponentially; and

② when $y(0) > 0$ & $k < 0$, we say y decays exponentially.

NEWTON'S LAW OF COOLING/WARMING (NS.26)

"Newton's Law of Cooling/Warming" states the rate of cooling/warming of an object is proportional to the temperature difference between the object and its surroundings; ie

$$T'(t) = k(K - T(t))$$

where $T(t)$ is the temperature of the object at time t, and K is the constant temperature of the surroundings.

MIXING PROBLEM (NS.29)

Imagine we have some solution with a given concentration c_1 of some solute, which enters a tank at a fixed rate r_1.

The mixture is stirred (to ensure equidistribution of the substance), and drained at another rate r_2.

Let $V(t)$ denote the volume of the tank at time t. Then

$$V(t) = V(0) + (r_1 - r_2)(t); \text{ and}$$

$$V'(t) = r_1 - r_2.$$

Proof. This follows from solving the DE

$$y'(t) = r_1c_1 - r_2c_2;$$

where $c_2 = \frac{y}{V}$.

TORICELLI'S LAW (NS.31)

"Toricelli's Law" states when a liquid drains through a hole in a tank of liquid, it flows through the hole at a speed of which is proportional to the square root of the depth of the water above the hole.

For a non-viscous liquid, the speed is

$$v \cong \sqrt{2gy},$$

where g is the gravitational constant, and y is the depth of the liquid.

Chapter 6: Sequences and Series

* this relies heavily on knowledge from MATH 147.

LIMIT SUPREMUM & INFEMUM (D6.13)

Let $(a_n)_{n\geq k} \subset \mathbb{R}$ be a sequence.

Then, the "limit supremum" of $(a_n)_{n\geq k}$ is defined to be the extended real number

$$\limsup_{n\to\infty}(a_n) = \lim_{n\to\infty} \sup(\{a_k : k \geq n\}),$$

and the "limit infimum" of $(a_n)_{n\geq k}$ is defined to be the extended real number

$$\liminf_{n\to\infty}(a_n) = \lim_{n\to\infty} \inf(\{a_k : k\geq n\}).$$

* the extended real numbers $= \mathbb{R} \cup \{-\infty, \infty\}$.

Then, note that

① (a_n) is bounded above if and only if $\limsup\limits_{n\to\infty} a_n < \infty$;

② (a_n) is bounded below if and only if $\liminf\limits_{n\to\infty} a_n > -\infty$; and

③ $\lim\limits_{n\to\infty} a_n = b$ if and only if $\limsup\limits_{n\to\infty} a_n = b = \liminf\limits_{n\to\infty} a_n$. (T6.14)

SERIES (D6.15)

For a sequence $(a_n)_{n\geq k}$, we define the "series" $\sum\limits_{n\geq k} a_n$ to be equal to the sequence $(S_\ell)_{\ell\geq k}$, where

$$S_\ell = \sum_{n=k}^{\ell} a_n = a_k + \cdots + a_\ell,$$

called the ℓ^{th} "partial sum" of the series $\sum\limits_{n\geq k} a_n$.

Then, we define the "sum" of the series to be the sum

$$S = \sum_{n=k}^{\infty} a_n = a_k + a_{k+1} + \cdots = \lim_{\ell\to\infty} S_\ell,$$

and if S exists and finite we say the series converges.

FIRST FINITELY MANY TERMS DO NOT AFFECT CONVERGENCE (T6.19)

Let $(a_n)_{n\geq k} \subset \mathbb{R}$ be a sequence.

Then for any $m \geq k$, the series $\sum\limits_{n\geq k} a_n$ converges if and only if $\sum\limits_{n\geq m} a_n$ converges, and in this case

$$\sum_{n=k}^{\infty} a_n = (a_k + \cdots + a_{m-1}) + \sum_{n=m}^{\infty} a_n.$$

Proof. Simple. ∎

Note that since the first finitely many terms of a series does not affect its convergence, we may opt to just write the sum $S = \sum\limits_{n\geq k} a_n$ as simply $\sum a_n$ if we just want to determine whether S converges. (N6.20)

ERROR (N6.22)

Let $\sum\limits_{n\geq k} a_n$ be convergent, and so by N6.20 for any $\ell \geq k$, $\sum\limits_{n\geq \ell+1} a_n$ is also convergent.

Then, if we approximate the sum $S = \sum\limits_{n=k}^{\infty} a_n$ by the ℓ^{th} partial sum $S_\ell = \sum\limits_{n=k}^{\ell} a_n$, the "error" in our approximation is

$$|S - S_\ell| = \left|\sum_{n=\ell+1}^{\infty} a_n\right|.$$

CONVERGENCE TESTS

INTEGRAL TEST (T6.29)

Let $f(x)$ be positive and decreasing $\forall x \geq k$, and let $a_n = f(n)$ $\forall n\geq k$, $n \in \mathbb{Z}$.

Then $\sum a_n$ converges if and only if $\int_k^\infty f(x)\,dx$ converges, and this case, for any $\ell \geq k$ we have

$$\int_{\ell+1}^{\infty} f(x)\,dx \leq \sum_{n=\ell+1}^{\infty} a_n \leq \int_{\ell}^{\infty} f(x)\,dx.$$

Proof. Fix $\ell \geq k$, and let $T_m = \sum\limits_{n=\ell+1}^{m} a_n$. Note that since $f(x)$ is decreasing, it is integrable on any closed interval. Also for each $n \geq \ell$ necessarily $f(n) \leq f(x)$ $\forall x \in [n-1, n]$, so that

$$\int_{n-1}^{n} f(x)\,dx \geq \int_{n-1}^{n} a_n\,dx = a_n.$$

It follows that

$$T_m = \sum_{n=\ell+1}^{m} a_n \leq \sum_{n=\ell+1}^{m} \int_{n-1}^{n} f(x)\,dx = \int_{\ell}^{m} f(x)\,dx \leq \int_{\ell}^{\infty} f(x)\,dx.$$

Since $f(n) = a_n > 0$, the sequence (T_m) is increasing. If $\int_k^\infty f(x)\,dx$ converges, then (T_n) is bounded above by $\int_\ell^\infty f(x)\,dx$, so (by MCT) it converges with $\lim\limits_{m\to\infty} T_m \leq \int_\ell^\infty f(x)\,dx$.

A similar argument can be used to prove $T_m \geq \int_{\ell+1}^{\infty} a_n\,dx$. ∎

P-SERIES (E6.30)

We can show $\sum\limits_{n\geq 1} \frac{1}{n^p}$ converges if and only if $p > 1$.

Proof. If $p < 0$, then $\lim\limits_{n\to\infty} \frac{1}{n^p} = \infty$, and if $p = 0$, then $\lim\limits_{n\to\infty} \frac{1}{n^p} = 1$, so, in either case, by the Divergence Test $\sum \frac{1}{n^p}$ diverges.

Then, suppose $p > 0$. Let $a_n = \frac{1}{n^p}$ $\forall n \geq 1$, $n \in \mathbb{Z}$, and let $f(x) = \frac{1}{x^p}$ $\forall x \geq 1$.

Note that $f(x)$ is positive and decreasing $\forall x \geq 1$, and $a_n = f(n)$ $\forall n \geq 1$.

Since we know $\int_1^\infty f(x)\,dx$ converges if and only if $p > 1$, it follows by the integral test that $\sum a_n$ converges if and only if $p > 1$, as needed. ∎

APPROXIMATE $\sum\limits_{n\geq 1} \frac{1}{2n^2}$ (E6.31)

We can approximate the sum $S = \sum\limits_{n\geq 1} \frac{1}{2n^2}$ so that the error is at most $\frac{1}{100}$.

Let $a_n = \frac{1}{2n^2}$ and $f(x) = \frac{1}{2x^2}$, so we can apply the Integral Test. If we choose to approximate S by the ℓ^{th} partial sum S_ℓ, the error is

$$E = S - S_\ell = \sum_{n=\ell+1}^{\infty} a_n \leq \int_\ell^\infty \frac{1}{2x^2}\,dx = \left[-\frac{1}{2x}\right]_\ell^\infty = \frac{1}{2\ell}.$$

So to ensure $E \leq \frac{1}{100}$ we can choose $\ell \geq 50$.

Since it would be tedious to add up the first 50 terms of the series, we instead take the upper & lower bounds of $S - S_\ell$ using the Integral Test:

$$\int_{\ell+1}^{\infty} f(x)\,dx \leq S - S_\ell \leq \int_\ell^\infty f(x)\,dx$$

$$\therefore \quad \frac{1}{2(\ell+1)} \leq S - S_\ell \leq \frac{1}{2\ell}$$

$$\Rightarrow \quad S_\ell + \frac{1}{2(\ell+1)} \leq S \leq S_\ell + \frac{1}{2\ell}.$$

If we approximate S using the midpoint of the upper and lower bounds, ie $S \approx \frac{1}{2}\left(S_\ell + \frac{1}{2(\ell+1)} + S_\ell + \frac{1}{2\ell}\right) = S_\ell + \frac{1}{2}\left(\frac{1}{2(\ell+1)} + \frac{1}{2\ell}\right)$, we get $E \leq \frac{1}{2}\left(\frac{1}{2\ell} - \frac{1}{2(\ell+1)}\right) = \frac{1}{4\ell(\ell+1)}$.

So, to get $E \leq \frac{1}{100}$, we want $\frac{1}{4\ell(\ell+1)} \leq \frac{1}{100}$, so we can take $\ell = 5$.

Then we estimate

$$S \approx S_5 + \frac{1}{2}\left(\frac{1}{10} + \frac{1}{12}\right) = \frac{5929}{7200}.$$

LIMIT COMPARISON TEST (T6.36)

Let $a_n \geq 0$ and $b_n > 0$ $\forall n \geq k$, and suppose that $\lim_{n\to\infty} \frac{a_n}{b_n} = r$.

Then:

1. If $r = \infty$ and $\sum a_n$ converges, $\sum b_n$ also converges;
2. If $r = 0$ and $\sum b_n$ converges, $\sum a_n$ also converges; and
3. If $0 < r < \infty$, then $\sum a_n$ converges if and only if $\sum b_n$ converges.

Proof. If $\lim_{n\to\infty} \frac{a_n}{b_n} = \infty$, then for large n necessarily $\frac{a_n}{b_n} > 1$, so that $a_n > b_n$, and the proof follows from comparison.

A similar proof exists for the case if $\lim_{n\to\infty} \frac{a_n}{b_n} = 0$.

Then, suppose $\lim_{n\to\infty} \frac{a_n}{b_n} = r$ with $0 < r < \infty$. Choose m so that when $n \geq m$, we have $\left| \frac{a_n}{b_n} - r \right| < \frac{r}{2}$;

this implies $\frac{r}{2} < \frac{a_n}{b_n} < \frac{3r}{2}$, so that $0 < \frac{r}{2} b_n \leq a_n \leq \frac{3r}{2} b_n$.

If $\sum a_n$ converges, then $\sum \frac{r}{2} b_n$ converges by comparison, and hence $\sum b_n$ converges by linearity.

If $\sum b_n$ converges, then $\sum \frac{3r}{2} b_n$ converges by linearity, and hence $\sum a_n$ converges by comparison. ■

RATIO TEST (T6.38)

Let $a_n > 0$ $\forall n \geq k$, and suppose $\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = r$.

Then,

1. If $r < 1$, $\sum a_n$ necessarily converges; and
2. If $r > 1$, $\sum a_n = \infty$.

Proof. This follows from the theorems of T6.19, using geometric series, and comparison.

*note that if $\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = 1$, $\sum a_n$ could converge or diverge.

eg If $a_n = \frac{1}{n}$, $\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = 1$ & $\sum a_n$ diverges; and

if $a_n = \frac{1}{n^2}$, $\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = 1$ & $\sum a_n$ converges.

ROOT TEST (T6.41)

Let $a_n \geq 0$ $\forall n \geq k$, and let $r = \limsup_{n\to\infty} \sqrt[n]{a_n}$.

Then

1. If $r < 1$, then $\sum a_n$ converges, and
2. If $r > 1$, then $\sum a_n = \infty$ (since $\lim_{n\to\infty} a_n = \infty$).

ALTERNATING SERIES (D6.43)

We say a sequence $(a_n)_{n\geq k}$ is "alternating" if either $a_n = (-1)^n |a_n|$ or $a_n = (-1)^{n+1} |a_n|$ $\forall n \geq k$.

ALTERNATING SERIES TEST (T6.44)

Let $(a_n)_{n\geq k}$ be an alternating series. Suppose the sequence $(|a_n|)$ is decreasing with $\lim_{n\to\infty} |a_n| = 0$.

Then $\sum_{n\geq k} a_n$ converges, and in this case we have $\left| \sum_{n=k}^{\infty} a_n \right| \leq |a_k|$.

Proof. We just give the proof in the case that $k=0$ and $a_n = (-1)^n |a_n|$.

Suppose $(|a_n|)$ is decreasing and $|a_n| \to 0$.

Let $S_\ell = \sum_{n=0}^{\ell} a_n$. Then, note that since $(|a_n|)$ is decreasing, necessarily

$$S_{2\ell} - S_{2\ell-1} = |a_{2\ell}| - |a_{2\ell-1}| \leq 0,$$

so that the sequence $(|S_{2\ell}|)$ is decreasing.

Moreover,

$$\begin{aligned} S_{2\ell} &= |a_0| - |a_1| + |a_2| - |a_3| + \cdots + |a_{2\ell-2}| - |a_{2\ell-1}| + |a_{2\ell}| \\ &= (|a_0| - |a_1|) + (|a_2| - |a_3|) + \cdots + (|a_{2\ell-2}| - |a_{2\ell-1}|) + |a_{2\ell}| \\ &\geq |a_0| - |a_1|, \end{aligned}$$

and so $S_{2\ell}$ is bounded below by $|a_0| - |a_1|$.

It follows $(S_{2\ell})$ converges by MCT.

Similarly, $(S_{2\ell-1})$ is increasing and bounded above by $|a_0|$, so it also converges by MCT, and $\lim_{\ell\to\infty} S_{2\ell-1} \leq |a_0|$.

Finally, since $|a_n| \to 0$, taking limits on both sides of the equality $|a_{2\ell}| = S_{2\ell} - S_{2\ell-1}$ gives us that $0 = \lim_{\ell\to\infty} S_{2\ell} - \lim_{\ell\to\infty} S_{2\ell-1}$, so we have $\lim_{\ell\to\infty} S_{2\ell} = \lim_{\ell\to\infty} S_{2\ell-1}$.

It follows that (S_ℓ) converges with $\lim_{\ell\to\infty} S_\ell = \lim_{\ell\to\infty} S_{2\ell} = \lim_{\ell\to\infty} S_{2\ell-1} \leq |a_0|$. ■

ABSOLUTE CONVERGENCE (D6.47)

We say a series $\sum_{n\geq k} a_n$ "converges absolutely" if $\sum_{n\geq k} |a_n|$ converges.

CONDITIONAL CONVERGENCE (D6.47)

We say a series $\sum_{n\geq k} a_n$ "converges conditionally" if $\sum_{n\geq k} a_n$ converges but $\sum_{n\geq k} |a_n|$ diverges.

eg $\sum \frac{(-1)^n}{n^p}$ converges conditionally for $0 < p \leq 1$.

(Follows from E6.30 and T6.44). (E6.48)

ABSOLUTE CONVERGENCE IMPLIES CONVERGENCE (T6.49)

Note that if $\sum |a_n|$ converges, necessarily $\sum a_n$ converges as well.

Proof. This follows from the fact that $0 \leq a_n + |a_n| \leq 2|a_n|$ for all n, and by linearity and comparison. ■

MULTIPLICATION OF SERIES (T6.51)

Suppose $\sum_{n\geq 0} |a_n|$ converges and $\sum_{n\geq 0} |b_n|$ converges.

Let $c_n = \sum_{k=0}^{n} a_k b_{n-k}$. Then $\sum_{n\geq 0} c_n$ converges, and

$$\sum_{n=0}^{\infty} c_n = \left(\sum_{n=0}^{\infty} a_n \right) \left(\sum_{n=0}^{\infty} b_n \right).$$

Proof. Let $A_\ell = \sum_{n=0}^{\ell} a_n$, $B_\ell = \sum_{n=0}^{\ell} b_n$, $C_\ell = \sum_{n=0}^{\ell} c_n$,

$A = \sum_{n=0}^{\infty} a_n$, $B = \sum_{n=0}^{\infty} b_n$, $K = \sum_{n=0}^{\infty} |a_n|$ and $E_\ell = B - B_\ell$.

Then

$$\begin{aligned} C_\ell &= a_0 b_0 + (a_0 b_1 + a_1 b_0) + (a_0 b_2 + a_1 b_1 + a_2 b_0) + \cdots \\ &\quad + (a_0 b_\ell + \cdots + a_\ell b_0) \\ &= a_0 B_\ell + a_1 B_{\ell-1} + \cdots + a_\ell B_0 \\ &= a_0 (B - E_\ell) + a_1 (B - E_{\ell-1}) + \cdots + a_\ell (B - E_0) \\ &= A_\ell B - (a_0 E_\ell + a_1 E_{\ell-1} + \cdots + a_\ell E_0). \end{aligned}$$

It follows that

$$|AB - C_\ell| \leq |(A - A_\ell) B| + |a_0 E_\ell + \cdots + a_\ell E_0|$$

by the Triangle Inequality.

Then, let $\varepsilon > 0$. Choose m so that $j > m$ implies $E_j < \frac{\varepsilon}{3K}$.

Let $E = \max\{|E_0|, \ldots, |E_m|\}$. Choose $L > m$ so that when $\ell > L$, we have $\sum_{n=\ell-m}^{\ell} |a_n| < \frac{\varepsilon}{3E}$ and $|A_\ell - A||B| < \frac{\varepsilon}{3}$.

Then for $\ell > L$,

$$\begin{aligned} |C_\ell - AB| &< |(A - A_\ell) B| + |a_0 E_\ell + \cdots + a_{\ell-m-1} E_{m+1}| \\ &\quad + |a_{\ell-m} E_m + \cdots + a_\ell E_0| \\ &\leq \frac{\varepsilon}{3} + \left(\sum_{n=0}^{\ell-m-1} |a_n| \right) \frac{\varepsilon}{3K} + \left(\sum_{n=\ell-m}^{\ell} |a_n| \right) E \\ &< \frac{\varepsilon}{3} + K \frac{\varepsilon}{3K} + \frac{\varepsilon}{3E} E \\ &< \varepsilon, \end{aligned}$$

thus showing that $\lim_{\ell\to\infty} C_\ell = C = AB$, as needed. ■

FUBINI'S THEOREM FOR SERIES (T6.53)

Let $a_{n,m} \in \mathbb{R}$ $\forall n,m \geq 0$, and suppose that $\sum_{m\geq 0} |a_{n,m}|$ converges for each $n \geq 0$ and that $\sum_{n\geq 0}\left(\sum_{m=0}^{\infty} |a_{n,m}|\right)$ converges.

Then necessarily

① $\sum_{m\geq 0} a_{n,m}$ converges $\forall n \geq 0$;

② $\sum_{n\geq 0}\left(\sum_{m=0}^{\infty} a_{n,m}\right)$ converges;

③ $\sum_{n\geq 0} a_{n,m}$ converges $\forall m \geq 0$;

④ $\sum_{m\geq 0}\left(\sum_{n=0}^{\infty} a_{n,m}\right)$ converges; and

⑤ $\sum_{n=0}^{\infty}\left(\sum_{m=0}^{\infty} a_{n,m}\right) = \sum_{m=0}^{\infty}\left(\sum_{n=0}^{\infty} a_{n,m}\right)$.

Proof. First, we claim $\sum_{n\geq 0} |a_{n,m}|$ converges $\forall m \geq 0$, $\sum_{m\geq 0}\left(\sum_{n=0}^{\infty} |a_{n,m}|\right)$ converges, and $\sum_{n=0}^{\infty}\left(\sum_{m=0}^{\infty} |a_{n,m}|\right) = \sum_{m=0}^{\infty}\left(\sum_{n=0}^{\infty} |a_{n,m}|\right)$.

For all n,m, we have $|a_{n,m}| \leq \sum_{k=0}^{\infty} |a_{n,k}|$, and since $\sum_{n\geq 0}\left(\sum_{k=0}^{\infty} |a_{n,k}|\right)$ converges, we know $\sum_{n\geq 0} |a_{n,m}|$ converges by the comparison test.

Let $k \geq 0$ and $\varepsilon > 0$ be arbitrary. Since each sum $\sum_{n\geq 0} |a_{n,m}|$ converges, we can choose L so that when $\ell > L$, we have $\sum_{n=\ell+1}^{\infty} |a_{n,m}| < \frac{\varepsilon}{k+1}$ $\forall m \in 0, 1, \dots, k$.

Then for $\ell > L$, we have

$$\begin{aligned}
\sum_{m=0}^{k}\left(\sum_{n=0}^{\infty} |a_{n,m}|\right) &= \sum_{m=0}^{k}\left(\sum_{n=0}^{\ell} |a_{n,m}| + \sum_{n=\ell+1}^{\infty} |a_{n,m}|\right) \\
&< \sum_{m=0}^{k}\left(\sum_{n=0}^{\ell} |a_{n,m}| + \frac{\varepsilon}{k+1}\right) \\
&= \sum_{m=0}^{k}\left(\sum_{n=0}^{\ell} |a_{n,m}|\right) + \varepsilon \\
&= \sum_{n=0}^{\ell}\left(\sum_{m=0}^{k} |a_{m,n}|\right) + \varepsilon \\
&\leq \sum_{n=0}^{\ell}\left(\sum_{m=0}^{\infty} |a_{m,n}|\right) + \varepsilon
\end{aligned}$$

$$\therefore \sum_{m=0}^{k}\left(\sum_{n=0}^{\infty} |a_{n,m}|\right) \leq \sum_{n=0}^{\ell}\left(\sum_{m=0}^{\infty} |a_{m,n}|\right) + \varepsilon.$$

Since ε was arbitrary, we have $\sum_{m=0}^{k}\left(\sum_{n=0}^{\infty} |a_{n,m}|\right) \leq \sum_{n=0}^{\ell}\left(\sum_{m=0}^{\infty} |a_{m,n}|\right)$.

Then, as the sequence of partial sums $\left(\sum_{m=0}^{k}\left(\sum_{n=0}^{\infty} |a_{n,m}|\right)\right)$ is increasing and bounded above by $\sum_{n=0}^{\infty}\left(\sum_{m=0}^{\infty} |a_{m,n}|\right)$, by MCT we have that $\sum_{m=0}^{\infty}\left(\sum_{n=0}^{\infty} |a_{n,m}|\right)$ converges and $\sum_{m=0}^{\infty}\left(\sum_{n=0}^{\infty} |a_{n,m}|\right) \leq \sum_{n=0}^{\infty}\left(\sum_{m=0}^{\infty} |a_{n,m}|\right)$.

By symmetry, we get the opposite inequality, thus proving the claim. ※

Subsequently, for all $n \geq 0$, since $\sum_{m\geq 0} |a_{n,m}|$ converges, we know that $\sum_{m\geq 0} a_{n,m}$ converges (since absolute convergence implies convergence) and that $\left|\sum_{m=0}^{\infty} a_{n,m}\right| \leq \sum_{m=0}^{\infty} |a_{n,m}|$.

Since $\sum_{n\geq 0}\left(\sum_{m=0}^{\infty} |a_{n,m}|\right)$ converges, necessarily $\sum_{n\geq 0}\left|\sum_{m=0}^{\infty} a_{n,m}\right|$ converges by the Comparison Test, so that $\sum_{n\geq 0}\left(\sum_{m=0}^{\infty} a_{n,m}\right)$ also converges (again, since absolute convergence implies convergence).

Similarly, $\sum_{n\geq 0} a_{n,m}$ converges for all $m \geq 0$, and $\sum_{m\geq 0}\left(\sum_{n=0}^{\infty} a_{n,m}\right)$ converges. ※

Finally, let $\varepsilon > 0$. Since $\sum_{n\geq 0}\left(\sum_{m=0}^{\infty} |a_{n,m}|\right)$ and $\sum_{m\geq 0}\left(\sum_{n=0}^{\infty} |a_{n,m}|\right)$ both converge, we can choose k and ℓ so that $\sum_{n=\ell+1}^{\infty}\left(\sum_{m=0}^{\infty} |a_{n,m}|\right) < \frac{\varepsilon}{4}$ and $\sum_{m=k+1}^{\infty}\left(\sum_{n=0}^{\infty} |a_{n,m}|\right) < \frac{\varepsilon}{4}$.

It follows that

$$\begin{aligned}
\sum_{n=0}^{\infty}\left(\sum_{m=0}^{\infty} a_{n,m}\right) &= \sum_{n=0}^{\infty}\left(\sum_{m=0}^{k} a_{n,m} + \sum_{m=k+1}^{\infty} a_{n,m}\right) \\
&= \sum_{n=0}^{\infty}\left(\sum_{m=0}^{k} a_{n,m}\right) + \sum_{n=0}^{\infty}\left(\sum_{m=k+1}^{\infty} a_{n,m}\right) \\
&= \sum_{n=0}^{\ell}\left(\sum_{m=0}^{k} a_{n,m}\right) + \sum_{n=\ell+1}^{\infty}\left(\sum_{m=0}^{k} a_{n,m}\right) + \sum_{n=0}^{\infty}\left(\sum_{m=k+1}^{\infty} a_{n,m}\right).
\end{aligned}$$

Hence

$$\begin{aligned}
\left|\sum_{n=0}^{\infty}\left(\sum_{m=0}^{\infty} a_{n,m}\right) - \sum_{n=0}^{\ell}\left(\sum_{m=0}^{k} a_{n,m}\right)\right| &\leq \left|\sum_{n=\ell+1}^{\infty}\left(\sum_{m=0}^{k} a_{n,m}\right)\right| + \left|\sum_{n=0}^{\infty}\left(\sum_{m=k+1}^{\infty} a_{n,m}\right)\right| \\
&\leq \sum_{n=\ell+1}^{\infty}\left|\sum_{m=0}^{k} a_{n,m}\right| + \sum_{n=0}^{\infty}\left|\sum_{m=k+1}^{\infty} a_{n,m}\right| \\
&\leq \sum_{n=\ell+1}^{\infty}\left(\sum_{m=0}^{k} |a_{n,m}|\right) + \sum_{n=0}^{\infty}\left(\sum_{m=k+1}^{\infty} |a_{n,m}|\right) \\
&= \sum_{n=\ell+1}^{\infty}\left(\sum_{m=0}^{k} |a_{n,m}|\right) + \sum_{m=k+1}^{\infty}\left(\sum_{n=0}^{\infty} |a_{n,m}|\right) \\
&\leq \sum_{n=\ell+1}^{\infty}\left(\sum_{m=0}^{\infty} |a_{n,m}|\right) + \sum_{m=k+1}^{\infty}\left(\sum_{n=0}^{\infty} |a_{n,m}|\right) \\
&< \frac{\varepsilon}{4} + \frac{\varepsilon}{4} = \frac{\varepsilon}{2}.
\end{aligned}$$

Similarly $\left|\sum_{m=0}^{\infty}\left(\sum_{n=0}^{\infty} a_{n,m}\right) - \sum_{m=0}^{\ell}\left(\sum_{n=0}^{k} a_{n,m}\right)\right| < \frac{\varepsilon}{2}$, and so

$$\left|\sum_{n=0}^{\infty}\left(\sum_{m=0}^{\infty} a_{n,m}\right) - \sum_{m=0}^{\infty}\left(\sum_{n=0}^{\infty} a_{n,m}\right)\right| < \varepsilon.$$

Since $\varepsilon > 0$ was arbitrary, it follows that $\sum_{n=0}^{\infty}\left(\sum_{m=0}^{\infty} a_{n,m}\right) = \sum_{m=0}^{\infty}\left(\sum_{n=0}^{\infty} a_{n,m}\right)$, as required. ∎

Chapter 7: Sequences and Series of Functions

POINTWISE CONVERGENCE (D7.1)

Let $A \subseteq \mathbb{R}$, $f: A \to \mathbb{R}$ and define a $f_n: A \to \mathbb{R}$ for each $n \geq p$, where $p \in \mathbb{Z}$.

Then, we say the sequence of functions $(f_n)_{n \geq p}$ "converges pointwise" to f on A when $\lim_{n\to\infty} f_n(x) = f(x)$.

In other words, $(f_n)_{n \geq p}$ converges pointwise to f on A if and only if for any $\varepsilon > 0$ and $x \in A$, there exists a $m \geq p$ such that

$$|f_n(x) - f(x)| < \varepsilon \quad \forall n \geq m.$$

In this case, we write "$f_n \to f$ pointwise on A".

CAUCHY DEFINITION FOR POINTWISE CONVERGENCE (D7.2)

Equivalently, we can also deduce $f_n \to f$ pointwise on A if and only if for any $\varepsilon > 0$, there exists a $m \geq p$ such that

$$|f_k(x) - f_\ell(x)| < \varepsilon \quad \forall x \in A, \ \forall k, \ell \geq m$$

by the Cauchy criterion for convergence.

EXAMPLE 1: $f_n(x) = x^n$ (E7.3)

7.3 Example: Find an example of a sequence of functions $(f_n)_{n\geq 1}$ and a function f with $f_n \to f$ pointwise on $[0,1]$ such that each f_n is continuous but f is not.

Let $f_n(x) = x^n$. Then observe that

$$\lim_{n\to\infty} f_n(x) = \begin{cases} 0, & x \neq 1 \\ 1, & x = 1. \end{cases}$$

EXAMPLE 2: $f_n(x) = \frac{1}{n}\tan^{-1}(nx)$ (E7.4)

7.4 Example: Find an example of a sequence of functions $(f_n)_{n\geq 1}$ and a function f with $f_n \to f$ pointwise on $[0,1]$ such that each f_n is differentiable and f is differentiable, but $\lim\limits_{n\to\infty} f_n' \neq f'$.

Let $f_n(x) = \frac{1}{n}\tan^{-1}(nx)$. Then $\lim_{n\to\infty} f_n(x) = 0$, and $f_n'(x) = \frac{1}{1+n^2x^2}$,

so

$$\lim_{n\to\infty} f_n'(x) = \begin{cases} 0, & x \neq 0 \\ 1, & x = 0. \end{cases}$$

EXAMPLE 3 (E7.5)

7.5 Example: Find an example of a sequence of functions $(f_n)_{n\geq 1}$ and a function f with $f_n \to f$ pointwise on $[0,1]$ such that each f_n is integrable but f is not.

Let $(a_n)_{n\geq 1} = \left(\frac{0}{1}, \frac{1}{1}, \frac{0}{2}, \frac{1}{2}, \frac{2}{2}, \frac{0}{3}, \frac{1}{3}, \frac{2}{3}, \frac{3}{3}, \frac{0}{4}, \dots, \frac{4}{4}, \dots\right)$.

For $x \in [0,1]$, let $f_n(x) = \begin{cases} 0, & x \notin \{a_1, \dots, a_n\} \\ 1, & x \in \{a_1, \dots, a_n\}. \end{cases}$

Clearly each f_n is integrable since it is only discontinuous at finitely many points, but $\lim_{n\to\infty} f_n(x) = \begin{cases} 0, & x \notin \mathbb{Q} \\ 1, & x \in \mathbb{Q}. \end{cases}$

EXAMPLE 4 (E7.6)

7.6 Example: Find an example of a sequence of functions $(f_n)_{n\geq 1}$ and a function f with $f_n \to f$ pointwise on $[0,1]$ such that each f_n is integrable and f is integrable but

$$\lim_{n\to\infty} \int_0^1 f_n(x)\,dx \neq \int_0^1 f(x)\,dx.$$

Let $f_1(x) = \begin{cases} 48(x-\frac{1}{2})(1-x), & \frac{1}{2} \leq x \leq 1 \\ 0, & \text{otherwise}. \end{cases}$ For $n \geq 1$, let $f_n(x) = n f_1(nx)$.

Then each f_n is continuous with $\int_0^1 f_n(x)\,dx = 1$, and

$$\lim_{n\to\infty} f_n(x) = 0 \quad \forall x \in [0,1].$$

UNIFORM CONVERGENCE (D7.7)

1. Let $A \subseteq \mathbb{R}$, $f: A \to \mathbb{R}$, and for each $n \geq p$, define a $f_n : A \to \mathbb{R}$.
Then, we say the sequence of functions $(f_n)_{n \geq p}$ "converges uniformly" to f on A if and only if for any $\varepsilon > 0$, there exists a $m \geq p$ such that for all $x \in A$, we have that

$$n \geq m \implies |f_n(x) - f(x)| < \varepsilon.$$

*note the difference in the wording between pointwise & absolute convergence!

2. In this case, we write "$f_n \to f$ uniformly on A".

3. Like pointwise convergence, a similar "Cauchy definition" exists for absolute convergence. (T7.8)

UNIFORM CONVERGENCE, LIMITS & CONTINUITY (T7.9)

Let $f_n \to f$ uniformly on A, and let x be a limit point of A.
Suppose $\lim_{y \to x} f_n(y)$ exists for each $n \in \mathbb{Z}^+$.
Then necessarily

$$\lim_{y \to x} \lim_{n \to \infty} f_n(y) = \lim_{n \to \infty} \lim_{y \to x} f_n(y).$$

In particular, if each f_n is continuous in A, then so is f.

Proof. Let $b_n = \lim_{y \to x} f_n(y)$. We need to show $\lim_{n \to \infty} b_n = \lim_{y \to x} f(y)$.
We claim first that $(b_n)_{n \geq 1}$ converges.
Proof. Let $\varepsilon > 0$. Choose m so that

$$k, \ell \geq m \implies |f_k(y) - f_\ell(y)| < \frac{\varepsilon}{3} \quad \forall y \in A.$$

Fix $k, \ell \geq m$, and choose a $y \in A$ so that

$$|f_k(y) - b_k| < \frac{\varepsilon}{3} \quad \text{and} \quad |f_\ell(y) - b_\ell| < \frac{\varepsilon}{3}.$$

Then

$$\begin{aligned} |b_k - b_\ell| &\leq |b_k - f_k(y)| + |f_k(y) - f_\ell(y)| + |f_\ell(y) - b_\ell| \\ &< \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} \\ &= \varepsilon, \end{aligned}$$

and so by the Cauchy Criterion for sequences, (b_n) must converge. ※

So, let $b = \lim_{n \to \infty} b_n$. Let $\varepsilon > 0$. Choose m so that when $n \geq m$, we have $|f_n(y) - f(y)| < \frac{\varepsilon}{3}$ $\forall y \in A$, and $|b_n - b| < \frac{\varepsilon}{3}$.
Fix $n \geq m$. Since $\lim_{y \to x} f_n(y) = b_n$, we can choose a $\delta > 0$ so that $0 < |y - x| < \delta \implies |f_n(y) - b_n| < \frac{\varepsilon}{3}$.
Then, when $0 < |y - x| < \delta$, we have

$$\begin{aligned} |f(y) - b| &\leq |f(y) - f_n(y)| + |f_n(y) - b_n| + |b_n - b| \\ &< \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} \end{aligned}$$

$\therefore |f(y) - b| < \varepsilon$,
proving that $\lim_{y \to x} f(y) = b$, and so $\lim_{y \to x} f(y) = \lim_{n \to \infty} b_n$, as needed. ☒

In particular, if $x \in A$ and each f_n is continuous at x, then we have

$$\lim_{y \to x} f(y) = \lim_{y \to x} \lim_{n \to \infty} f_n(y) = \lim_{n \to \infty} \lim_{y \to x} f_n(y) = \lim_{n \to \infty} f_n(x) = f(x),$$

so f is continuous at x by definition. ☒

UNIFORM CONVERGENCE & INTEGRATION (T7.10)

Let $f_n \to f$ uniformly on $[a, b]$.
Then, if each f_n is integrable, f is necessarily also integrable.
In this case, if we denote $g_n(x) = \int_a^x f_n(t)\,dt$ and $g(x) = \int_a^x f(t)\,dt$, then necessarily $g_n \to g$ uniformly on $[a, b]$.
In particular, we have that

$$\lim_{n \to \infty} \int_a^b f_n(x)\,dx = \int_a^b \lim_{n \to \infty} f_n(x)\,dx.$$

Proof. Let each f_n be integrable on $[a, b]$.
Let $\varepsilon > 0$. Choose N so $n \geq N \Rightarrow |f_n(x) - f(x)| < \frac{\varepsilon}{4(b-a)}$ $\forall x \in [a, b]$.
Fix $n \geq N$. Choose a partition X of $[a, b]$ so that $U(f_n, X) - L(f_n, X) < \frac{\varepsilon}{2}$.
Note that since $|f_n(x) - f(x)| < \frac{\varepsilon}{4(b-a)}$, we have $M_i(f) < M_i(f_n) + \frac{\varepsilon}{4(b-a)}$
and $m_i(f) > m_i(f_n)$ $\forall i \in \{1, 2, \ldots, n\}$, and so

$$\begin{aligned} U(f, X) - L(f, X) &= \sum_{i=1}^{n} (M_i(f) - m_i(f)) \Delta_i x \\ &< \sum_{i=1}^{n} \left(M_i(f_n) - m_i(f_n) + \frac{\varepsilon}{2(b-a)}\right) \Delta_i x \\ &= U(f_n, X) - L(f_n, X) + \frac{\varepsilon}{2} \\ &< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} \\ &= \varepsilon. \end{aligned}$$

It follows that f is integrable on $[a, b]$. ☒
Next, define $g_n(x) = \int_a^x f_n(t)\,dt$ and $g(x) = \int_a^x f(t)\,dt$. Let $\varepsilon > 0$.
Choose N so that $n \geq N \Rightarrow |f_n(t) - f(t)| < \frac{\varepsilon}{2(b-a)}$ $\forall t \in I$.
Let $n \geq N$, and $x \in [a, b]$. Then observe that

$$\begin{aligned} |g_n(x) - g(x)| &= \left| \int_a^x f_n(t)\,dt - \int_a^x f(t)\,dt \right| \\ &= \left| \int_a^x (f_n(t) - f(t))\,dt \right| \\ &\leq \int_a^x |f_n(t) - f(t)|\,dt \quad \text{(by estimation)} \\ &\leq \int_a^x \frac{\varepsilon}{2(b-a)}\,dt \\ &= \frac{\varepsilon}{2(b-a)}(x - a) \\ &\leq \frac{\varepsilon}{2} \\ &< \varepsilon, \end{aligned}$$

so that $g_n \to g$ uniformly on $[a, b]$, as needed.
In particular, since $\lim_{n \to \infty} g_n(b) = g(b)$, it follows that

$$\lim_{n \to \infty} \int_a^b f_n(x)\,dx = \int_a^b \lim_{n \to \infty} f_n(x)\,dx. \quad ☒$$

UNIFORM CONVERGENCE & DIFFERENTIATION

(T7.11)

Let (f_n) be a sequence of functions on $[a,b]$.
Suppose each f_n is differentiable on $[a,b]$, and that (f_n') converges uniformly on $[a,b]$.
Suppose further that $(f_n(c))$ converges for some $c \in [a,b]$.
Then necessarily

1. (f_n) converges uniformly on $[a,b]$;
2. $\lim_{n\to\infty} f_n(x)$ is differentiable; and
3. $\frac{d}{dx} \lim_{n\to\infty} f_n(x) = \lim_{n\to\infty} \frac{d}{dx} f_n(x)$.

Proof. Let $\varepsilon > 0$. Choose N so that when $n, m \geq N$, we have $|f_n'(t) - f_m'(t)| < \frac{\varepsilon}{2(b-a)}$ $\forall t \in [a,b]$, and $|f_n(c) - f_m(c)| < \frac{\varepsilon}{2}$.

Fix $m, n \geq N$, and $x \in [a,b]$. Then, by the Mean Value Theorem applied to the function $f_n(x) - f_m(x)$, we can choose t between c and x so that

$$(f_n(x) - f_m(x) - f_n(c) + f_m(c)) = (f_n'(t) - f_m'(t))(x-c).$$

Hence

$$\begin{aligned}
|f_n(x) - f_m(x)| &\leq |f_n(x) - f_m(x) - f_n(c) + f_m(c)| + |f_n(c) - f_m(c)| \\
&= |f_n'(t) - f_m'(t)||x-c| + |f_n(c) - f_m(c)| \\
&< \frac{\varepsilon}{2(b-a)}(b-a) + \frac{\varepsilon}{2} \\
&= \varepsilon,
\end{aligned}$$

showing (f_n) converges uniformly on $[a,b]$. ✱

Let $f(x) = \lim_{n\to\infty} f_n(x)$. Fix $x \in [a,b]$, and note that

$$\begin{aligned}
f'(x) = \lim_{n\to\infty} f_n'(x) &\iff \lim_{y\to x} \frac{f(y) - f(x)}{y-x} = \lim_{n\to\infty} \lim_{y\to x} \frac{f_n(y) - f_n(x)}{y-x} \\
&\iff \lim_{y\to x} \lim_{n\to\infty} \frac{f_n(y) - f_n(x)}{y-x} = \lim_{n\to\infty} \lim_{y\to x} \frac{f_n(y) - f_n(x)}{y-x},
\end{aligned}$$

so we just need to show (g_n) converges uniformly on $[a,b] \setminus \{x\}$, where $g_n(y) = \frac{f_n(y) - f_n(x)}{y-x}$, since the rest follows from T7.9.

Let $\varepsilon > 0$. Choose N so that $n, m \geq N \Rightarrow |f_n'(t) - f_m'(t)| < \varepsilon$ $\forall t \in [a,b]$.

Let $n, m \geq N$, and fix $y \in [a,b] \setminus \{x\}$.

Then, by the Mean Value Theorem, we can choose t between x and y so that

$$(f_n(y) - f_m(y) - f_n(x) + f_m(x)) = (f_n'(t) - f_m'(t))(y-x).$$

Then

$$|g_n(y) - g_m(y)| = \left| \frac{f_n(y) - f_m(y) - f_n(x) + f_m(x)}{y-x} \right| = |f_n'(t) - f_m'(t)| < \varepsilon,$$

showing (g_n) converges uniformly on $[a,b] \setminus \{x\}$, as required. ☒

SERIES OF FUNCTIONS (D7.12)

1. Let $(f_n)_{n\geq p}$ be a sequence of functions on $A \subseteq \mathbb{R}$.
Then, we define the "series of functions" $\sum_{n\geq p} f_n(x)$ is defined to be the sequence
$$(S_\ell(x)) = \left(\sum_{n=p}^{\ell} f_n(x)\right).$$

2. "$S_\ell(x)$" is called the "ℓ^{th} partial sum" of the series.

CONVERGENCE OF SERIES OF FUNCTIONS (D7.12)

1. We say the series $\sum_{n\geq p} f_n(x)$ "converges pointwise" on $A \subseteq \mathbb{R}$ when the sequence $(S_\ell(x))$ converges pointwise on A, and "converges uniformly" on $A \subseteq \mathbb{R}$ when $(S_\ell(x))$ converges uniformly on A.

2. In this case, the "sum" of the series of functions is defined to be the function
$$f(x) = \sum_{n=p}^{\infty} f_n(x) = \lim_{\ell\to\infty} S_\ell(x).$$

CAUCHY CONVERGENCE FOR A SERIES OF FUNCTIONS (T7.13)

Let (f_n) be a sequence of functions on $A \subseteq \mathbb{R}$.
Then, the series $\sum_{n\geq p} f_n(x)$ converges uniformly on A if and only if for every $\varepsilon > 0$, there exists $N \geq p$ such that for all $x \in A$ and $k, \ell \geq p$, we have
$$\ell > k \geq N \Rightarrow \left|\sum_{n=k+1}^{\ell} f_n(x)\right| < \varepsilon.$$

Proof. This follows from the analogous theorem for sequences of functions. ∎

UNIFORM CONVERGENCE, LIMITS & CONTINUITY FOR SERIES (T7.14)

Let (f_n) be a sequence of functions on $A \subseteq \mathbb{R}$.
Suppose $\sum_{n\geq p} f_n(x)$ converges uniformly on A, and let x be a limit point of A.
Suppose further that $\lim_{y\to x} f_n(y)$ exists for all $n \geq p$.
Then necessarily $\lim_{y\to x} \sum_{n=p}^{\infty} f_n(y) = \sum_{n=p}^{\infty} \lim_{y\to x} f_n(y)$.
In particular, if each $f_n(x)$ is continuous on A, then so is $\sum_{n=p}^{\infty} f_n(x)$.

Proof. This follows from the analogous theorem for sequences of functions. ∎

UNIFORM CONVERGENCE & INTEGRATION FOR SERIES (T7.15)

Let (f_n) be a sequence of functions on $[a,b]$, such that $\sum_{n\geq p} f_n(x)$ converges uniformly on $[a,b]$.
Suppose each $f_n(x)$ is integrable on $[a,b]$.
Then necessarily so is $\sum_{n=p}^{\infty} f_n(x)$.
In this case, if we define $g_n(x) = \int_a^x f_n(t)\,dt$ and $g(x) = \int_a^x \sum_{n=p}^{\infty} f_n(t)\,dt$, then $\sum_{n\geq p} g_n(x)$ converges uniformly to $g(x)$ on A.
In particular, we have
$$\int_a^b \sum_{n=p}^{\infty} f_n(x)\,dx = \sum_{n=p}^{\infty} \int_a^b f_n(x)\,dx.$$

Proof. This follows from the analogous theorem for sequences of functions. ∎

UNIFORM CONVERGENCE & DIFFERENTIATION (T7.16)

Let (f_n) be a sequence of functions, so that each $f_n(x)$ is differentiable on $[a,b]$.
Suppose further that $\sum_{n\geq p} f_n'(x)$ converges uniformly on $[a,b]$, and $\sum_{n\geq p} f_n(c)$ converges for some $c \in [a,b]$.
Then necessarily $\sum_{n\geq p} f_n(x)$ converges uniformly on $[a,b]$, and
$$\frac{d}{dx}\sum_{n=p}^{\infty} f_n(x) = \sum_{n=p}^{\infty} \frac{d}{dx} f_n(x).$$

Proof. This follows from the analogous theorem for sequences of functions. ∎

WEIERSTRASS M-TEST (T7.17)

Suppose that $|f_n(x)| \leq M_n$ $\forall x \in A$ for each f_n, where $n \geq p$, such that $\sum_{n\geq p} M_n$ converges.
Then necessarily $\sum_{n\geq p} f_n(x)$ converges uniformly on A.

Proof. Let $\varepsilon > 0$. Choose N so that $\ell > k \geq N \Rightarrow \sum_{n=k+1}^{\ell} M_n < \varepsilon$.
Fix $\ell > k \geq N$ and $x \in A$. Then observe that
$$\left|\sum_{n=k+1}^{\ell} f_n(x)\right| \leq \sum_{n=k+1}^{\ell} |f_n(x)| \leq \sum_{n=k+1}^{\ell} M_n < \varepsilon,$$
which by T7.13, is sufficient to show $\sum_{n\geq p} f_n(x)$ converges uniformly on A. ∎

POWER SERIES (D7.19)

A "power series centred at a" is a series of the form
$$\sum_{n\geq 0} a_n (x-a)^n$$
for some $a_n \in \mathbb{R}$.

ABEL'S FORMULA (L7.21)

Let $\{a_n\}$ and $\{b_n\}$ be sequences.
Then necessarily
$$\sum_{n=m}^{\ell} a_n b_n + \sum_{p=m}^{\ell-1}\left(\sum_{n=m}^{p} a_n\right)(b_{p+1}-b_p) = \left(\sum_{n=m}^{\ell} a_n\right) b_\ell .$$

Proof.
$$\begin{aligned}\sum_{p=m}^{\ell-1}\left(\sum_{n=m}^{p} a_n\right)(b_{p+1}-b_p) &= a_m(b_{m+1}-b_m)\\ &\quad + (a_m+a_{m+1})(b_{m+2}-b_{m+1})\\ &\quad + (a_m+a_{m+1}+a_{m+2})(b_{m+3}-b_{m+2})\\ &\quad + \cdots\\ &\quad + (a_m+a_{m+1}+\cdots+a_{\ell-1})(b_\ell-b_{\ell-1})\\ &= -a_mb_m - a_{m+1}b_{m+1} - \cdots - a_{\ell-1}b_{\ell-1}\\ &\quad + (a_m+a_{m+1}+\cdots+a_{\ell-1})b_\ell - a_\ell b_\ell + a_\ell b_\ell\end{aligned}$$
$$\therefore \sum_{p=m}^{\ell-1}\left(\sum_{n=m}^{p} a_n\right)(b_{p+1}-b_p) = \left(\sum_{n=m}^{\ell} a_n\right) b_\ell - \sum_{n=m}^{\ell} a_n b_n . \quad \square$$

INTERVAL & RADIUS OF CONVERGENCE (T7.23)

Let $\sum_{n\geq 0} a_n(x-a)^n$ be a power series, and let
$$R = \frac{1}{\limsup_{n\to\infty} \sqrt[n]{|a_n|}} = \frac{1}{\lim_{n\to\infty} \sup\{\sqrt[k]{|a_k|} : k\geq n\}} \in [0,\infty].$$
Then the set of $x\in\mathbb{R}$ for which the power series converges is necessarily an interval I centred at a of radius R.
Indeed
① If $|x-a|>R$, then $\lim_{n\to\infty} a_n(x-a) \neq 0$, so $\sum_{n\geq 0} a_n(x-a)^n$ diverges;
② If $|x-a|<R$, then $\sum_{n\geq 0} a_n(x-a)^n$ converges absolutely; and
③ If $0<r<R$, then $\sum_{n\geq 0} a_n(x-a)^n$ converges uniformly in $[a-r, a+r]$.

Proof. To prove ①, suppose $|x-a|>R$. Then
$$\limsup_{n\to\infty} \sqrt[n]{|a_n(x-a)^n|} = |x-a| \limsup_{n\to\infty}\sqrt[n]{|a_n|} > R\cdot\frac{1}{R} = 1,$$
and so by the Root Test, $\lim_{n\to\infty} a_n(x-a)^n \neq 0$ and so $\sum a_n(x-a)^n$ diverges.
To prove ②, suppose $|x-a|<R$. Then
$$\limsup_{n\to\infty} \sqrt[n]{|a_n(x-a)^n|} = |x-a| \limsup_{n\to\infty}\sqrt[n]{|a_n|} < R\cdot\frac{1}{R} = 1,$$
and so $\sum|a_n(x-a)^n|$ converges, by the Root Test.
To prove ③, fix $0<r<R$. By ②, $\sum|a_n(x-a)^n|$ converges when $x=a+r$; ie $\sum|a_nr^n|$ converges.
Fix $x\in[a-r, a+r]$. Then since $|a_n(x-a)^n| \leq |a_nr^n|$ and $\sum|a_nr^n|$ converges, so by the Weierstrass M-Test necessarily $\sum|a_n(x-a)^n|$ converges uniformly. $\square$

In the above theorem,
① "R" is called the "radius of convergence" of the power series; and
② "I" is called the "interval of convergence" of the power series. (D7.24)

ABEL'S THEOREM (T7.23 (4))

Let $\sum_{n\geq 0} a_n(x-a)^n$ be a power series, with radius of convergence R and interval of convergence I.
Then, if $\sum_{n\geq 0} a_n(x-a)^n$ converges when $x=a+R$, then the convergence is necessarily uniform on $[a, a+R]$.
Similarly, if $\sum_{n\geq 0} a_n(x-a)^n$ converges when $x=a-R$, then the convergence is necessarily uniform on $[a-R, a]$.

Proof. Suppose $\sum a_n(x-a)^n$ converges when $x=a+R$, ie $\sum a_nR^n$ converges.
Let $\varepsilon>0$. Choose N so that $\ell>m>N \Rightarrow \left|\sum_{n=m}^{\ell} a_nR^n\right| < \varepsilon$.
Then by Abel's Formula and using telescoping, we have
$$\begin{aligned}\left|\sum_{n=m}^{\ell} a_n(x-a)^n\right| &= \left|\sum_{n=m}^{\ell} a_nR^n\left(\frac{x-a}{R}\right)^n\right|\\ &= \left|\left(\sum_{n=m}^{\ell} a_nR^n\right)\left(\frac{x-a}{R}\right)^\ell - \sum_{p=m}^{\ell-1}\left(\sum_{n=m}^{p} a_nR^n\right)\left(\left(\frac{x-a}{R}\right)^{p+1} - \left(\frac{x-a}{R}\right)^p\right)\right|\\ &\leq \left|\sum_{n=m}^{\ell} a_nR^n\right|\left(\frac{x-a}{R}\right)^\ell - \sum_{p=m}^{\ell-1}\left|\sum_{n=m}^{p} a_nR^n\right|\left(\left(\frac{x-a}{R}\right)^{p+1} - \left(\frac{x-a}{R}\right)^p\right)\\ &< \varepsilon\left(\frac{x-a}{R}\right)^\ell + \varepsilon\left(\left(\frac{x-a}{R}\right)^m - \left(\frac{x-a}{R}\right)^\ell\right) = \varepsilon\left(\frac{x-a}{R}\right)^m < \varepsilon,\end{aligned}$$
proving the series uniformly converges. $\square$

CONTINUITY OF POWER SERIES (T7.26)

Suppose the power series $\sum a_n(x-a)^n$ converges in an interval I.
Then the sum $f(x) = \sum_{n=0}^{\infty} a_n(x-a)^n$ is continuous in I.

Proof. This follows from uniform convergence of $\sum a_n(x-a)^n$ in closed subintervals of I.

ADDITION & SUBTRACTION OF POWER SERIES (T7.27)

Suppose $\sum a_n(x-a)^n$ and $\sum b_n(x-a)^n$ both converge on I.
Then $\sum(a_n+b_n)(x-a)^n$ and $\sum(a_n-b_n)(x-a)^n$ both converge in I, and for all $x\in I$, we have
$$\sum_{n=0}^{\infty} a_n(x-a)^n \pm \sum_{n=0}^{\infty} b_n(x-a)^n = \sum_{n=0}^{\infty}(a_n\pm b_n)(x-a)^n.$$

Proof. This follows from Linearity.

MULTIPLICATION OF POWER SERIES (T7.28)

Suppose $\sum a_n(x-a)^n$ and $\sum b_n(x-a)^n$ both converge in an open interval I, and suppose $a\in I$.
Let $c_n = \sum_{k=0}^{n} a_k b_{n-k}$.
Then necessarily $\sum c_n(x-a)^n$ converges in I, and for all $x\in I$, we have
$$\sum_{n=0}^{\infty} c_n(x-a)^n = \left(\sum_{n=0}^{\infty} a_n(x-a)^n\right)\left(\sum_{n=0}^{\infty} b_n(x-a)^n\right).$$

Proof. This follows from the Multiplication of Series Theorem, since the power series converges absolutely in I.

DIVISION OF POWER SERIES (T7.29)

Suppose $\sum a_n(x-a)^n$ and $\sum b_n(x-a)^n$ both converge in an open interval I, with $a\in I$, and that $b_0\neq 0$.
Define c_n by
$$c_0 = \frac{a_0}{b_0}, \quad \text{and for } n>0, \quad c_n = \frac{a_n}{b_0} - \frac{b_n}{b_0}c_0 - \frac{b_{n-1}}{b_0}c_0 - \cdots - \frac{b_1}{b_0}c_{n-1}.$$
Then there exists an open interval J with $a\in J$ such that $\sum c_n(x-a)^n$ converges in J, and for all $x\in J$, we have that
$$\sum_{n=0}^{\infty} c_n(x-a)^n = \frac{\sum_{n=0}^{\infty} a_n(x-a)^n}{\sum_{n=0}^{\infty} b_n(x-a)^n}$$

Proof. Choose $r>0$ so that $a+r\in I$.
Note that $\sum|a_nr^n|$ and $\sum|b_nr^n|$ both converge.
Since $|a_nr^n|\to 0$ and $|b_nr^n|\to 0$ and $b_0\neq 0$, we can choose M so that $M \geq \left|\frac{a_nr^n}{b_0}\right|$ and $M\geq\left|\frac{b_nr^n}{b_0}\right|$ for all n.
Note that $|c_0| = \left|\frac{a_0}{b_0}\right| \leq M$, and since $c_1 = \frac{a_1}{b_0} + \frac{b_1c_0}{b_0}$, we have
$$|c_1r| \leq \left|\frac{a_1r}{b_0}\right| + \left|\frac{b_1r}{b_0}\right||c_0| \leq M + M^2 = M(1+M).$$
Suppose, inductively, that $|c_kr^k| \leq M(1+M)^k \ \forall k<n$.
Then, since
$$a_n = b_nc_0 + b_{n-1}c_1 + \cdots + b_1c_{n-1} + b_0c_n,$$
we have
$$\begin{aligned}|c_nr^n| &\leq \left|\frac{a_nr^n}{b_0}\right| + \left|\frac{b_nr^n}{b_0}\right||c_0| + \left|\frac{b_{n-1}r^{n-1}}{b_0}\right||c_1r| + \cdots + \left|\frac{b_1r}{b_0}\right||c_{n-1}r^{n-1}|\\ &\leq M + M^2 + M^2(1+M) + M^2(1+M)^2 + \cdots + M^2(1+M)^{n-1}\\ &= M + M^2\left(\frac{(1+M)^n-1}{M}\right)\end{aligned}$$
$$\therefore |c_nr^n| = M(1+M)^n.$$
So, by induction, we have $|c_nr^n| \leq M(1+M)^n \ \forall n\geq 0$.
Let $J_1 = \left(a-\frac{r}{1+M}, a+\frac{r}{1+M}\right)$, and let $x\in J_1$ so that $|x-a| < \frac{r}{1+M}$.
Then for all n we have
$$\begin{aligned}|c_n(x-a)^n| &= |c_nr^n|\cdot\frac{1}{(1+M)^n}\cdot\left|\frac{x-a}{r/(1+M)}\right|^n\\ &\leq M\left|\frac{x-a}{r/(1+M)}\right|^n,\end{aligned}$$
and so $\sum|c_n(x-a)^n|$ converges by comparison.
Note that from the definition of c_n, we have $a_n = \sum_{k=0}^{n} c_kb_{n-k}$, and so by multiplying power series, we have
$$\left(\sum_{n=0}^{\infty} c_n(x-a)^n\right)\left(\sum_{n=0}^{\infty} b_n(x-a)^n\right) = \sum_{n=0}^{\infty} a_n(x-a)^n \quad \forall x\in I\cap J_1.$$
Finally, note that $f(x) = \sum_{n=0}^{\infty} b_n(x-a)^n$ is continuous on I, and we have $f(0) = b_0\neq 0$, So there exists an interval $J\subset I\cap J_1$ with $a\in J$ such that $f(x)\neq 0 \ \forall x\in J$. $\square$

COMPOSITION OF POWER SERIES (T7.30)

Let $f(x) = \sum_{n=0}^{\infty} a_n(x-a)^n$ in an open interval I with $a \in I$, and let $g(y) = \sum_{m=0}^{\infty} b_m(y-b)^m$ in an open interval J with $b \in J$ and $a_0 \in J$.

Let K be an open interval with $a \in K$ such that $f(K) \subset J$.

For each $m \geq 0$, let $c_{n,m}$ be the coefficients of the product

$$\sum_{n=0}^{\infty} c_{n,m}(x-a)^n = b_n\left(\sum_{n=0}^{\infty} a_n(x-a)^n - b\right)^m.$$

Then $\sum_{m\geq 0} c_{n,m}$ for all $m \geq 0$, and for all $x \in K$, $\sum_{n\geq 0}\left(\sum_{m=0}^{\infty} c_{n,m}\right)(x-a)^n$ converges and

$$\sum_{n=0}^{\infty}\left(\sum_{m=0}^{\infty} c_{n,m}\right)(x-a)^n = g(f(x)).$$

Proof. This follows from Fubini's Theorem for Series, since

$$g(f(x)) = \sum_{m=0}^{\infty} b_m(f(x)-b)^m$$
$$= \sum_{m=0}^{\infty} b_m\left(\sum_{n=0}^{\infty} a_n(x-a)^n - b\right)$$
$$\therefore g(f(x)) = \sum_{m=0}^{\infty}\left(\sum_{n=0}^{\infty} c_{n,m}(x-a)^n\right) \quad \blacksquare$$

INTEGRATION OF POWER SERIES (T7.31)

Suppose $\sum a_n(x-a)^n$ converges in the interval I.

Then for all $x \in I$, the sum $f(x) = \sum_{n=0}^{\infty} a_n(x-a)^n$ is integrable on $[a,x]$ (or $[x,a]$) and

$$\int_a^x \sum_{n=0}^{\infty} a_n(t-a)^n\,dt = \sum_{n=0}^{\infty}\int_a^x a_n(t-a)^n\,dt = \sum_{n=0}^{\infty}\frac{a_n}{n+1}(x-a)^{n+1}.$$

Proof. This follows from uniform convergence. $\blacksquare$

DIFFERENTIATION OF POWER SERIES (T7.32)

Suppose $\sum a_n(x-a)^n$ converges in the open interval I.

Then the sum $f(x) = \sum_{n=0}^{\infty} a_n(x-a)^n$ is differentiable in I, and

$$f'(x) = \sum_{n=1}^{\infty} na_n(x-a)^{n-1}.$$

Proof. We claim the radius of convergence of $\sum a_n(x-a)^n$ is equal to the radius of convergence of $\sum na_n(x-a)^{n-1}$.

Let R be the radius of convergence of $\sum a_n(x-a)^n$, and let S be the radius of convergence of $\sum na_n(x-a)^{n-1}$.

Fix $x \in (a-R, a+R)$, so $|x-a| < R$ and $\sum |a_n(x-a)^n|$ converges.

Choose r, s with $|x-a| < r < s < R$.

Since $\lim_{n\to\infty}\frac{(r/s)^n}{1/n} = 0$, we can choose N so that $n \geq N$ we have

$$|na_n(x-a)^n| = \left|n\left(\tfrac{r}{s}\right)^n\left(\tfrac{x-a}{r}\right)^n a_n s^n\right| \leq 1\cdot 1\cdot |a_n s^n|.$$

Since $\sum |a_n s^n|$ converges, necessarily $\sum |na_n(x-a)^n|$ converges by comparison, and so by linearity $\sum |na_n(x-a)^{n-1}|$ converges.

Hence $R \leq S$.

Now, fix $x \in (a-S, a+S)$ so that $|x-a| < S$ and $\sum |na_n(x-a)^{n-1}|$ converges.

Then $\sum |na_n(x-a)^n|$ converges by linearity, and since $|a_n(x-a)^n| \leq |na_n(x-a)^n|$, hence $\sum |a_n(x-a)^n|$ converges by comparison. Thus $S \leq R$ and so $R = S$ as claimed.

The theorem now follows from the uniform convergence of $\sum na_n(x-a)^{n-1}$.