MATH 247
Personal Notes

Marcus Chan (UW 25)

Taught by Blake Madill

Module 1.1: Normed Vector Spaces

1. A "normed vector space", or "NVS", is a vector space V over $\mathbb{R}$ equipped with a function

$$\|\cdot\| : V \to [0, \infty),$$

and the function satisfies the following:

① $\|v\| = 0 \iff v = 0$;

② $\|\alpha v\| = |\alpha| \cdot \|v\| \quad \forall \alpha \in \mathbb{R}$; and

③ $\|u+v\| \leq \|u\| + \|v\|$. (Triangle Inequality).

2. Such a function $\|\cdot\|$ is called a "norm".

3. We generally use the notation "$(V, \|\cdot\|)$" to indicate a normed vector space.

4. Geometrically,

① $\|v\|$ refers to the "length" of v, or the distance between v & 0; and

② $\|v-w\|$ refers to the "distance" between v & w.

5. Normed vector spaces are useful in real analysis because the "notion" of "distance" in NVS helps us talk about "approximating" real numbers with more well-behaved ones (eg $\mathbb{Q}$).

p-NORMS : $\|v\|_p$

1. Let $p \geq 1$, and let $v = \begin{pmatrix} v_1 \\ \vdots \\ v_n \end{pmatrix} \in \mathbb{R}^n$.

Then, the "p-norm" of v, denoted as "$\|v\|_p$", is equal to

$$\|v\|_p = \left(\sum_{i=1}^{n} |v_i|^p \right)^{\frac{1}{p}}.$$

2. We can show that the p-norm is indeed a norm of $\mathbb{R}^n$.

(see A1.)

3. In particular, the 2-norm,

$$\|v\|_2 = \left(\sum_{i=1}^{n} |v_i|^2 \right)^{\frac{1}{2}},$$

is called the "Euclidean norm" on $\mathbb{R}^n$;

In this course, we equip $\mathbb{R}^n$ with $\|\cdot\|_2$, unless stated otherwise.

INFINITY NORM : $\|v\|_\infty$

Let $v = \begin{pmatrix} v_1 \\ \vdots \\ v_n \end{pmatrix} \in \mathbb{R}^n$.

Then, the "infinity norm" of v, denoted as "$\|v\|_\infty$", is defined to be

$$\|v\|_\infty = \max\{|v_1|, |v_2|, \ldots, |v_n|\}.$$

$(\ell^p, \|\cdot\|_p)$ & $(\ell^\infty, \|\cdot\|_\infty)$ ARE NVS OF $\mathbb{R}^N$

1. Let $v = (v_1, v_2, \ldots) \in \mathbb{R}^N$ (ie let v be a sequence of reals).

Then, we let ℓ^p be defined by

$$\ell^p = \{ v \in \mathbb{R}^N : \|v\|_p < \infty \},$$

where

$$\|v\|_p = \left(\sum_{i=1}^{\infty} |v_i|^p \right)^{\frac{1}{p}}.$$

2. Similarly, we let ℓ^∞ be defined by

$$\ell^\infty = \{ v \in \mathbb{R}^N : \|v\|_\infty < \infty \},$$

where

$$\|v\|_\infty = \max\{|v_1|, |v_2|, \ldots\}.$$

3. Then, $(\ell^p, \|\cdot\|_p)$ & $(\ell^\infty, \|\cdot\|_\infty)$ are NVS.

UNIFORM NORM ON $\mathcal{C}([a,b])$: $\|f\|_\infty$

1. Let $V = \mathcal{C}([a,b])$, ie the set of all continuous functions $f : [a,b] \to \mathbb{R}$.

Then, the "uniform norm" of a $f \in V$, denoted as "$\|f\|_\infty$", is defined to be

$$\|f\|_\infty = \sup\{ |f(x)| : x \in [a,b] \}$$
$$= \max\{ |f(x)| : x \in [a,b] \} \quad \text{(by EVT)}$$

Unless otherwise stated, we assume the uniform norm is used if working with $\mathbb{R}^N$ as a NVS.

2. An alternative norm to $\mathbb{R}^N$ is the "integration-based" norm $\|f\|_p$, where $p \geq 1$ and $f \in V$, which is defined as

$$\|f\|_p = \left(\int_a^b |f(x)|^p \, dx \right)^{\frac{1}{p}}.$$

Module 1.2: Convergence

CONVERGENCE / DIVERGENCE

Let V be a NVS, and let the sequence $(a_n) \subseteq V$.
Then, we say (a_n) "converges" to some $a \in V$, denoted "$a_n \to a$", if for all $\varepsilon > 0$, there exists a $N \in \mathbb{N}$ such that $\|a_n - a\| < \varepsilon$ $\forall n \geq N$.

Otherwise, we say that (a_n) diverges.

eg[1] $V = \ell^\infty$, $(a_n) \subseteq V$

$a_n = (1, \frac{1}{2}, \dots, \frac{1}{n}, 0, 0, \dots)$

$a = (1, \frac{1}{2}, \frac{1}{3}, \dots)$

Claim: $a_n \to a$

Proof. Let $\varepsilon > 0$, and choose $N \in \mathbb{N}$ such that $\frac{1}{N} < \varepsilon$.
Then, for $n \geq N$, note that

$$\|a_n - a\|_\infty = \|(\underbrace{0, 0, \dots, 0}_{n}, \tfrac{1}{n+1}, -\tfrac{1}{n+2}, \dots)\|_\infty$$
$$= \sup\{0, \tfrac{1}{n+1}, \tfrac{1}{n+2}, \dots\}$$
$$= \tfrac{1}{n+1} < \tfrac{1}{n} < \tfrac{1}{N} < \varepsilon,$$

showing that the sequence converges. ▨

eg[2] $V = \ell^\infty$, $(a_n) \subseteq V$

$a_n = (\underbrace{1, \dots, 1}_{n}, 0, 0, \dots)$

$a = (1, 1, \dots, 1, \dots)$

Claim: $a_n \not\to a$.

Proof. $\forall n \in \mathbb{N}$: $\|a_n - a\|_\infty = 1$
(since always 1 present in the sequence $(a_n - a)$).

BOUNDED (SUBSET)

Let $A \subseteq V$, where V is a NVS.
Then, we say A is "bounded" if there exists a $M > 0$ such that $\|a\| \leq M$ for all $a \in A$.

BOUNDED (SEQUENCES)

Let $(a_n) \subseteq V$, where V is a NVS.
Then, we say (a_n) is "bounded" if $\{a_1, \dots, a_n, \dots\}$ is itself bounded.

(a_n) IS CONVERGENT $\Rightarrow$ (a_n) IS BOUNDED

Let $(a_n) \subseteq V$ be so that (a_n) is convergent.
Then necessarily (a_n) is bounded.

Proof. Suppose $a_n \to a \in V$.
$\Rightarrow \exists N \in \mathbb{N}$ such that if $n \geq N$, then $\|a_n - a\| < 1$.
Then, notice that for $n \geq N$,

$$\|a_n\| = \|a_n - a + a\|$$
$$\leq \|a_n - a\| + \|a\|$$
$$\leq 1 + \|a\|.$$

Let $M = \max\{\|a_1\|, \dots, \|a_{N-1}\|, 1 + \|a\|\}$,
we have that $\|a_n\| \leq M$ $\forall n \in \mathbb{N}$, as needed. ▨

Note the converse is not necessarily true!
eg $(a_n) = (1, -1, 1, -1, \dots)$

$a_n \to a$, $b_n \to b \Rightarrow a_n + b_n \to a + b$

Let $(a_n), (b_n) \subseteq V$, and let $a_n \to a$ & $b_n \to b$.
Then necessarily $a_n + b_n \to a + b$.

$a_n \to a \Rightarrow \alpha a_n \to \alpha a$

Let $(a_n) \subseteq V$, and let $a_n \to a$.
Then necessarily $\alpha a_n \to \alpha a$.

Module 1.3: Completeness

CAUCHY SEQUENCE

Let $(a_n) \subseteq V$, where V is a NVS.
Then, we say (a_n) is a "Cauchy sequence" if for all $\varepsilon > 0$, there exists a $N \in \mathbb{N}$ such that $\|a_n - a_m\| < \varepsilon$ for all $n, m \geq N$.

(a_n) IS CONVERGENT $\Rightarrow$ (a_n) IS CAUCHY

Let $(a_n) \subseteq V$ be convergent.
Then necessarily (a_n) is also Cauchy.

Proof. Let $\varepsilon > 0$.
We know $N \in \mathbb{N}$, $a \in V$ such that $\|a_n - a\| < \frac{\varepsilon}{2}$ $\forall n \geq N$.
Also, for $n, m \geq N$, we know

$$\|a_n - a_m\| \leq \|a_n - a\| + \|a_m - a\| < \varepsilon,$$

as needed. ∎

(a_n) IS CAUCHY $\not\Rightarrow$ (a_n) IS CONVERGENT

Note that $(a_n) \subseteq V$ is Cauchy does not necessarily imply it is also convergent.

For example, take the NVS $(c_{00}, \|\cdot\|_\infty)$, where

$$c_{00} = \{(x_n) \in \ell^\infty : \exists N \ni x_n = 0 \ \forall n \geq N\},$$

and let $(a_n) \subseteq c_{00}$ be defined by

$$a_n = (1, \tfrac{1}{2}, \ldots, \tfrac{1}{n}, 0, 0, \ldots),$$

and let a be equal to

$$a = (1, \tfrac{1}{2}, \tfrac{1}{3}, \ldots) \notin c_{00}.$$

We know $a_n \to a$ in ℓ^∞, so $(a_n) \subseteq \ell^\infty$ is Cauchy.
$\Rightarrow (a_n) \subseteq c_{00}$ is still Cauchy.
However, since $a_n \to a \notin c_{00}$, and limits are unique, it follows that $(a_n) \subseteq c_{00}$ diverges.

COMPLETE

Let $A \subseteq V$, where V is a NVS.
Then, we say A is "complete" if whenever $(a_n) \subseteq A$ is Cauchy, it follows that there exists an $a \in A$ such that $a_n \to a$.

BANACH SPACE

Let V be a NVS.
Then, we say V is a "Banach space" if V is complete.

Module 1.4: Banach Spaces

$\mathbb{R}^n$ IS A BANACH SPACE WRT $\|\cdot\|_p$ & $\|\cdot\|_\infty$

1. First, let $v = \begin{pmatrix} v_1 \\ \vdots \\ v_n \end{pmatrix} \in \mathbb{R}^n$ & $1 \leq p \leq \infty$.

Then, note the following:

① $\|v\|_p^p = |v_1|^p + \dots + |v_n|^p$

$\leq n \max\{|v_1|, \dots, |v_n|\}^p$

$\therefore \|v\|_p^p = n\|v\|_\infty^p$;

② $\|v\|_\infty^p \leq |v_1|^p + \dots + |v_n|^p = \|v\|_p^p$.

③ So, $\|v\|_p \leq \sqrt[p]{n}\, \|v\|_\infty$, and

$\|v\|_\infty \leq \|v\|_p$.

2. We can also show $(\mathbb{R}^n, \|\cdot\|_\infty)$ is a Banach space.

<u>Proof</u>. Suppose $(a_k) \subseteq \mathbb{R}^n$ is Cauchy, say

$a_k = (a_k^{(1)}, \dots, a_k^{(n)})$,

for each $k \in \mathbb{N}$, where $a_i^{(j)} \in \mathbb{R}$.

Let $\varepsilon > 0$. We know $\exists N \in \mathbb{N}$ such that $\|a_k - a_\ell\|_\infty < \varepsilon$ for all $k, \ell \geq N$.

Then, for all $k, \ell \geq N$ and $1 \leq i \leq n$, we have that

$|a_k^{(i)} - a_\ell^{(i)}| \leq \|a_k - a_\ell\|_\infty < \varepsilon$,

and so it follows that $(a_k^{(i)})_{k=1}^\infty \subseteq \mathbb{R}$ is Cauchy.

Finally, since $\mathbb{R}$ is complete, it follows that

$a_k^{(i)} \to b_i \in \mathbb{R} \quad \forall 1 \leq i \leq n$.

To finish, we want to prove $a_k \to (b_1, \dots, b_n)$, which is sufficient to prove the statement in question.

Let $\varepsilon > 0$. Fix $1 \leq i \leq n$.

We know there exists a $N_i \in \mathbb{N}$ such that $|a_k^{(i)} - b_i| < \varepsilon \;\; \forall k \geq N_i$.

Let $N = \max\{N_1, N_2, \dots, N_n\}$, so that for $k \geq N$, we have that

$\left\|a_k - \begin{pmatrix} b_1 \\ \vdots \\ b_n \end{pmatrix}\right\|_\infty = \max\{|a_k^{(i)} - b_i| : 1 \leq i \leq n\}$

$\leq \varepsilon$,

completing the proof (as $a_k \to \begin{pmatrix} b_1 \\ \vdots \\ b_n \end{pmatrix}$). ▨

3. This is sufficient to prove $(\mathbb{R}^n, \|\cdot\|_p)$ is a Banach space.

<u>Proof</u>. Assume $(\mathbb{R}^n, \|\cdot\|_\infty)$ is a Banach space.

Let $1 \leq p < \infty$, and let $(a_k) \subseteq \mathbb{R}^n$ be Cauchy wrt $\|\cdot\|_p$.

Thus, (a_k) is Cauchy wrt $\|\cdot\|_\infty$, and so

$a_k \to a \in \mathbb{R}^n$ wrt $\|\cdot\|_\infty$.

Hence $a_k \to a$ wrt $\|\cdot\|_p$, and so $(\mathbb{R}^n, \|\cdot\|_p)$ is also a Banach space. ▨

ℓ^∞ IS A BANACH SPACE

We can prove ℓ^∞ is a Banach space.

<u>Proof</u>. Let $(a_n) \subseteq \ell^\infty$ be Cauchy.

For all $n \in \mathbb{N}$, we can write

$a_n = (a_n^{(1)}, a_n^{(2)}, \dots)$,

where $a_n^{(i)} \in \mathbb{R}$.

We claim for an $i \in \mathbb{N}$, that $(a_n^{(i)})$ is Cauchy.

<u>Proof</u>. Let $\varepsilon > 0$ be given.

Then, there exists a $N \in \mathbb{N}$ such that $\|a_n - a_m\|_\infty < \varepsilon$ $\forall n, m \geq N$.

Next, fix $i \in \mathbb{N}$. For $n, m \geq N$, note that

$|a_n^{(i)} - a_m^{(i)}| \leq \sup\{|a_n^{(i)} - a_m^{(i)}| : i \in \mathbb{N}\}$

$= \|a_n - a_m\|_\infty$

$< \varepsilon$,

proving the claim. #

So, by the completeness of $\mathbb{R}$, we have that

$a_n^{(i)} \to b_i \quad (\text{as } n \to \infty)$

for all $i \in \mathbb{N}$.

Next, we claim that $a_n \to b$, where $b = (b_1, b_2, \dots)$.

<u>Proof</u>. Let $\varepsilon > 0$ be given.

Then, we know $\exists N \in \mathbb{N}$ such that $\|a_n - a_m\|_\infty < \varepsilon$ $\forall n, m \geq N$.

We also have that

$|a_n^{(i)} - a_m^{(i)}| < \varepsilon \quad \forall n, m \geq N$ and $i \in \mathbb{N}$

by definition.

Next, taking $m \to \infty$, we note for $n \geq N$ that

$|a_n^{(i)} - b_i| \leq \varepsilon \quad \forall i \in \mathbb{N}$.

Thus $\|a_n - b\|_\infty < \varepsilon$ for all $n \geq N$, and so indeed

$a_n \to b$. #

It follows that ℓ^∞ is a Banach space, as needed. ▨

Module 2.1: Closed and Open Sets

CLOSED SET

Let V be a NVS, and let $C \subseteq V$.
Then, we say C is "closed" if whenever $(a_n) \subseteq C$ with $a_n \to a \in V$, then $a \in C$.

OPEN SET

Let V be a NVS, and let $U \subseteq V$.
Then, we say U is "open" if $V \setminus U$ is closed.

TOPOLOGY ON A SPACE

Let V be a NVS.
Then, the "topology" on V is defined to be the set
$$\mathcal{T} = \{U \subseteq V \mid U \text{ is open}\}.$$

$\emptyset, V$ ARE OPEN & CLOSED IN V

Note $\emptyset$ and V are always open and closed in V.

CLOSED BALL: $\overline{B_r(a)}$

1. Let $r > 0$ and $a \in V$, where V is a NVS.
Then, the "closed ball" of radius r centred at a, denoted as "$\overline{B_r(a)}$", is defined to be the set
$$\overline{B_r(a)} = \{x \in V : \|x - a\| \leq r\}.$$

2. We can prove $\overline{B_r(a)}$ is closed.

Proof. Let $(a_n) \subseteq \overline{B_r(a)} \ni a_n \to b \in V$.
By def^n, $\|a_n - a\| \leq r \quad \forall n \in \mathbb{N}$.
But since $a_n \to a$, it follows that
$$\|a_n - a\| \to \|b - a\|,$$
and so $\|b - a\| \leq \max\{\|a_n - a\| : n \in \mathbb{N}\} \leq r$,
and so $b \in \overline{B_r(a)}$, which is sufficient to prove the statement. ∎

OPEN BALL: $B_r(a)$

1. Let $r > 0$ and $a \in V$, where V is a NVS.
Then, the "open ball" of radius r centred at a, denoted as "$B_r(a)$", is defined to be the set
$$B_r(a) = \{x \in V : \|x - a\| < r\}.$$

2. We can show $B_r(a)$ is open.

Proof. By a similar proof to the above, we can show $\{x \in V : \|x - a\| \geq r\}$ is closed.
Hence
$$B_r(a) = V \setminus \{x \in V : \|x - a\| \geq r\}$$
is open. ∎

$V = \ell^\infty$, $c_0 = \{(x_n) \in \ell^\infty \mid x_n \to 0\}$ IS CLOSED

We can show $c_0 = \{(x_n) \in \ell^\infty \mid x_n \to 0\}$ is closed in $V = \ell^\infty$.

Proof. Let $(a_n) \subseteq c_0 \ni a_n \to a \in \ell^\infty$.
Let $a_n = (a_n^{(1)}, a_n^{(2)}, \dots) \quad \forall n \in \mathbb{N}$.
Hence, we know
$$\lim_{k \to \infty} a_n^{(k)} = 0 \quad \forall n \in \mathbb{N}.$$
Then, say $a = (b_1, b_2, \dots)$, and let $\varepsilon > 0$.
We know there exists $N_1, N_2 \in \mathbb{N} \ni$

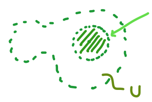
1. $\|a_n - a\|_\infty < \frac{\varepsilon}{2} \quad \forall n \geq N_1$; and
2. $|a_{N_1}^{(k)}| < \frac{\varepsilon}{2} \quad \forall k \geq N_2$.

Finally, for $k \geq N_2$, note that
$$\begin{aligned} |b_k| &= |a_{N_1}^{(k)} - b_k - a_{N_1}^{(k)}| \\ &\leq |a_{N_1}^{(k)} - b_k| + |a_{N_1}^{(k)}| \\ &\leq \|a_{N_1} - a\|_\infty + |a_{N_1}^{(k)}| \\ &< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon, \end{aligned}$$
showing $b_k \to 0$ and so $a = (b_1, b_2, \dots) \in c_0$, and so c_0 is closed. ∎

U IS OPEN $\iff \forall a \in U: \exists r > 0 \ni B_r(a) \subseteq U$

Let V be a NVS, and let $U \subseteq V$.
Then, U is open if and only if for any $a \in U$, there exists a $r > 0$ such that $B_r(a) \subseteq U$.

eg in $\mathbb{R}^2$



For any arbitrary point, you can draw a "circle" that is contained in U.

Proof. ($\Rightarrow$) Assume U is open, so $V \setminus U$ is closed.
Suppose, for a contradiction, that
$$\exists a \in U \ni (\nexists r > 0 \ni B_r(a) \subseteq U).$$
In particular, $\forall n \geq \mathbb{N}$, there exists some $a_n \in B_{\frac{1}{n}}(a)$ such that $a_n \notin U$.
Note that

1. $\|a_n - a\| < \frac{1}{n} \implies a_n \to a$ (since $\frac{1}{n} \to 0$); and
2. $(a_n) \subseteq V \setminus U$, and $V \setminus U$ is closed
$\implies a \in V \setminus U$.

But $a \in U$ by assumption, a contradiction.
Thus $\forall a \in U: \exists r > 0 \ni B_r(a) \subseteq U$, as needed. #

($\Leftarrow$) Assume $\forall a \in U: \exists r > 0 \ni B_r(a) \subseteq U$.
We claim $V \setminus U$ is closed. Indeed, let
$$(a_n) \subseteq V \setminus U \ni a_n \to a \in V.$$
Suppose $a \in U$, so in particular $\exists r > 0 \ni B_r(a) \subseteq U$.
But since $a_n \to a$, $\exists N \in \mathbb{N} \ni \|a_N - a\| < r$,
implying $a_N \in B_r(a) \subseteq U$, and so $a_N \in U$.
However we assumed $a_N \in V \setminus U$, so this is a contradiction!
Hence $V \setminus U$ is closed, and so U is open, as needed.

Module 2.2-2.4: Closure and Interior

UNION OF OPEN SETS IS OPEN

Let V be a NVS, and let $\{U_\alpha\}_{\alpha\in I}$ be a collection of open sets in V.
Then necessarily $U = \bigcup_{\alpha\in I} U_\alpha$ is open.

INTERSECTION OF CLOSED SETS IS CLOSED

Let V be a NVS, and let $\{C_\alpha\}_{\alpha\in I}$ be a collection of closed sets in V.
Then necessarily $C = \bigcap_{\alpha\in I} C_\alpha$ is closed.

FINITE INTERSECTION OF OPEN SETS IS OPEN

Let $U_1, \dots, U_n \subseteq V$ be open.
Then necessarily $U = U_1 \cap \dots \cap U_n$ is open.

FINITE UNION OF CLOSED SETS IS CLOSED

Let $C_1, \dots, C_n \subseteq V$ be closed.
Then necessarily $C = C_1 \cup \dots \cup C_n$ is closed.

CLOSURE OF A SUBSET: $\overline{A}$

Let $A \subseteq V$.
Then, we define the "closure" of A, denoted as "$\overline{A}$", to be the set

$$\overline{A} = \bigcap_{\substack{A\in C, \\ C \text{ closed}}} C.$$

* $\overline{A}$ is the smallest closed set containing A.

INTERIOR OF A SUBSET: Int(A)

Let $A \subseteq V$.
Then, we define the "interior" of A, denoted as "$\mathrm{int}(A)$", to be the set

$$\mathrm{Int}(A) = \bigcup_{\substack{U\subseteq A, \\ U \text{ open}}} U.$$

* $\mathrm{Int}(A)$ is the largest open set contained in A.

LIMIT POINT

Let V be a NVS, and let $A \subseteq V$.
Then, we say $a \in V$ is a "limit point" of A if there exists a $(a_n) \subseteq V$ with $a_n \to a$.

INTERIOR POINT

Let V be a NVS, and let $A \subseteq V$.
Then, we say $a \in V$ is an "interior point" of A if there exists a $r>0$ such that $B_r(a) \subseteq A$.

$\overline{A} = \{\text{limit points of } A\}$

We can show that $\overline{A}$ is the set of limit points of A for any $A \subseteq V$.

Proof. Let $X = \{\text{limit points of } A\}$.

Claim: X is closed.

↳ Proof: Let $(a_n) \subseteq X \ni a_n \to a \in V$.
In particular, we know

$$\forall n \in \mathbb{N},\ \exists b_n \in A \ni |a_n - b_n| < \tfrac{1}{n}.$$

But we also know

$$b_n = b_n - a_n + a_n \to 0 + a = a,$$

and so $a \in X$, showing X is closed. #

By definition, we know $\overline{A} \subseteq X$, since $\overline{A}$ is the smallest closed set that contains A.
Now, let $x \in X$, so that $\exists (a_n) \subseteq A \ni a_n \to x$.
Let $C \subseteq V$ be closed, so that $A \subseteq C$.
Then, note $(a_n) \subseteq C$, and so $x \in C$ (since C is closed).
Thus x is in any closed set containing A, and so $X \subseteq \overline{A}$ (and thus $X = \overline{A}$, as needed). ▣

$\mathrm{Int}(A) = \{\text{interior points of } A\}$

Similarly, we can show $\mathrm{int}(A)$ is the set of interior points of A for any $A \subseteq V$.

Proof. Similar to above.

$A = \{(a_n) \in \ell^1 : a_n \in \mathbb{Q}\}$: $\overline{A} = \ell^1$

We claim $\overline{A} = \ell^1$.

Note: for $x \in \ell^1$, assume $\forall \varepsilon>0,\ \exists a \in A \ni \|x-a\|_1 < \varepsilon$.
Then, $x \in \overline{A}$.
Why? $\because \forall n \in \mathbb{N} : \exists a_n \in A \ni \|x - a_n\|_1 < \frac{1}{n}$.
$\Rightarrow (a_n) \subseteq A$ and so $a_n \to x$.

Proof. Let $x = (x_1, x_2, \dots) \in \ell^1$ and let $\varepsilon > 0$.
By the density of $\mathbb{Q}$,

$$\forall n \in \mathbb{N} : \exists y_n \in \mathbb{Q} \ni |x_n - y_n| < \frac{\varepsilon}{2^n}.$$

Consider $y = (y_1, y_2, \dots)$.
Then, note that

$$\|x-y\|_1 = \sum_{n=1}^{\infty} |x_n - y_n| < \sum_{n=1}^{\infty} \frac{\varepsilon}{2^n} = \varepsilon,$$

which is sufficient to show $x \in \overline{A}$, and so $\ell^1 \subseteq \overline{A}$.
As $\overline{A} \subseteq \ell^1$ by definition, it follows that $\overline{A} = \ell^1$, as needed. ▣

$V = \ell^\infty$; $\overline{c_{00}} = c_0$

We can show $\overline{c_{00}} = c_0$ in $V = \ell^\infty$.

Proof. We know $c_{00} \subseteq c_0$, and since c_0 is closed,
it follows that $\overline{c_{00}} \subseteq c_0$.
Now, let $x = (x_1, x_2, \dots) \in c_0$, and let $\varepsilon > 0$.
Since $x_n \to 0$, it follows that

$$\exists N \in \mathbb{N} \ni |x_n| < \tfrac{\varepsilon}{2}\ \forall n \geq N.$$

Let $y = (x_1, \dots, x_{N-1}, 0, 0, \dots) \in c_{00}$.
It follows that

$$\|x-y\|_\infty = \|(0, \dots, 0, x_N, x_{N+1}, \dots)\|_\infty = \sup\{|x_k| : k \geq N\} \leq \frac{\varepsilon}{2} < \varepsilon,$$

and so necessarily $x \in \overline{c_{00}}$, and so $c_0 \subseteq \overline{c_{00}}$.
Thus $c_0 = \overline{c_{00}}$, as needed. ▣

$\overline{A\cup B} = \overline{A} \cup \overline{B}$

Let $A, B \subseteq V$.
Then necessarily $\overline{A\cup B} = \overline{A} \cup \overline{B}$.

Proof. Since $\overline{A}\cup\overline{B}$ is closed & $A\cup B \subseteq \overline{A}\cup\overline{B}$,
it follows $\overline{A\cup B} \subseteq \overline{A}\cup\overline{B}$.
Then, since $A, B \subseteq A\cup B$, it follows that
$\overline{A}, \overline{B} \subseteq \overline{A\cup B}$.
Thus $\overline{A}\cup\overline{B} \subseteq \overline{A\cup B}$, and so $\overline{A}\cup\overline{B} = \overline{A\cup B}$. ■

$\mathrm{Int}(A\cap B) = \mathrm{Int}(A) \cap \mathrm{Int}(B)$

Let $A, B \subseteq V$.
Then necessarily $\mathrm{Int}(A\cap B) = \mathrm{Int}(A) \cap \mathrm{Int}(B)$.

$\overline{A\cap B} \subseteq \overline{A} \cap \overline{B}$

Let $A, B \subseteq V$.
Then necessarily $\overline{A\cap B} \subseteq \overline{A} \cap \overline{B}$.

$\mathrm{Int}(A\cup B) \supseteq \mathrm{Int}(A) \cup \mathrm{Int}(B)$

Let $A, B \subseteq V$.
Then necessarily $\mathrm{Int}(A\cup B) \supseteq \mathrm{Int}(A) \cup \mathrm{Int}(B)$.

$A = (0,1)$, $B = (1,2)$, $V = \mathbb{R}$: $\overline{A\cap B} \neq \overline{A}\cap\overline{B}$

Let $V = \mathbb{R}$, and let $A = (0,1)$, $B = (1,2)$.
Then note that $\overline{A\cap B} \neq \overline{A}\cap\overline{B}$.

Proof. See that
$\overline{A\cap B} = \overline{\emptyset} = \emptyset$ but $\overline{A}\cap\overline{B} = [0,1]\cap[1,2] = \{1\}$. ■

$A = [0,1]$, $B = [1,2]$, $V = \mathbb{R}$: $\mathrm{Int}(A\cup B) \neq \mathrm{Int}(A) \cup \mathrm{Int}(B)$

Let $V = \mathbb{R}$, and let $A = [0,1]$, $B = [1,2]$.
Then note that $\mathrm{Int}(A\cup B) \neq \mathrm{Int}(A) \cup \mathrm{Int}(B)$.

Proof. See that
$\mathrm{Int}(A\cup B) = (0,2)$ but $\mathrm{Int}(A)\cup\mathrm{Int}(B) = (0,1)\cup(1,2) = (0,2)\setminus\{1\}$.

$\mathrm{Int}(V\setminus A) = V\setminus\overline{A}$

Let $A \subseteq V$.
Then necessarily $\mathrm{Int}(V\setminus A) = V\setminus\overline{A}$.

Proof. Since $V\setminus\overline{A} \subseteq V\setminus A$ and $V\setminus\overline{A}$ is open,
it follows (by the largeness of the interior)
that $V\setminus\overline{A} \subseteq \mathrm{Int}(V\setminus A)$.
Then, observe that
$V\setminus\mathrm{Int}(V\setminus A) \supseteq V\setminus(V\setminus A) = A$,
and since $\mathrm{Int}(V\setminus A)$ is open, hence $V\setminus\mathrm{Int}(V\setminus A)$
is closed; thus A is closed, and so
$\overline{A} \subseteq V\setminus\mathrm{Int}(V\setminus A)$.
Hence
$V\setminus\overline{A} \supseteq V\setminus(V\setminus\mathrm{Int}(V\setminus A)) = \mathrm{Int}(V\setminus A)$,
and so necessarily $V\setminus\overline{A} = \mathrm{Int}(V\setminus A)$, as needed. ■

$\overline{V\setminus A} = V\setminus\mathrm{Int}(A)$

Let $A \subseteq V$.
Then necessarily $\overline{V\setminus A} = V\setminus\mathrm{Int}(A)$.

BOUNDARY OF A SUBSET: $\partial(A)$

Let $A \subseteq V$.
Then, the "boundary" of A, denoted as "$\partial(A)$",
is defined to be the set
$\partial(A) = \overline{A} \setminus \mathrm{Int}(A)$.

$\partial(A)$ IS CLOSED

Let $A \subseteq V$.
Then necessarily $\partial(A)$ is closed.

Proof. $\partial(A) = \overline{A}\setminus\mathrm{Int}(A)$
$= \underbrace{\overline{A}}_{\text{closed}} \cap \underbrace{(V\setminus\mathrm{Int}(A))}_{\text{closed}}$,
and so $\partial(A)$ is closed. ■

A IS CLOSED $\Leftrightarrow$ $\partial(A) \subseteq A$

Let $A \subseteq V$.
Then necessarily A is closed if and only if $\partial(A) \subseteq A$.

Proof. ($\Rightarrow$) A is closed $\Rightarrow$ $\partial(A) \subseteq \overline{A} = A$.
($\Leftarrow$) Suppose $\partial(A) \subseteq A$.
Recall $\partial(A) = \overline{A}\setminus\mathrm{Int}(A)$; in particular, we can write
$\overline{A} = \partial(A) \cup \mathrm{Int}(A)$,
and since $\partial(A), \mathrm{Int}(A) \subseteq A$ it follows that
$\overline{A} \subseteq A$.
As $\overline{A} \supseteq A$ it follows that $A = \overline{A}$, and so A is closed. ■

Module 3: Compactness & Open Covers

BOUNDED (SETS)

Let $A \subseteq V$, where V is a NVS.
Then, we say that A is "bounded" if there exists a $M \in \mathbb{N}$ such that

$\|a\| < M \quad \forall a \in A$.

COMPACT (SUBSETS)

Let $C \subseteq V$, where V is a NVS.
Then, we say C is "compact" if every $(a_n) \subseteq C$ has a subsequence $(a_{n_k}) \subseteq (a_n)$ with $a_{n_k} \to a \in C$.

$A \subseteq \mathbb{R}^n$ IS CLOSED & BOUNDED $\Rightarrow$ A IS COMPACT

Let $A \subseteq \mathbb{R}^n$ be closed and bounded.
Then necessarily A is compact.

Proof. Let $(a_k) \subseteq A$.
Since A is bounded, $\Rightarrow (a_k)$ is bounded.
By A2, we know $\exists (a_{k_l}) \subseteq (a_k) \ni a_{k_l} \to a \in \mathbb{R}^n$.
Since A is closed, it follows that $a \in A$,
and so A is compact. ∎

$A = \{e_1 = (1,0,\dots,0),\ e_2 = (0,1,0,\dots 0),\ \dots\} \subseteq \ell^\infty$: A IS NOT COMPACT

Let A be the above set. We can show A is not compact.

Proof. $\forall n \neq m$, $\|e_n - e_m\|_\infty = 1$
$\Rightarrow (e_n)$ is not Cauchy
$\Rightarrow (e_n)$ is not convergent
$\Rightarrow A$ is not compact. ∎

Then, notice that
$A = (e_1, e_2, \dots) \subseteq \overline{B_1(0)}$.
The RHS is trivially closed and bounded, but since A is not compact, necessarily $\overline{B_1(0)}$ cannot be compact!

Therefore, closed & bounded does not automatically imply compactness.

C IS COMPACT $\Rightarrow$ C IS CLOSED & BOUNDED

Let $C \subseteq V$ be compact.
Then necessarily C is closed and bounded.

Proof. Let $C \subseteq V$ be compact.
① We claim C is closed.
Proof. Let $(a_n) \subseteq C \ni a_n \to a \in V$.
$\Rightarrow \exists (a_{n_k}) \subseteq (a_n) \ni a_{n_k} \to b \in C$.
However, we must have $a = b \in C$,
and so C is closed. #
② We claim C is bounded.
Proof. Suppose this was not the case.
$\Rightarrow \exists n \in \mathbb{N}$, $\exists a_n \in C \ni \|a_n\| \geq n$.
Then consider $(a_n) \subseteq C$.
$\Rightarrow \exists (a_{n_k}) \subseteq (a_n) \ni a_{n_k} \to a \in C$.
But $\|a_{n_k}\| \geq n_k$! So it is unbounded,
and so has to be divergent.
This is a contradiction!
Hence C is bounded, as needed. ∎

HEINE-BOREL THEOREM: $C \in \mathbb{R}^n$ IS COMPACT $\Leftrightarrow$ C IS CLOSED & BOUNDED

Let $C \in \mathbb{R}^n$.
Then C is compact if and only if it is also closed & bounded.
(This has been proved by previous observations!)

$C \subseteq V$ IS COMPACT, $A \subseteq C$ IS CLOSED $\Rightarrow$ A IS COMPACT

Let $C \subseteq V$ be compact, and let $A \subseteq C$ be closed.
Then necessarily A is compact.

Proof. Let $(a_n) \subseteq A$. Since $A \subseteq C$,
$\exists (a_{n_k}) \subseteq (a_n) \ni a_{n_k} \to a \in C$.
But as A is closed $\Rightarrow a \in A$, and
so A is compact. ∎

OPEN COVERS (OF SUBSETS)

Let $A \subseteq V$, where V is a NVS.
Then, an "open cover" of A is a collection of open sets $\{U_\alpha : \alpha \in I\}$ such that

$$A \subseteq \bigcup_{\alpha \in I} U_\alpha.$$

We say the open cover is "finite" if $|I| < |\mathbb{N}|$.

Examples:
① $V = \mathbb{R}$, $A = [0,1]$, $A \subseteq (-\frac{1}{4}, \frac{1}{4}) \cup (0, \frac{1}{2}) \cup \dots \cup (\frac{3}{4}, \frac{5}{4})$
② $V = \mathbb{R}^2$, $A = \mathbb{Z} \times \mathbb{Z}$, $A \subseteq \bigcup_{\alpha \in \mathbb{Z}\times\mathbb{Z}} B_{\frac{1}{2}}(\alpha)$
③ $V = \mathbb{R}$, $A = (0,1]$, $A \subseteq \bigcup_{n \in \mathbb{N}} (\frac{1}{n}, 2)$.

SUBCOVERS

Let $\{U_\alpha : \alpha \in I\}$ be an open cover of $A \subseteq V$.
Then, we say a subset of $\{U_\alpha : \alpha \in I\}$ is a "subcover" of $\{U_\alpha : \alpha \in I\}$.

Note that subcovers are also open covers of A.

$A \subseteq V$ COMPACT, $A \subseteq \bigcup_{\alpha \in I} U_\alpha$ IS AN OPEN COVER $\Rightarrow \exists R > 0 \ni \forall a \in A$: $B_R(a) \subseteq U_\alpha$ FOR SOME $\alpha \in I$

Let $A \subseteq V$ be compact, and let $A \subseteq \bigcup_{\alpha \in I} U_\alpha$ be an open cover of A.

Then, there necessarily exists some $R > 0$ such that for each $a \in A$, we have

$$B_R(a) \subseteq U_\alpha$$

for some $\alpha \in I$.

Proof. Suppose no such $R > 0$ exists.

In particular, $\forall n \in \mathbb{N}$: $\exists a_n \in A \ni B_{\frac{1}{n}}(a_n) \nsubseteq U_\alpha$ $\forall \alpha \in I$.

Since $(a_n) \subseteq A$ and A is compact,

$$\Rightarrow \exists (a_{n_k}) \subseteq (a_n) \ni a_{n_k} \to a \in A.$$

Say $a \in U_{\alpha_0}$, where $\alpha_0 \in I$ (since the union is an open cover).

Pick $M \in \mathbb{N} \ni B_{\frac{2}{M}}(a) \subseteq U_{\alpha_0}$

Moreover, since $a_{n_k} \to a$, we may find $N \in \mathbb{N}$

$\ni a_{n_k} \in B_{\frac{1}{M}}(a)$ $\forall k \geq N$.

Then, for $k \geq N$ such that $n_k > M$, take

$$x \in B_{\frac{1}{M}}(a_{n_k}).$$

But

$$\begin{aligned} \Rightarrow \|x - a\| &= \|x - a_{n_k} + a_{n_k} - a\| \\ &\leq \|x - a_{n_k}\| + \|a_{n_k} - a\| \\ &< \tfrac{1}{M} + \tfrac{1}{M} = \tfrac{2}{M}, \end{aligned}$$

and so $x \in B_{\frac{2}{M}}(a)$.

$$\therefore B_{\frac{1}{M}}(a_{n_k}) \subseteq B_{\frac{2}{M}}(a) \subseteq U_{\alpha_0}.$$

But since $n_k > M$, it follows that

$$B_{\frac{1}{n_k}}(a_{n_k}) \subseteq B_{\frac{1}{M}}(a_{n_k}) \subseteq U_{\alpha_0},$$

which is a contradiction to our earlier assumption that $B_{\frac{1}{n}}(a_n) \nsubseteq U_\alpha$ $\forall \alpha \in I$. ∎

$A \subseteq V$ IS COMPACT $\Rightarrow$ EVERY OPEN COVER OF A HAS A FINITE SUBCOVER

Let $A \subseteq V$ be compact.

Then necessarily every open cover of A has a finite subcover.

Proof. Suppose $A \subseteq V$ is compact.

Let $A \subseteq \bigcup_{\alpha \in I} U_\alpha$ be an open cover of A.

Since A is compact, by the above lemma,

$$\exists R > 0 \ni \forall a \in A: B_R(a) \subseteq U_\alpha \text{ for some } \alpha \in I.$$

If $\exists a_1, \dots, a_n \in A \ni A \subseteq B_R(a_1) \cup \dots \cup B_R(a_n)$, by the lemma we are done.

So, suppose no such covering existed.

Then, we can find a $a_1 \in A$, $a_2 \in A \ni a_2 \notin B_R(a_1)$, $a_3 \in A \ni a_3 \notin B_R(a_1) \cup B_R(a_2)$, ...

Since $(a_n) \subseteq A$ & A is compact,

$$\exists (a_{n_k}) \subseteq (a_n) \ni a_{n_k} \to a.$$

However, for $n < m$, we have that

$$a_m \notin B_R(a_n),$$

or in other words,

$$\|a_m - a_n\| \geq R.$$

$\Rightarrow (a_n)$ has no Cauchy subsequences,

$\Rightarrow (a_n)$ has no convergent subsequences,

giving us our contradiction. ∎

EVERY OPEN COVER OF $A \subseteq V$ HAS A FINITE SUBCOVER, & $A \subseteq \bigcup_{\alpha \in I} U_\alpha$ WHERE EACH U_α IS RELATIVELY OPEN IN $A \Rightarrow \exists \alpha_1, \dots, \alpha_n \in I \ni A \subseteq U_{\alpha_1} \cup \dots \cup U_{\alpha_2}$

Why is the above lemma true?

Proof. $A \subseteq \bigcup_{\alpha \in I} U_\alpha$, where $U_\alpha = A \cap O_\alpha$, $O_\alpha \subseteq V$ open

$$\begin{aligned} \Rightarrow A &\subseteq \bigcup_\alpha (A \cap O_\alpha) \\ &= A \cap \left(\bigcup_\alpha O_\alpha\right) \\ &\subseteq \bigcup_\alpha O_\alpha, \end{aligned}$$

$$\Rightarrow A \subseteq O_{\alpha_1} \cup \dots \cup O_{\alpha_n}$$

$$\Rightarrow A \subseteq \underbrace{U_{\alpha_1} \cup \dots \cup U_{\alpha_n}}_{\text{the relatively open cover.}}$$ ∎

EVERY OPEN COVER OF A HAS A FINITE SUBCOVER $\Rightarrow$ $A \subseteq V$ IS COMPACT

Let $A \subseteq V$, and suppose every open cover of A has a finite subcover.

Then necessarily $A \subseteq V$ is compact.

Proof. Let $(a_n) \subseteq A$.

For $k \in \mathbb{N}$, consider $C_k = \overline{(a_n)_{n=k}^{\infty}} \cap A$.

We want to show $\bigcap_{k=1}^{\infty} C_k \neq \emptyset$.

Each C_k is relatively closed in A. Hence every $U_k = A \setminus C_k$ is relatively open.

For contradiction, assume $\bigcap_{k=1}^{\infty} C_k = \emptyset$.

$$\begin{aligned} \Rightarrow A &= A \setminus \emptyset \\ &= A \setminus \left(\bigcap_{k=1}^{\infty} C_k\right) \\ &= \bigcup (A \setminus C_k) \\ &= \bigcup_{k=1}^{\infty} U_k \leftarrow \text{the relatively open subcover.} \end{aligned}$$

By the lemma above,

$$\exists i_1, \dots, i_\ell \ni A \subseteq U_{i_1} \cup \dots \cup U_{i_\ell}.$$

Since $C_1 \supseteq C_2 \supseteq \cdots$, we have that $U_1 \subseteq U_2 \subseteq \cdots$;

$$\therefore A \subseteq U_{i_\ell} \subseteq A,$$

and so $A = U_{i_\ell}$.

$$\Rightarrow C_{i_\ell} = A \setminus U_{i_\ell} = A \setminus A = \emptyset.$$

But since $a_{i_\ell} \in C_{i_\ell} = \emptyset$, this is a contradiction!

Hence, we may find some $a \in \bigcap_{k=1}^{\infty} C_k$.

$$\therefore \exists n_1 < n_2 < \dots \ni \|a_{n_k} - a\| < \tfrac{1}{k} \quad \forall k \in \mathbb{N},$$

and so $(a_{n_k}) \subseteq A \ni a_{n_k} \to a \in A$, showing A is compact, as needed. ∎

Module 4.1: Limits

LIMIT

1. Let $f: A \to W$, where $A \subseteq V$, and let $a \in V$.
Then, the "limit" of $f(x)$ as x approaches a is $w \in W$ if
(1) $a \in \overline{A \setminus \{a\}}$; and
(2) $\forall \varepsilon > 0 : \exists \delta > 0$ such that if $x \in A$ with $0 < \|x - a\| < \delta$, then $\|f(x) - w\| < \varepsilon$.

2. In this case, we write
$\lim_{x \to a} f(x) = w$.

* note that w is unique.

ISOLATED POINT

1. Let $a \in A$, where $A \subseteq V$.
Then, we call a an "isolated point with respect to A" if $a \notin \overline{A \setminus \{a\}}$.

2. If $a \notin \overline{A \setminus \{a\}}$, then there exists $r > 0$ such that $B_r(a) \cap A = \{a\}$ or $\emptyset$.
In other words, there does not exist $x \in A$ with $0 < \|x - a\| < r$.

LIMITS PRESERVE ORDER

Let $A \subseteq V$, and let $f, g, h : A \to \mathbb{R}$ and $a \in \overline{A \setminus \{a\}}$.
Suppose $\lim_{x \to a} f(x)$ & $\lim_{x \to a} g(x)$ exist and $f(x) \leq g(x)$ $\forall x \in A$.
Then necessarily $\lim_{x \to a} f(x) \leq \lim_{x \to a} g(x)$.

SQUEEZE THEOREM

Let $A \subseteq V$, and let $f, g, h : A \to \mathbb{R}$ and $a \in \overline{A \setminus \{a\}}$.
Suppose $f(x) \leq g(x) \leq h(x)$ $\forall x \in A$ and
$\lim_{x \to a} f(x) = \lim_{x \to a} h(x) = L$.
Then necessarily $\lim_{x \to a} g(x) = L$ as well.

LIMITS OF MULTIVARIABLE FUNCTIONS

eg[1] Evaluate

$$\lim_{(x,y,z) \to (0,0,0)} \frac{xy^2 + x^2 z + xyz}{\sqrt{x^2 + y^2 + z^2}}.$$

Sol$^{\underline{n}}$. If $x \neq 0$, then observe that

$$0 \leq \left| \frac{xy^2 + x^2 z + xyz}{\sqrt{x^2 + y^2 + z^2}} \right| \leq \frac{|xy^2 + x^2 z + xyz|}{\sqrt{x^2}}$$
$$= \frac{|xy^2 + x^2 z + xyz|}{|x|}$$
$$\leq \frac{|x||y|^2 + x^2 |z| + |x||y||z|}{|x|}$$
$$= y^2 + |x||z| + |y||z|.$$

If $x = 0$, then $f(x,y,z) = 0$.
Since
$$\lim_{(x,y,z) \to (0,0,0)} y^2 + |x||z| + |y||z| = 0,$$
by S.T it follows that $\lim_{(x,y,z) \to (0,0,0)} f(x,y,z) = 0$. #

eg[2] Evaluate

$$\lim_{(x,y) \to (0,0)} \frac{xy^2}{x^2 + y^4}. \quad \} f(x,y)$$

Sol$^{\underline{n}}$. As $(\frac{1}{n}, 0) \to (0,0)$, we see that
$f(\frac{1}{n}, 0) = 0 \to 0$.
As $(\frac{1}{n^2}, \frac{1}{n}) \to (0,0)$, we see that
$$f(\tfrac{1}{n^2}, \tfrac{1}{n}) = \frac{\frac{1}{n^4}}{\frac{1}{n^4} + \frac{1}{n^4}} = \frac{1}{2} \to \frac{1}{2}.$$
Since $0 \neq \frac{1}{2}$, the limit does not exist. #

Module 4.2: Continuity

CONTINUOUS (FUNCTIONS)

1. Let $f: A \to W$, where $A \subseteq V$.
Then, we say f is "continous" at $a \in A$ if for any $\varepsilon > 0$, there exists $\delta > 0$ such that if $x \in A$ with $\|x - a\| < \delta$, then $\|f(x) - f(a)\| < \varepsilon$.

2. Note that f is continuous at $a \in \overline{A \setminus \{a\}}$ if and only if $\lim\limits_{x \to a} f(x) = f(a)$.

3. However, if $a \notin \overline{A \setminus \{a\}}$, then f is automatically continuous at a.
Why? $\Rightarrow$ $\exists r > 0$, $B_r(a) \cap A = \{a\}$
Let $\varepsilon > 0$, & chose $\delta = r$.
If $x \in A$ & $\|x - a\| < \delta$, then $x = a$.
$\therefore$ $\|f(x) - f(a)\| = \|f(a) - f(a)\| = 0 < \varepsilon$

4. We say f is continuous if f is continuous at all $a \in A$.

f IS CTS $\Leftrightarrow$ f PRESERVES CONVERGENCE $\Leftrightarrow$ $\forall$ OPEN $U \subseteq W$, $f^{-1}(U)$ IS RELATIVELY OPEN IN A

Let $f: A \to W$, where $A \subseteq V$. Then, the following are equivalent:
① f is continuous;
② f preserves convergence; and
③ $\forall$ open $U \subseteq W$, $f^{-1}(U)$ is relatively open in A.

Proof. ③ $\Leftrightarrow$ ③ from Assignment 2.
①$\Rightarrow$②: Suppose f is cts, and let $(a_n) \subseteq A \ni a_n \to a \in A$.
Let $\varepsilon > 0$. There exists $\delta > 0 \ni x \in A$ & $\|x - a\| < \delta$,
then $\|f(x) - f(a)\| < \varepsilon$.
Take $N \in \mathbb{N} \ni \|a_n - a\| < \delta$ $\forall n \geq N$.
But then, for $n \geq N$, we see that $\|f(a_n) - f(a)\| < \varepsilon$,
showing that $f(a_n) \to f(a)$. #

②$\Rightarrow$①: Assume f preserves convergence, and suppose f is discontinuous at a.
$\Rightarrow$ $\exists \varepsilon > 0$ & $(a_n) \subseteq A \ni \|a_n - a\| < \frac{1}{n}$ but $\|f(a_n) - f(a)\| \geq \varepsilon$.
Then, $a_n \to a$ but $f(a_n) \not\to f(a)$ — contradiction! ☒

PROJECTION MAP IS CTS

1. The "i^{th} projection map" $P_i: \mathbb{R}^n \to \mathbb{R}$, where $1 \leq i \leq n$, is defined by
$P_i(x_1, \dots, x_n) = x_i$.

2. We can prove the projection map is continuous for any $1 \leq i \leq n$.

Proof. Let $(a_k) \subseteq \mathbb{R}^n \ni a_k \to a \in \mathbb{R}^n$, say $a_k = (a_k^{(1)}, \dots, a_k^{(n)})$ and $a = (b_1, \dots, b_n)$.
We know $a_k^{(i)} \to b_i$ $\forall 1 \leq i \leq n$ as $k \to \infty$;
$\Rightarrow$ $P_i(a_i) \to P_i(a)$
$\therefore$ P_i is cts. ☒

$f, g: A \to W$ ARE CTS $\Rightarrow$ $f+g$, αf $(\alpha \in \mathbb{R})$ ARE CTS

Let $f, g: A \to W$ be continuous, and let $\alpha \in \mathbb{R}$.
Then $f+g$ and αf are necessarily also continuous.

Proof. Let $(a_n) \subseteq A \ni a_n \to x$.
$\Rightarrow$ (Since f, g are cts) $\Rightarrow$ $f(a_n) \to f(a)$ & $g(a_n) \to g(a)$.
$\Rightarrow$ $f(a_n) + g(a_n) \to f(a) + g(a)$ & $\alpha f(a_n) \to \alpha f(a)$. ☒

$f: A \to W_1$, $g: B \to W_2$, $B \subseteq W_1$ ARE CTS $\Rightarrow$ $g \circ f$ IS CTS

Let $f: A \to W_1$ and $g: B \to W_2$ be continuous, where $B \subseteq W_1$.
Then necessarily $(g \circ f)$ is continuous.

Proof. Let $(a_n) \subseteq A \ni a_n \to a$.
Since f is cts, $\Rightarrow$ $f(a_n) \to f(a)$.
Since g is cts, $\Rightarrow$ $g(f(a_n)) \to g(f(a))$.
$\Rightarrow$ $g \circ f$ preserves convergence
$\Rightarrow$ $g \circ f$ is continuous. ☒

Module 4.3: Uniform Continuity

UNIFORM CONTINUITY

Let $f: A \to V$, where $f: A \to W$.
Then, we say f is "uniformly continuous"
if for any $\varepsilon > 0$, there exists a $\delta > 0$ such
that if $x, a \in A$ with $\|x - a\| < \delta$, then
$\|f(x) - f(a)\| < \varepsilon$.

Note that uniform continuity implies continuity.

LIPSCHITZ (FUNCTIONS)

Let $f: A \to W$.
We say f is "Lipschitz" if there exists a $M > 0$
such that

$$\|f(a) - f(b)\| \leq M\|a - b\| \quad \forall a, b \in A.$$

LIPSCHITZ ⇒ UNIFORM CONTINUITY

Let $f: A \to W$ be Lipschitz.
Then necessarily f is uniformly continuous.

Proof. Let $\varepsilon > 0$. Choose $\delta = \frac{\varepsilon}{M}$.
If $a, b \in A$ with $\|a - b\| < \delta$, then

$$\begin{aligned} \|f(a) - f(b)\| &\leq M\|a - b\| \\ &< M\delta = M\left(\frac{\varepsilon}{M}\right) = \varepsilon, \end{aligned}$$

showing f is uniformly cts. ☒

C IS COMPACT, $f: C \to W$ IS CTS ⇒ f IS UNIF CTS

Let $C \subseteq V$ be compact & $f: C \to W$ be continuous.
Then f is uniformly continuous.

Why? → Suppose for contradiction that f is not unif. cts.

$\Rightarrow \exists (a_n), (b_n) \subseteq C \ni \|a_n - b_n\| < \frac{1}{n}, \|f(a_n) - f(b_n)\| \geq \varepsilon.$

By compactness,

$\exists (a_{n_k}) \subseteq (a_n) \ni a_{n_k} \to a \in C.$

$$\Rightarrow b_{n_k} = \underbrace{b_{n_k} - a_{n_k}}_{\to 0} + \underbrace{a_{n_k}}_{\to a}$$

$\Rightarrow b_{n_k} \to a.$

By continuity, $f(a_{n_k}) \to f(a)$, $f(b_{n_k}) \to f(b)$.

$\Rightarrow \|f(a_{n_k}) - f(b_{n_k})\| \to 0.$

But this is a contradiction $\because \|f(a_n) - f(b_n)\| \geq \varepsilon$ by earlier assumption! ☒

Module 4.4: Extreme Value Theorem

$C \subseteq W$ IS COMPACT, $f: C \to W$ IS CTS $\Rightarrow$ $f(C)$ IS COMPACT

Let $C \subseteq W$ be compact, and let $f: C \to W$ be continuous.
Then necessarily $f(C)$ is compact.

Why? Take $(f(a_n)) \subseteq f(C)$, $a_n \in C$.

$\Rightarrow (a_n) \subseteq C$.

(compact) $\exists (a_{n_k}) \subseteq (a_n) \ni a_{n_k} \to a$.

(continuity) $\Rightarrow f(a_{n_k}) \to f(a) \in f(C)$

$\Rightarrow f(C)$ is compact. $\boxtimes$

$\emptyset \neq A \subseteq \mathbb{R}$ IS BOUNDED $\Rightarrow$ inf A, sup $A \in \overline{A}$

Let $\emptyset \neq A \subseteq \mathbb{R}$ be bounded.
Then necessarily $\inf A, \sup A \in \overline{A}$.

Proof. We prove the claim for $\sup A$; $\inf A$ is similar.
$\forall n \in \mathbb{N}$, we know

$$\sup A - \tfrac{1}{n} < \underset{\in A}{a_n} \leq \sup A.$$

$\Rightarrow$ (By S.T) $a_n \to \sup A$.

$\therefore \sup A \in \overline{A}$.

$\emptyset \neq C \subseteq V$ IS COMPACT, $f: C \to \mathbb{R}$ IS CTS $\Rightarrow$ $\exists a, b \in C \ni f(a) = \min f(C)$ & $f(b) = \max f(C)$ (EXTREME VALUE THEOREM (EVT))

Let $\emptyset \neq C \subseteq V$ be compact, and let $f: C \to \mathbb{R}$ be continuous.
Then, there must exist some $a, b \in C$ such that
$f(a) = \min f(C)$ & $f(b) = \max f(C)$.

Proof. $f(C)$ is compact.

$f(C) \subseteq \mathbb{R} \Rightarrow f(C)$ is closed & bounded.

$\Rightarrow$ (by bounded) $\sup f(C), \inf f(C) \in \overline{f(C)}$.

(since $f(C)$ is compact) $\therefore \overline{f(C)} = f(C)$.

$\therefore \exists a, b \in C \ni f(a) = \inf f(C) = \min f(C)$
& $f(b) = \sup f(C) = \max f(C)$. $\boxtimes$

UNIFORM NORM (FOR $C(K, W)$)

Let $K \subseteq V$ be compact, and let W be a NVS.
Then, $C(K, W) = \{ f: K \to W \text{ cts} \}$ is a NVS when equipped with the uniform norm

$$\|f\|_\infty = \max \{ \|f(x)\| : x \in K \}.$$

Module 5: Sequences of Functions

POINTWISE CONVERGENCE [OF FUNCTIONS]

Let $A \subseteq V$, $f_n : A \to W$ and $f : A \to W$.
Then, we say f_n converges to f "pointwise" if

$$f_n(x) \to f(x) \quad \forall x \in A.$$

UNIFORM CONVERGENCE [OF FUNCTIONS]

1. Let $A \subseteq V$, $f_n : A \to W$ and $f : A \to W$.
Then, we say f_n converges to f "uniformly" if for any $\varepsilon > 0$, there exists a $N \in \mathbb{N}$ such that

$$\| f_n(x) - f(x) \| < \varepsilon \quad \forall n \geq N,\ x \in A.$$

2. In this case, note that the same N works uniformly for all $x \in A$.

$\| f_n - f \|_\infty := \sup \{ \| f_n(x) - f(x) \| : x \in A \}$

1. Let $f_n, f : A \to W$, where $A \subseteq V$.
Then, we define

$$\| f_n - f \|_\infty := \sup \{ \| f_n(x) - f(x) \| : x \in A \}.$$

2. Note that

$f_n \to f$ uniformly $\iff$ $\| f_n - f \|_\infty < \infty$ eventually
& $\| f_n - f \|_\infty \to 0$.

*note that since A may not be compact & f may not be cts, $\| f_n - f \|_\infty$ could be infinite.

3. For example, take $f_n : \mathbb{R} \to \mathbb{R}$, with $f_1(x) = x$
& $f_n(x) = 0 \quad \forall n > 1$.

Then $f_n \to 0$ uniformly, even though $\| f_1 - 0 \|_\infty = \infty$.

EXAMPLES

For each sequence of functions, find the pointwise limit and determine whether the convergence is uniform.

eg¹ $f_n : (0,1) \to \mathbb{R}$, $f_n(x) = \dfrac{nx}{1+nx}$

Pointwise limit: for $x \in (0,1)$,

$$f_n(x) = \frac{nx}{1+nx} \to 1.$$

$\therefore f_n \to 1$ pointwise.

For $n > 1$, we have

$$\left| f_n\left(\tfrac{1}{n}\right) - 1 \right| = \tfrac{1}{2}.$$

$\therefore \| f_n - f \|_\infty \not\to 0$

$\Rightarrow$ convergence is not uniform.

eg² $f_n : c_0 \to \mathbb{R}$, $f_n((a_k)) \to a_n$.

Pointwise: For $(a_k) \in c_0$, see that

$$f_n((a_k)) = a_n \to 0.$$

$\therefore f_n \to 0$ pointwise.

Uniform? For $n \in \mathbb{N}$, see that

$$| f_n((\underbrace{1, \dots, 1}_{n}, 0, 0, \dots)) - 0 | = |1 - 0| = 1.$$

$\therefore \| f_n - 0 \| \geq 1 \quad \Rightarrow \| f_n - 0 \|_\infty \not\to 0.$

$\Rightarrow$ convergence is not uniform.

eg³ $f_n : [0,1] \times [0,1] \to \mathbb{R}$, $f_n(a,b) = \dfrac{a^n}{n} + \dfrac{1}{b+n}$

Pointwise: For $(a,b) \in [0,1] \times [0,1]$,

$$f_n(a,b) = \frac{a^n}{n} + \frac{1}{b+n} \in [0, \tfrac{1}{n}]$$

$\therefore$ (by ST) $f_n(a,b) \to 0$.

$\Rightarrow f_n(a,b) \to 0$ pointwise.

Uniform? Note that

$$| f_n(a,b) - 0 | = \frac{a^n}{n} + \frac{1}{b+n} \leq \frac{1}{n} + \frac{1}{n} = \frac{2}{n}$$

$\therefore \| f_n - 0 \|_\infty \leq \frac{2}{n} \to 0$,

$\therefore f_n \to 0$ uniformly.

$f_n : A \to W$, $A \subseteq V$: f_n IS CTS $\forall n \in \mathbb{N}$, $f_n \to f$ UNIFORMLY $\Rightarrow$ f IS CTS

Let $f_n : A \to W$, where $A \subseteq V$.
Suppose each f_n is continuous, and $f_n \to f$ uniformly.
Then necessarily f is continuous.

Proof. Let $(a_n) \subseteq A \ni a_n \to a$ & let $\varepsilon > 0$.
We know we may find $N \in \mathbb{N} \ni$

$$\| f_N - f \|_\infty < \boxed{\varepsilon/3}.$$

Since f_N is cts, we know $\exists M \in \mathbb{N} \ni$

$$| f_N(a_n) - f_N(a) | < \boxed{\varepsilon/3} \quad \forall n \geq M.$$

Then, for $n \geq M$, see that

$$\begin{aligned} | f(a_n) - f(a) | &= | f(a_n) - f_N(a_n) + f_N(a_n) - f_N(a) + f_N(a) - f(a) | \\ &\leq | f(a_n) - f_N(a_n) | + | f_N(a_n) - f_N(a) | + | f_N(a) - f(a) | \\ &\leq \| f_n - f \|_\infty + | f_n(a_n) - f_N(a) | + \| f_n - f \|_\infty \\ &\leq \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon, \end{aligned}$$

and so $f(a_n) \to f(a)$, and so f is cts. ∎

$A \subseteq V$ IS COMPACT, W IS A BANACH SPACE $\Rightarrow$ $(C(A,W), \| \cdot \|_\infty)$ IS A BANACH SPACE

Let $A \subseteq V$ be compact, and let W be a Banach space.
Then necessarily $(C(A,W), \| \cdot \|_\infty)$ is a Banach space.

Proof. Let $(f_n) \subseteq C(A,W)$ be Cauchy, and let $\varepsilon > 0$.
We know $\exists N \in \mathbb{N} \ni$

$$\| f_n - f_m \|_\infty < \varepsilon \quad \forall n, m \geq N.$$

For $x \in A$ & $n, m \geq N$, see that

$$\| f_n(x) - f_m(x) \| \leq \| f_n - f_m \|_\infty < \varepsilon,$$

and so $(f_n(x)) \subseteq W$ is Cauchy.
Since W is a Banach space, it is complete, and so $f_n(x) \to f(x) \in W$ for some $f(x) \in W$.
Thus, we have constructed a fn $f : A \to W \ni$
$f_n \to f$ pointwise.
For $x \in A$ and $n \geq N$, we have that

$$\lim_{m \to \infty} \| f_n(x) - f_m(x) \| \leq \varepsilon,$$

and so

$$\| f_n(x) - f(x) \| \leq \varepsilon \quad \text{(limits preserve order);}$$

$\Rightarrow \| f_n - f \|_\infty \leq \varepsilon$ since $x \in A$ was arbitrary.

$\Rightarrow f_n \to f$ uniformly.

So, by the previous theorem, it follows that $f \in C(A,W)$,
and so $f_n \to f$ in $C(A,W)$.

$\Rightarrow C(A,W)$ is a Banach space (since (f_n) was an arbitrary Cauchy sequence). ∎

Module 6.1: Partial Derivatives

SCALAR FUNCTION

1. A "scalar function" is any function of the form
$$f: A \to \mathbb{R}, \quad A \subseteq \mathbb{R}^n.$$

2. Note for any $f: A \to \mathbb{R}^m$, $A \subseteq \mathbb{R}^n$, there exist scalar functions $f_1, \dots, f_m : A \to \mathbb{R}$ such that
$$f = (f_1, f_2, \dots, f_m).$$
eg $f: \mathbb{R}^3 \to \mathbb{R}^2$ by $f(x,y,z) = (xze^y, \; x^2+z^2)$.
Then $f_1(x,y,z) = xze^y$ & $f_2(x,y,z) = x^2+z^2$.
Then $f = (f_1, f_2)$.

i^{th} PARTIAL DERIVATIVE [OF SCALAR FUNCTIONS]:
$$\frac{\partial f}{\partial x_i}(a) = f_{x_i}(a)$$

1. Let $f: A \to \mathbb{R}$, where $A \subseteq \mathbb{R}^n$.
Let $\{e_1, \dots, e_n\}$ be the standard basis for $\mathbb{R}^n$.
Then, for $1 \le i \le n$, we define the "i^{th} partial derivative" of f at $a = (a_1, \dots, a_n) \in A$, denoted as $\frac{\partial f}{\partial x_i}(a)$ or $f_{x_i}(a)$, to be equal to
$$f_{x_i}(a) = \frac{\partial f}{\partial x_i}(a) := \lim_{h \to 0} \frac{f(a+he_i) - f(a)}{h},$$
provided the limit exists.

2. We use the notation "$f(x_1, \dots, x_n)$" when talking about functions $f: A \to \mathbb{R}$, $A \subseteq \mathbb{R}^n$.

3. Note that $f_{x_i}(a)$ is the derivative of f at a wrt x_i, treating the other x_j, $j \ne i$ as constants.

4. Moreover, $f_{x_i}(a)$ is the slope of the tangent line to the surface $y = f(x_1, x_2, \dots, x_n)$ which is parallel to e_i.

5. For example, for $f: \mathbb{R}^2 \to \mathbb{R}$, find $\frac{\partial f}{\partial x}(a)$ & $\frac{\partial f}{\partial y}(a)$:
$$f_x(a) = \lim_{h \to 0} \frac{f(a+he_1) - f(a)}{h} = \lim_{h \to 0} \frac{f(a_1+h, a_2) - f(a_1, a_2)}{h},$$
and similarly
$$f_y(a) = \lim_{h \to 0} \frac{f(a_1, a_2+h) - f(a_1, a_2)}{h}.$$

6. We also treat $\frac{\partial f}{\partial x_i}$ as a function, and write
$$f_{x_i}(x_1, \dots, x_n) \quad \text{or} \quad \frac{\partial}{\partial x_i} f(x_1, \dots, x_n).$$

EXAMPLE 1: $f: \mathbb{R}^3 \to \mathbb{R}$, $f(x,y,z) = xy^2z + e^{xy}$, FIND PARTIAL DERIVATIVES

Sol^n.
$$f_x(x,y,z) = y^2z + ye^{xy}$$
$$f_y(x,y,z) = 2xyz + xe^{xy}$$
$$f_z(x,y,z) = xy^2.$$

i^{th} PARTIAL DERIVATIVE [OF FUNCTIONS]:
$$\frac{\partial f}{\partial x_i}(a) = f_{x_i}(a)$$

1. Let $A \subseteq \mathbb{R}^n$ and $f: A \to \mathbb{R}^m$, where $f = (f_1, \dots, f_m)$.
For $a \in A$, we define the "i^{th} partial derivative" of f at a, denoted as $\frac{\partial f}{\partial x_i}(a)$ or $f_{x_i}(a)$, to be equal to
$$f_{x_i}(a) = \frac{\partial f}{\partial x_i}(a) := \left(\frac{\partial f_1}{\partial x_i}(a), \dots, \frac{\partial f_m}{\partial x_i}(a)\right) \in \mathbb{R}^m.$$
provided it exists.

EXAMPLE 2: $f: \mathbb{R}^2 \to \mathbb{R}^3$, $f(x,y) = (2x^2y, 4x, e^{xy})$, FIND $\frac{\partial f}{\partial x}$ & $\frac{\partial f}{\partial y}$

Sol^n.
$$f_x(x,y) = (4xy, \; 4, \; ye^{xy}) \; ; \; \&$$
$$f_y(x,y) = (2x^2, \; 0, \; xe^{xy}).$$

Module 6.2: Differentiability

DIFFERENTIABLE [FUNCTIONS AT $a \in A$]

Let $a \in A \subseteq \mathbb{R}^n$, and let $f: A \to \mathbb{R}^m$.

Then, we say f is "differentiable" at $a \in A$ if

① $a \in \text{Int}(A)$; and

② There exists a $T \in \mathcal{L}(\mathbb{R}^n, \mathbb{R}^m)$ such that

$$\lim_{h \to 0} \frac{f(a+h) - f(a) - T(h)}{\|h\|} = 0.$$

(recall $\mathcal{L}(\mathbb{R}^n, \mathbb{R}^m) = \{T: \mathbb{R}^n \to \mathbb{R}^m \mid T \text{ is linear}\}$)

Note that by ①, $f(a+h)$ is defined for small enough h.

OPERATOR NORM [ON $M_{m \times n}(\mathbb{R})$]: $\|A\|_{op}$

Let $A \in M_{m \times n}(\mathbb{R})$.

Then, the "operator norm" of A, denoted as "$\|A\|_{op}$", is defined to be equal to

$$\|A\|_{op} = \sup\{\|Ax\| : x \in \mathbb{R}^n, \|x\| = 1\}.$$

$\|Ax\| \leq \|A\|_{op} \|x\|$

Note that for any $A \in M_{m \times n}(\mathbb{R})$ and $x \in \mathbb{R}^n$, we have

$$\|Ax\| \leq \|A\|_{op} \|x\|.$$

Proof. Clear if $x = 0$.

otherwise, see that

$$\left\| \frac{x}{\|x\|} \right\| = 1.$$

$$\Rightarrow \|A\|_{op} \geq \left\|A \frac{x}{\|x\|}\right\| = \frac{\|Ax\|}{\|x\|}.$$

Proof follows. ☒

DIFFERENTIABILITY $\Rightarrow$ CONTINUITY

Let $a \in A \subseteq \mathbb{R}^n$, and $f: A \to \mathbb{R}^m$.

Then, if f is diff at a, then necessarily f is cts at a.

Proof. f is diff $\Rightarrow \exists T \in \mathcal{L}(\mathbb{R}^n, \mathbb{R}^m) \ni$

$$\lim_{h \to 0} \frac{f(a+h) - f(a) - T(h)}{\|h\|} = 0$$

$$\Rightarrow \lim_{h \to 0} \frac{f(a+h) - f(a) - Bh}{\|h\|} = 0 \quad (\text{where } B \in M_{m \times n}(\mathbb{F}) \ni T(x) = Bx)$$

$\Rightarrow$ we can find $\delta > 0 \ni$ if $0 < \|h\| < \delta$, then

$$\left\| \frac{f(a+h) - f(a) - Bh}{\|h\|} \right\| < 1$$

$$\Rightarrow \|f(a+h) - f(a) - Bh\| < \|h\|$$

$$\Rightarrow \|f(a+h) - f(a)\| - \|Bh\| < \|h\|$$

$$\Rightarrow \|f(a+h) - f(a)\| < \|Bh\| + \|h\| \leq \|B\|_{op} \|h\| + \|h\|$$

As $h \to 0$, $\|B\|_{op} \|h\| + \|h\| \to 0$.

$\Rightarrow$ (by ST) $\lim_{h \to 0} f(a+h) = f(a)$.

Letting $x = a+h$,

$$\Rightarrow \lim_{x \to a} f(x) = f(a).$$ ☒

DIFFERENTIABLE [FUNCTIONS ON OPEN $U \subseteq \mathbb{R}^n$]

Let $U \subseteq \mathbb{R}^n$ be open, and let $f: U \to \mathbb{R}^m$.

Then, we say f is "differentiable" on U if f is differentiable at every point in U.

Module 6.3: Total Derivatives

TOTAL DERIVATIVE [OF f AT a]

Let $a \in A \subseteq \mathbb{R}^n$, and $f: A \to \mathbb{R}^m$.
Then, the "total derivative" of f at a, denoted as "$D_f(a)$", is defined to be the matrix

$$D_f(a) = \left(\frac{\partial f_i}{\partial x_j}(a)\right) \in M_{m \times n}(\mathbb{R}),$$

provided it exists.

f IS DIFFERENTIABLE $\Rightarrow$ "B" $= D_f(a)$

Let $a \in A \subseteq \mathbb{R}^n$, $f: A \to \mathbb{R}^m$.
Suppose f is differentiable at a, so that there exists a $B \in M_{m \times n}(\mathbb{R})$ such that

$$\lim_{h \to 0} \frac{f(a+h) - f(a) - Bh}{\|h\|} = 0.$$

Then necessarily $B = D_f(a)$.

Proof. It suffices to show that

$$b_j = \frac{\partial f}{\partial x_j} = \left(\frac{\partial f_1}{\partial x_j}, \ldots, \frac{\partial f_m}{\partial x_j}\right),$$

So let's do so.
Observe that as $t \in \mathbb{R}$, $t \to 0$, hence $te_j \to 0$, where $\{e_1, \ldots, e_n\}$ is the std basis for $\mathbb{R}^n$.
Then,

$$\lim_{h \to 0} \frac{f(a+h) - f(a) - Bh}{\|h\|} = 0 \iff \lim_{t \to 0} \frac{f(a+te_j) - f(a) - B(te_j)}{|t|} = 0$$

$$\iff \lim_{t \to 0^+} \frac{f(a+te_j) - f(a)}{t} = Be_j \;\; \& \;\; \lim_{t \to 0^-} \frac{f(a+te_j) - f(a)}{-t} = -Be_j$$

$$\iff \lim_{t \to 0} \frac{f(a+te_j) - f(a)}{t} = Be_j$$

$$\iff \frac{\partial f}{\partial x_j}(a) = Be_j = b_j,$$

as needed. ∎

In particular, if f is diff at a, then

1. $\frac{\partial f}{\partial x_i}$ exists $\forall 1 \leq i \leq n$; and
2. $\lim_{h \to 0} \frac{f(a+h) - f(a) - D_f(a)\,h}{\|h\|} = 0$.

GRADIENT [OF $f: A \to \mathbb{R}$ AT a]: $\nabla f(a)$

Let $a \in A \subseteq \mathbb{R}^n$, and $f: A \to \underline{\underline{\mathbb{R}}}$.
Then, the "gradient" of f at a, denoted as $\nabla f(a)$, is defined to be equal to

$$\nabla f(a) = D_f(a) = \left(\frac{\partial f}{\partial x_1}(a), \ldots, \frac{\partial f}{\partial x_n}(a)\right).$$

Module 6.4: Continuous Partials

$U \subseteq \mathbb{R}^n$ OPEN, $f: U \to \mathbb{R}$; $\frac{\partial f}{\partial x_j}$ EXISTS $\forall 1 \leq j \leq n$ AND IS CTS AT $a \in U$ $\Rightarrow$ f IS DIFF AT a

Let $U \subseteq \mathbb{R}^n$ be open, and let $f: U \to \mathbb{R}$.
Suppose, for some $a \in U$, that $\frac{\partial f}{\partial x_j}$ exists on U
and is cts at a for each $1 \leq j \leq n$.
Then necessarily f is diff at a.

Proof. Let $a = (a_1, \ldots, a_n)$. Since U is open, $\exists r > 0 \ni B_r(a) \subseteq U$.

Then, for any $h = (h_1, \ldots, h_n) \neq 0 \ni a + h \in B_r(a)$, we have that

$$\begin{aligned} f(a+h) - f(a) &= f(a_1+h_1, \ldots, a_n+h_n) - f(a_1, \ldots, a_n) \\ &= f(a_1+h_1, \ldots, a_n+h_n) - f(a_1, a_2+h_2, \ldots, a_n+h_n) \\ &\quad + f(a_1, a_2+h_2, \ldots, a_n+h_n) - f(a_1, a_2, a_3+h_3, \ldots, a_n+h_n) \\ &\quad + \cdots \\ &\quad + f(a_1, \ldots, a_{n-1}, a_n+h_n) - f(a_1, \ldots, a_n). \end{aligned}$$

By the single variable MVT on x_i, $\forall 1 \leq i \leq j$, $\exists c_j$ b/w $a_j, a_j + h_j \ni$

$$\frac{f(a_1, \ldots, a_{j-1}, a_j + h_j, \ldots, a_n + h_n) - f(a_1, \ldots, a_j, a_{j+1} + h_{j+1}, \ldots, a_n + h_n)}{a_j + h_j - a_j}$$

$$= \frac{\partial f}{\partial x_j}(a_1, \ldots, a_{j-1}, c_j, a_{j+1} + h_{j+1}, \ldots, a_n + h_n).$$

Thus

$$f(a+h) - f(a) = \sum_{j=1}^{n} h_j \frac{\partial f}{\partial x_j}(a_1, \ldots, a_{j-1}, c_j, a_{j+1} + h_{j+1}, \ldots, a_n + h_n).$$

Next, for $1 \leq j \leq n$, let

$$\delta_j = \frac{\partial f}{\partial x_j}(a_1, \ldots, a_{j-1}, c_j, a_{j+1} + h_{j+1}, \ldots, a_n + h_n),$$

and $\delta = (\delta_1, \ldots, \delta_n)$, so that

$$f(a+h) - f(a) - \nabla f(a) \cdot h = h \delta.$$

Since each partial is cts at a, as $h \to 0$, each $\delta_j \to 0$, and so $\delta \to 0$ in $\mathbb{R}^n$. Thus

$$\begin{aligned} 0 &\leq \lim_{h \to 0} \frac{|f(a+h) - f(a) - \nabla f(a) \cdot h|}{\|h\|} \\ &= \lim_{h \to 0} \frac{|\delta \cdot h|}{\|h\|} && (\cdot \text{ is the dot product}) \\ &\leq \lim_{h \to 0} \frac{\|\delta\| \|h\|}{\|h\|} && (\text{by Cauchy-Schwartz}) \\ &= 0. \end{aligned}$$

Hence

$$\lim_{h \to 0} \frac{|f(a+h) - f(a) - \nabla f(a) \cdot h|}{\|h\|} = 0,$$

and so

$$\lim_{h \to 0} \frac{f(a+h) - f(a) - \nabla f(a) \cdot h}{\|h\|} = 0,$$

showing f is diff at a. ∎

Note that the **converse** is **not necessarily true**!

eg Let $f: \mathbb{R}^2 \to \mathbb{R}$ by

$$f(x,y) = \begin{cases} (x^2+y^2)\sin\left(\frac{1}{\sqrt{x^2+y^2}}\right), & (x,y) \neq (0,0) \\ 0, & (x,y) = (0,0) \end{cases}$$

We know f is diff at $(0,0)$ (using the Theorem).
But

$$f_x(x,y) = 2x \sin\left(\frac{1}{\sqrt{x^2+y^2}}\right) - \cos\left(\frac{1}{\sqrt{x^2+y^2}}\right) \frac{x}{\sqrt{x^2+y^2}} \quad \forall (x,y) \neq (0,0).$$

See that $(\frac{1}{n}, 0) \to (0,0)$, but

$$f_x\left(\frac{1}{n}, 0\right) = \frac{2}{n}\sin(n) - \cos(n)$$

diverges, so f_x is not cts at $(0,0)$. #

Module 7.1-7.2: Differentiation Rules

$D_{(f+\alpha g)}(a) = D_f(a) + \alpha D_g(a)$

« SUM & SCALAR MULTIPLICATION RULE »

Let $f, g : A \to \mathbb{R}^m$ be diff at $a \in A \subseteq \mathbb{R}^n$.
Then necessarily for any $\alpha \in \mathbb{R}$, $f + \alpha g$ is diff at a and

$$D_{(f+\alpha g)}(a) = D_f(a) + \alpha D_g(a).$$

Why? Let $p = f + \alpha g$.

Then,

$$\lim_{h\to 0} \frac{p(a+h) - p(a) - (D_f(a) + \alpha D_g(a))}{\|h\|}$$

$$= \lim_{h\to 0} \frac{f(a+h) - f(a) - D_f(a)}{\|h\|} + \alpha \lim_{h\to 0} \frac{g(a+h) - g(a) - D_g(a)}{\|h\|}$$

$$= 0 + \alpha(0) = 0,$$

so in particular

$$D_p(a) = D_f(a) + \alpha D_g(a)$$

as needed. ☑

$D_{(f\cdot g)}(a) = g(a) D_f(a) + f(a) D_g(a)$

« DOT PRODUCT RULE »

Let $f, g : A \to \mathbb{R}^m$ be diff at $a \in A \subseteq \mathbb{R}^n$.
Consider $f \cdot g : A \to \mathbb{R}$ defined by

$$(f \cdot g)(x) = f(x) \cdot g(x) \quad \text{(dot product)}.$$

Then necessarily $f \cdot g$ is diff at a and

$$D_{f\cdot g}(a) = g(a) D_f(a) + f(a) D_g(a).$$

Proof. We need to prove that

$$\lim_{h\to 0} \frac{(f\cdot g)(a+h) - (f\cdot g)(a) - Xh}{\|h\|} = 0,$$

where $X = g(a) D_f(a) + f(a) D_g(a)$, so let's do so.

Let

$$\varepsilon(h) = f(a+h) - f(a) - D_f(a)h$$

&

$$\delta(h) = g(a+h) - g(a) - D_g(a)h.$$

Since f and g are diff at a, thus

$$\lim_{h\to 0} \frac{\varepsilon(h)}{\|h\|} = \lim_{h\to 0} \frac{\delta(h)}{\|h\|} = 0.$$

Then

$$\lim_{h\to 0} \frac{(f\cdot g)(a+h) - (f\cdot g)(a) - Xh}{\|h\|} = \lim_{h\to 0} \frac{(f\cdot g)(a+h) - (f\cdot g)(a) - g(a) D_f(a) - f(a) D_g(a)}{\|h\|}$$

$$= \lim_{h\to 0} g(a) f(a+h) - g(a) f(a)$$

$$= \lim_{h\to 0} \frac{g(a)\cdot\varepsilon(h) + f(a)\cdot\delta(h)}{\|h\|} + \lim_{h\to 0} \frac{f(a)\cdot g(a) - g(a)\cdot f(a+h) - f(a)\cdot g(a+h) + f(a+h)\cdot g(a+h)}{\|h\|}$$

$$= 0 + \lim_{h\to 0} \frac{f(a)\cdot g(a) - g(a)\cdot f(a+h) - f(a)\cdot g(a+h) + f(a+h)\cdot g(a+h)}{\|h\|}$$

$$= \lim_{h\to 0} \frac{g(a)\cdot(f(a) - f(a+h)) - g(a+h)\cdot(f(a) - f(a+h))}{\|h\|}$$

$$= \lim_{h\to 0} \frac{(g(a) - g(a+h))\cdot(f(a) - f(a+h))}{\|h\|}$$

By Cauchy-Schwartz,

$$\frac{|(g(a) - g(a+h))\cdot(f(a) - f(a+h))|}{\|h\|} \leq \frac{\|g(a) - g(a+h)\|\cdot\|f(a) - f(a+h)\|}{\|h\|},$$

we're taking the Euclidean norm!

and so

$$0 \leq \lim_{h\to 0} \frac{|(f\cdot g)(a+h) - (f\cdot g)(a) - Xh|}{\|h\|}$$

$$\leq \lim_{h\to 0} \frac{\|g(a) - g(a+h)\|\cdot\|f(a) - f(a+h)\|}{\|h\|}$$

$$= \lim_{h\to 0} \frac{\|g(a) - g(a+h)\|}{\|h\|}\cdot\frac{\|f(a) - f(a+h)\|}{\|h\|}\cdot\|h\|$$

$$= \lim_{h\to 0} \frac{\|g(a) - g(a+h) - D_g(a)h + D_g(a)h\|}{\|h\|}\cdot\frac{\|f(a) - f(a+h) - D_f(a)h + D_f(a)h\|}{\|h\|}\|h\|$$

$$\leq \lim_{h\to 0} \frac{\|g(a) - g(a+h) - D_g(a)h\| + \|D_g(a)h\|}{\|h\|}\cdot\frac{\|f(a) - f(a+h) - D_f(a)h\| + \|D_f(a)h\|}{\|h\|}\|h\|$$

$$\leq \lim_{h\to 0} \frac{\|g(a) - g(a+h) - D_g(a)h\| + \|D_g(a)\|_{op}\|h\|}{\|h\|}\cdot\frac{\|f(a) - f(a+h) - D_f(a)h\| + \|D_f(a)\|_{op}\|h\|}{\|h\|}\cdot\|h\|$$

$$= \lim_{h\to 0} (0 + \underbrace{\|D_g(a)\|_{op}}_{\text{constant}})(0 + \underbrace{\|D_f(a)\|_{op}}_{\text{constant}})\|h\|$$

$$= 0,$$

as needed. ☑

$D_{(g\circ f)}(a) = D_g(f(a)) D_f(a)$

« THE CHAIN RULE »

1. Let $A \subseteq \mathbb{R}^n$, $B \subseteq \mathbb{R}^m$, and let $f : A \to \mathbb{R}^m$, $g : B \to \mathbb{R}^k$, where $f(A) \subseteq B$.
Suppose f is diff at $a \in A$ & g is diff at $f(a) \in B$.
Then necessarily $g \circ f$ is diff at a, and

$$D_{g\circ f}(a) = D_g(f(a))\cdot D_f(a).$$

Proof. Let $X = D_g(f(a)) D_f(a)$.
Let
$b = f(a)$,
$\varepsilon(h) = f(a+h) - f(a) - D_f(a)h$, &
$\delta(k) = g(b+k) - g(b) - D_g(b)k$,

so that

$$\lim_{h\to 0} \frac{\varepsilon(h)}{\|h\|} = 0 \quad \& \quad \lim_{k\to 0} \frac{\delta(k)}{\|k\|} = 0.$$

Consider $k = f(a+h) - f(a)$. By continuity of f at a, $h \to 0 \Rightarrow k \to 0$. So

$$\lim_{h\to 0} \frac{(g\circ f)(a+h) - (g\circ f)(a) - D_g(f(a)) D_f(a)h}{\|h\|}$$

$$= \lim_{h\to 0} \frac{g(k+b) - g(b) - D_g(b) D_f(a)h}{\|h\|}$$

$$= \lim_{h\to 0} \frac{D_g(b)\varepsilon(h) + \delta(k)}{\|h\|} \quad (\text{as } k = f(a+h) - f(a))$$

$$= \lim_{h\to 0} D_g(b) \frac{\varepsilon(h)}{\|h\|} + \frac{\delta(k)}{\|h\|}.$$

Then, since

$$0 \leq \frac{\|D_g(b)\varepsilon(h)\|}{\|h\|} \leq \|D_g(b)\|_{op} \frac{\|\varepsilon(h)\|}{\|h\|} \to 0,$$

as $h \to 0$ it follows that

$$\lim_{h\to 0} D_g(b) \frac{\varepsilon(h)}{\|h\|} = 0.$$

Next, see that

$$\lim_{h\to 0} \frac{\delta(k)}{\|h\|} = \lim_{h\to 0} \frac{\delta(k)}{\|k\|}\cdot\frac{\|k\|}{\|h\|}.$$

However

$$\|k\| = \|Df(a)h + \varepsilon(h)\| \leq \|D_f(a)\|_{op}\|h\| + \|\varepsilon(h)\|,$$

from which it follows that $\frac{\|k\|}{\|h\|}$ is bounded.

By ST,

$$\lim_{h\to 0} \frac{\delta(k)}{\|h\|} = \lim_{h\to 0} \frac{\delta(k)}{\|k\|}\cdot\frac{\|k\|}{\|h\|} = 0,$$

and so the "entire" limit evaluates to 0, as needed. ☑

2. We can apply this in a specific context, say $\mathbb{R}^3$:

eg' Let $f(x, y, z)$ be real-valued & diff.
Suppose $x(t_1, t_2)$, $y(t_1, t_2)$, $z(t_1, t_2)$ are diff real-valued functions themselves.
Let $p(t_1, t_2) = (x(t_1, t_2), y(t_1, t_2), z(t_1, t_2))$. By asmt, this is diff.
Moreover,

$$D(f\circ p)(t_1, t_2) = Df(p(t_1, t_2)) Dp(t_1, t_2) \quad \text{(chain rule)}.$$

So

$$\nabla f(t_1, t_2) = \nabla f(x, y, z)\cdot D_p(t_1, t_2),$$

or in other words

$$\begin{pmatrix} \frac{\partial f}{\partial t_1} & \frac{\partial f}{\partial t_2} \end{pmatrix} = \begin{pmatrix} \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} & \frac{\partial f}{\partial z} \end{pmatrix} \begin{pmatrix} \frac{\partial x}{\partial t_1} & \frac{\partial x}{\partial t_2} \\ \frac{\partial y}{\partial t_1} & \frac{\partial y}{\partial t_2} \\ \frac{\partial z}{\partial t_1} & \frac{\partial z}{\partial t_2} \end{pmatrix}.$$

Equating components, it follows that

$$\frac{\partial f}{\partial t_1} = \frac{\partial f}{\partial x}\cdot\frac{\partial x}{\partial t_1} + \frac{\partial f}{\partial y}\cdot\frac{\partial y}{\partial t_1} + \frac{\partial f}{\partial z}\cdot\frac{\partial z}{\partial t_1}, \quad \text{and}$$

$$\frac{\partial f}{\partial t_2} = \frac{\partial f}{\partial x}\cdot\frac{\partial x}{\partial t_2} + \frac{\partial f}{\partial y}\cdot\frac{\partial y}{\partial t_2} + \frac{\partial f}{\partial z}\cdot\frac{\partial z}{\partial t_2}.$$

Module 7.3: Mean Value Theorem

A naive approach to the MVT might look like this:
Say if $U \subseteq \mathbb{R}^n$ is open, and $f: U \to \mathbb{R}^m$ is diff.
If $a, b \in U$, then there exists a $c \in L(a,b) := \{(1-t)a + tb : t \in [0,1]\}$ such that

$$f(b) - f(a) = D_f(c)(b-a).$$

But this doesn't work!
eg consider $f: \mathbb{R} \to \mathbb{R}^2$, $f(x) = (\cos(x), \sin(x))$.
see that $f(0) = f(2\pi) = (1,0)$.
But

$$D_f(x) = \begin{pmatrix} -\sin(x) \\ \cos(x) \end{pmatrix} \neq 0 \quad \forall x \in \mathbb{R}.$$

$$\therefore \quad 0 \neq 2\pi \begin{pmatrix} -\sin(x) \\ \cos(x) \end{pmatrix} \quad \forall x \in \mathbb{R}.$$

$\varphi: \mathbb{R} \to \mathbb{R}^n$, $\varphi(t) = (1-t)a + tb$ IS DIFF, WITH $D_\varphi(t) = b - a$

Let $a, b \in \mathbb{R}^n$, and let $\varphi: \mathbb{R} \to \mathbb{R}^n$ be defined by
$\varphi(t) = (1-t)a + tb$.
Then φ is diff with
$D_\varphi(t) = b - a$.

Proof.

$$\lim_{h \to 0} \frac{\varphi(t+h) - \varphi(t) - (b-a)h}{\|h\|}$$

$$= \lim_{h \to 0} \frac{(1-t-h)a + (t+h)b - (1-t)a - tb - (b-a)h}{\|h\|}$$

$$= \lim_{h \to 0} \frac{-ha + hb - (b-a)h}{\|h\|}$$

$$= 0. \ \#$$

$f: U \to \mathbb{R}^m$ IS DIFF, $L(a,b) \subseteq U \Rightarrow \forall x \in \mathbb{R}^m: \exists c \in L(a,b) \ni x \cdot (f(b) - f(a)) = x \cdot (D_f(c)(b-a))$ ≪ THE MEAN VALUE THEOREM / MVT ≫

Let $U \subseteq \mathbb{R}^n$ be open, and let $f: U \to \mathbb{R}^m$ be diff and $a, b \in U$ such that $L(a,b) \subseteq U$.
$L(a,b)$:= "line" connecting a & b.
Then necessarily for any $x \in \mathbb{R}^m$, there exists a $c \in L(a,b)$ such that

$$x \cdot (f(b) - f(a)) = x \cdot (D_f(c)(b-a)).$$

Proof. Fix $x \in \mathbb{R}^m$. Consider $\varphi(t) = (1-t)a + tb$.
Note $\varphi([0,1]) = L(a,b) \subseteq U$.
$\Rightarrow \exists \delta > 0 \ni \varphi(0-\delta, 1+\delta) \subseteq U$.

Then, for $t \in (-\delta, 1+\delta)$, consider

$$D_{(f \circ \varphi)}(t) = D_f(\varphi(\ \ ))D_\varphi(t) \quad \text{(chain rule)}$$

$$\Rightarrow D_{(f \circ \varphi)}(t) = D_f(\varphi(t))(b-a) \quad \text{(by the lemma above).}$$

Now, let $F: (-\delta, 1+\delta) \to \mathbb{R}$ by $F(t) = x \cdot (f \circ \varphi)(t)$.
By the dot product rule:

$$F'(t) = x \cdot D_f(\varphi(t)) D_\varphi(t) = x \cdot D_f(\varphi(t))(b-a).$$

Then, by the single var MVT:

$$\exists t_0 \in (0,1) \ni F(1) - F(0) = F'(t_0)(1-0).$$

$$\Rightarrow x \cdot f(\varphi(1)) - x \cdot f(\varphi(0)) = F'(t_0)(1-0).$$

$$\Rightarrow x \cdot f(b) - x \cdot f(a) = x \cdot D_f(\varphi(t_0))(b-a).$$

$$\Rightarrow x \cdot (f(b) - f(a)) = x \cdot D_f(\varphi(t_0))(b-a).$$

So, if we let $\varphi(t_0) = c$, we see this is exactly what MVT is asking for, and we're done! ☒

Module 7.4: Tangent Hyperplanes

HYPERPLANE

A "hyperplane" in $\mathbb{R}^n$ is a set of the form

$$P = \{(x_1, \dots, x_n) \in \mathbb{R}^n : a_1x_1 + \dots + a_nx_n = d\}$$

for some fixed $a_1, \dots, a_n \in \mathbb{R}$ (not all zero) and $d \in \mathbb{R}$.

We note that
1. $n=2$ hyperplanes = "lines", &
2. $n=3$ hyperplanes = "planes".

NORMAL [VECTOR OF A HYPERPLANE]

Let $P = \{(x_1, \dots, x_n) : a_1x_1 + \dots + a_nx_n = d\}$ be a hyperplane in $\mathbb{R}^n$.

We call $n = (a_1, \dots, a_n)$ the "normal" vector of P.

Why is the normal vector important geometrically?

Let be $(b_1, \dots, b_n) \in P$, so that

$d = a_1b_1 + \dots + a_nb_n$.

$\Rightarrow x = (x_1, \dots, x_n) \in P \iff d = a_1x_1 + \dots + a_nx_n$

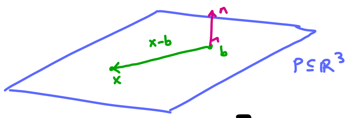
$\iff 0 = d - d$

$= a_1(x_1 - b_1) + \dots + a_n(x_n - b_n)$

$= n \cdot (x - b)$. ← dot product.

$\therefore P = \{x \in \mathbb{R}^n : n \cdot (x-b) = 0\}$;

ie $x \in P \iff n$ is orthogonal/perpendicular to $x - b$.



TANGENT [HYPERPLANES]

Let $A \subseteq \mathbb{R}^n$ and $a \in A$, and let P be a hyperplane with $a \in P$, with normal n.

Then, we say P is "tangent" to A at a if

$$n \cdot \frac{a_k - a}{\|a_k - a\|} \to 0$$

for any sequences $(a_k) \subseteq A \setminus \{a\}$ with $a_k \to a$.

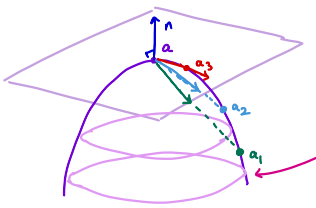
Why is this a good definition?

Recall that $a, b \in \mathbb{R}^n$ are ortho $\iff a \cdot b = 0$.

Then

$$n \cdot \frac{a_k - a}{\|a_k - a\|} \to 0$$

says that unit vectors in the direction of $a_k - a$ becomes closer and closer to being ortho to n as $k \to \infty$.



As $k \to \infty$, the a_k's become more orthogonal to the surface.

$f: U \to \mathbb{R}$, f IS DIFF AT $a \in U \Rightarrow S = \{(x, z) \in \mathbb{R}^{n+1} : z = f(x), x \in U\}$ HAS A TANGENT HYPERPLANE AT $(a, f(a))$ WITH NORMAL $n = (\nabla f(a), -1)$

Let $U \subseteq \mathbb{R}^n$ be open, and let $a \in U$ and $f: U \to \mathbb{R}$.

Suppose f is diff at a.

Then the surface

$$S = \{(\dots x \dots, z) \in \mathbb{R}^{n+1} : z \in f(x), x \in U\}$$

has a tangent hyperplane at $(a, f(a))$ with normal

$$n = (\nabla f(a), -1).$$

Proof. Let $(x_k, f(x_k)) \subseteq S \setminus \{(a, f(a))\}$ be a sequence such that $(x_k, f(x_k)) \to (a, f(a))$.

So $x_k \to a$. We need to prove

$$\lim_{k \to \infty} n \cdot \frac{(x_k, f(x_k)) - (a, f(a))}{\|(x_k, f(x_k)) - (a, f(a))\|} = 0,$$

so let's do so. Since f is diff at a we have that

$$\lim_{h \to 0} \frac{f(a+h) - f(a) - \nabla f(a)h}{\|h\|} = 0.$$

Let $\varepsilon(h) = f(a+h) - f(a) - \nabla f(a)h$, so that this implies that

$$\lim_{h \to 0} \frac{\varepsilon(h)}{\|h\|} = 0.$$

Moreover, see that

$$\|(x_k, f(x_k)) - (a, f(a))\|^2 = \|(x_k - a, f(x_k) - f(a))\|^2 \geq \|x_k - a\|^2.$$

Since $x_k - a \to 0$, thus

$$0 \leq \left| \lim_{k \to \infty} n \cdot \frac{(\dots x_k \dots, f(x_k)) - (\dots a \dots, f(a))}{\|(x_k, f(x_k)) - (a, f(a))\|} \right| \qquad (n = (\nabla f(a), -1))$$

$$= \lim_{k \to \infty} \frac{|\nabla f(a)(x_k - a) - (f(x_k) - f(a))|}{\|(x_k, f(x_k)) - (a, f(a))\|}$$

$$\leq \lim_{k \to \infty} \frac{|\nabla f(a)(x_k - a) - (f(x_k) - f(a))|}{\|x_k - a\|}$$

$$= \lim_{k \to \infty} \frac{|\varepsilon(x_k - a)|}{\|x_k - a\|}$$

$$= 0,$$

and the result follows.

eg: Find the tangent plane to the surface $z = 2x^2 + y^2$ at $(1, 1, 3)$.

Sol^n. Consider $f: \mathbb{R}^2 \to \mathbb{R}$ by

$$f(x, y) = 2x^2 + y^2.$$

Note that f_x, f_y exist & are cts on $\mathbb{R}^2$.

$\therefore f$ is diff on $\mathbb{R}^2$.

Then

$\nabla f(x, y) = (4x, 2y)$.

$\therefore \nabla f(1, 1) = (4, 2)$.

$\therefore n = (4, 2, -1)$.

$\Rightarrow$ eq^n of P is

$P: \ 4x + 2y - z = d$,

where $d = 4(1) + 2(1) - 3 = 3$.

$\therefore$ eq^n of P is

$4x + 2y - z = 3$.

Module 8.1-8.2: Higher Order Total Derivatives

k^{th} ORDER TOTAL DERIVATIVE : $D^k f(a)$

Let $U \subseteq \mathbb{R}^n$ be open, and let $f: U \to \mathbb{R}$.
Assume all partials of order $\leq k$ exist at $a \in U$.
Then, the "k^{th} order total derivative" of f at a, denoted by "$D^k f(a)$", is defined by

$$D^k f(a) : \mathbb{R}^n \to \mathbb{R}$$

by

$$D^k f(a)(h_1, \dots, h_n) = \sum_{i_1=1}^{n} \cdots \sum_{i_k=1}^{n} \frac{\partial^k}{\partial x_{i_1} \cdots \partial x_{i_k}}(a)\, h_{i_1} \cdots h_{i_k}.$$

eg ($f: \mathbb{R}^2 \to \mathbb{R}$)

$$D^2 f(a)(h_1, h_2) = f_{xx}(a) h_1^2 + f_{xy}(a) h_1 h_2 + f_{yx}(a) h_2 h_1 + f_{yy}(a) h_2^2.$$

$f \in C^p(U)$, $L(x,a) \subseteq U \Rightarrow \exists c \in L(x,a) \ni$

$$f(x) = f(a) + \sum_{k=1}^{p-1} \frac{1}{k!} D^k f(a)(x-a) + \frac{1}{p!} D^p f(c)(x-a)$$

<<TAYLOR'S THEOREM>>

Let $p \in \mathbb{N}$, $U \subseteq \mathbb{R}^n$ be open, and $f \in C^p(U)$.
Suppose $x, a \in U$ are such that $L(x,a) \subseteq U$.
Then, there necessarily exists a $c \in L(x,a)$ such that

$$f(x) = f(a) + \sum_{k=1}^{p-1} \frac{1}{k!} D^k f(a)(x-a) + \frac{1}{p!} D^p f(c)(x-a).$$

Proof. Let $h = x - a = (h_1, \dots, h_n)$. As $L(x,a) \subseteq U$ & U is open, $\exists \delta > 0 \ni a + th \in U$ $\forall t \in I := (-\delta, 1+\delta)$.
Then, by the chain rule, the fn $g: I \to \mathbb{R}$ by $g(t) = f(a+th)$ is diff and

$$g'(t) = Df(a+th)h = \sum_{i=1}^{n} \frac{\partial f}{\partial x_i}(a+th) h_i.$$

We can also show by induction that for $1 \leq j \leq p$,

$$g^{(j)}(t) = \sum_{i_1=1}^{n} \cdots \sum_{i_j=1}^{n} \frac{\partial^j f}{\partial x_{i_1} \cdots \partial x_{i_j}}(a+th) h_{i_1} \cdots h_{i_j}.$$

In particular, for $1 \leq j \leq p-1$ we have

$$g^{(j)}(0) = D^j f(a) h$$

and

$$g^{(p)}(t) = D^p f(a+th) h.$$

Hence $g: I \to \mathbb{R}$ is p-times differentiable and so by the 1D version of Taylor's Theorem,

$$g(1) - g(0) = \sum_{k=1}^{p-1} \frac{1}{k!} g^{(k)}(0) + \frac{1}{p!} g^{(p)}(t)$$

for some $0 < t < 1$. Thus

$$f(x) - f(a) = f(a+h) - f(a) = \sum_{k=1}^{p-1} \frac{1}{k!} D^k f(a)(h) + \frac{1}{p!} D^p f(a+th)(h),$$

as needed. $\boxed{}$

Module 8.3: Optimization

LOCAL MAXIMUM / MINIMUM (EXTREMA)

1. Let $U \subseteq \mathbb{R}^n$ be open, and let $f: U \to \mathbb{R}$ and $a \in U$. Then, we say $f(a)$ is a "local maximum" of f if there exists a $r > 0$ such that $f(x) \leq f(a)$ $\forall x \in B_r(a)$.

2. Note that "local minimum" is defined similarly, but with $f(x) \geq f(a)$ $\forall x \in B_r(a)$ instead.

3. We say $f(a)$ is a "local extrema" if it is a local minimum or maximum.

$f(a)$ IS A LOCAL EXTREMA $\Rightarrow \nabla f(a) = 0$

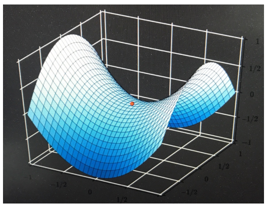
Let $f: U \to \mathbb{R}$ be diff, and let $a \in U$ so that $f(a)$ is a local extrema.

Then necessarily $\nabla f(a) = 0$.

Proof. Say $a = (a_1, \ldots, a_n)$.
Then
$$g_i(t) = f(a_1, \ldots, a_{i-1}, t, a_{i+1}, \ldots, a_n)$$
has a local extrema at $t = a_i$; ie $g_i'(a_i) = 0$.
ie $\frac{\partial f}{\partial x_i} = 0$.
$$\therefore \nabla f(a) = \left(\frac{\partial f}{\partial x_1}, \ldots, \frac{\partial f}{\partial x_n}\right) = 0. \quad \boxtimes$$

SADDLE POINT

1. Consider
$$f(x,y) = x^2 - y^2.$$
So
$$\nabla f(x,y) = (2x, 2y),$$
and so
$$\nabla f(0,0) = (0,0).$$
But the point isn't a local extrema!



2. In fact, we say $f(a)$ is a "saddle point" if $\nabla f(a) = 0$ but $f(a)$ is not a local extrema.

2^{nd} DERIVATIVE TEST

Let $U \subseteq \mathbb{R}^n$ be open, let $f: C^2(U)$, and let $a \in U$. Suppose $\nabla f(a) = 0$. Then:

1. If for all $h \neq 0$, $D^2 f(a)(h) > 0 \Rightarrow f(a)$ is a local min;
2. If for all $h \neq 0$, $D^2 f(a)(h) < 0 \Rightarrow f(a)$ is a local max;
3. If $\exists h, k \in \mathbb{R}^n$ such that $D^2 f(a)(h) > 0$ & $D^2 f(a)(k) < 0$
   $\Rightarrow a$ is a saddle point.

Proof

Lemma #1: Let $U \subseteq \mathbb{R}^n$ be open & let $f \in C^2(U)$. If $a \in U \ni D^2 f(a)(h) > 0$ $\forall 0 \neq h \in \mathbb{R}^n$, then $\exists m > 0 \ni D^2 f(a)(x) \geq m \|x\|^2$ $\forall x \in \mathbb{R}^n$.

Proof. Consider the compact set $K = \{x \in \mathbb{R}^n : \|x\| = 1\}$.
As $f \in C^2(U)$, thus $D^2 f(a)$ is cts and +ve on K.
By EVT, $\exists m > 0 \ni m = \min\{D^2 f(a)(x) : x \in K\}$.
For $0 \neq x \in \mathbb{R}^n$, we see $\frac{x}{\|x\|} \in K$ and so
$$D^2 f(a)\left(\frac{x}{\|x\|}\right) = \frac{1}{\|x\|^2} D^2 f(a)(x) \geq m,$$
and the proof follows. #

Lemma #2: Let $U \subseteq \mathbb{R}^n$ be open and let $f \in C^2(U)$. Suppose $a \in U \ni \nabla f(a) = 0$. Let $r > 0 \ni B_r(a) \subseteq U$. Then $\exists$ a fn $\varepsilon : B_r(0) \to \mathbb{R} \ni \lim_{h \to 0} \varepsilon(h) = 0$ & $f(a+h) - f(a) = \frac{1}{2} D^2 f(a)(h) + \|h\|^2 \varepsilon(h)$ for sufficiently small $\|h\|$.

Proof. Consider
$$\varepsilon(h) := \frac{f(a+h) - f(a) - \frac{1}{2} D^2 f(a)(h)}{\|h\|^2} \quad \text{for } 0 \neq h \in B_r(0) \text{ \& } \varepsilon(0) := 0.$$
We just need to prove $\lim_{h \to 0} \varepsilon(h) = 0$.
Since $\nabla f(a) = 0$, by Taylor's Theorem it follows that
$$f(a+h) - f(a) = \frac{1}{2} D^2 f(c) h \quad \text{for some } c \in L(a, a+h).$$
Then
$$\begin{aligned} 0 &\leq |\varepsilon(h)| \, \|h\|^2 \\ &= \left|\tfrac{1}{2} D^2 f(c)(h) - \tfrac{1}{2} D^2 f(a) h\right| \\ &\leq \frac{1}{2} \sum_i \sum_j \left| \frac{\partial^2 f}{\partial x_i \partial x_j}(c) - \frac{\partial^2 f}{\partial x_i \partial x_j}(a) \right| |h_i h_j| \\ &\leq \frac{1}{2} \sum_i \sum_j \left| \frac{\partial^2 f}{\partial x_i \partial x_j}(c) - \frac{\partial^2 f}{\partial x_i \partial x_j}(a) \right| \|h\|^2, \end{aligned}$$
which $\to 0$ as $c \to a$ as $h \to 0$ & $f \in C^2(U)$ (so $\frac{\partial^2 f}{\partial x_i x_j}$ is cts).
Proof follows. $\boxtimes$

We can now prove the main result.
Let $r > 0$ such that $B_r(a) \subseteq U$. By lemma #2, $\exists \varepsilon : B_r(0) \to \mathbb{R} \ni$
$$\lim_{h \to 0} \varepsilon(h) = 0$$
&
$$f(a+h) - f(a) = \frac{1}{2} D^2 f(a)(h) + \|h\|^2 \varepsilon(h)$$
for sufficiently small h.

1. Then, suppose $D^2 f(a)(h) > 0$ $\forall 0 \neq h \in \mathbb{R}^m$. Let $m > 0 \ni$
$$D^2 f(a)(x) \geq m \|x\|^2 \quad \forall x \in \mathbb{R}^n,$$
which we can do by lemma #1.
Then
$$f(a+h) - f(a) = \frac{1}{2} D^2 f(a)(h) + \|h\|^2 \varepsilon(h) \geq \left(\frac{m}{2} + \varepsilon(h)\right) \|h\|^2 > 0$$
as needed.
2. If $D^2 f(a)(h) < 0$ $\forall 0 \neq h \in \mathbb{R}^n$, the result follows by replacing f with $-f$ in 1.
3. Let $h \in \mathbb{R}^n$. For small $t \in \mathbb{R}$,
$$\begin{aligned} f(a+th) - f(a) &= \frac{1}{2} D^2 f(a)(th) + \|th\|^2 \varepsilon(th) \quad \text{(by lemma \#2)} \\ &= t^2 \left(\frac{1}{2} D^2 f(a)(h) + \|h\|^2 \varepsilon(th)\right). \end{aligned}$$
Letting $t \to 0$, we see $\varepsilon(th) \to 0$ and so $f(a+th) - f(a)$ takes on the same sign as $D^2 f(a)(h)$, which can be both +ve & -ve.
Thus a is a saddle point, as needed. $\boxtimes$

$\varphi(h,k) = ah^2 + 2bhk + ck^2$, $a,b,c \in \mathbb{R}$, $D = b^2 - ac$; $D<0 \Rightarrow \operatorname{sign}(a) = \operatorname{sign}(\varphi(h,k))$; $D>0 \Rightarrow \varphi(h,k)$ TAKES +VE & -VE VALUES

Let $a,b,c \in \mathbb{R}$, with $\varphi(h,k) = ah^2 + 2bhk + ck^2$ and $D = b^2 - ac$.

Then,

① If $D<0$, then a and $\varphi(h,k)$ share the same sign;

② If $D>0$, then $\varphi(h,k)$ can take positive & negative values.

$f \in C^2(U)$, $\nabla f(a,b) = 0$, $D = f_{xy}(a,b)^2 - f_{xx}(a,b) f_{yy}(a,b)$; $D<0$, $f_{xx}(a,b) > 0 \Rightarrow f(a,b)$ IS A LOCAL MIN; $D<0$, $f_{xx}(a,b) < 0 \Rightarrow f(a,b)$ IS A LOCAL MAX; $D>0 \Rightarrow (a,b)$ IS A SADDLE POINT

Let $U \subseteq \mathbb{R}^2$ be open, with $f \in C^2(U)$ and let $\nabla f(a,b) = 0$.

Let $D := f_{xy}(a,b)^2 - f_{xx}(a,b) f_{yy}(a,b)$ (this is called the "discriminant").

Then,

① If $D<0$ & $f_{xx}(a,b) > 0$, then $f(a,b)$ is a local minimum;

② If $D<0$ & $f_{xx}(a,b) < 0$, then $f(a,b)$ is a local maximum; and

③ If $D>0$, then (a,b) is a saddle point.

Proof: Follows from the above lemma by setting $a = f_{xx}(a,b)$, $b = f_{xy}(a,b)$ & $c = f_{yy}(a,b)$, so that

$$\varphi(h,k) = D^2 f(a)(h,k).$$ ☐

EX: $f(x,y) = x^4 + y^4 - 4xy + 2$

Q. Classify all local extrema and/or saddle points of $f(x,y) = x^4 + y^4 - 4xy + 2$.

Sol$^{\text{n}}$. See that

$\nabla f(x,y) = (4x^3 - 4y,\ 4y^3 - 4x)$

$\therefore \nabla f(x,y) = 0 \iff \begin{cases} x^3 - y = 0 \\ y^3 - x = 0 \end{cases} \iff \begin{matrix} x^3 = y \\ y = x^3 \end{matrix} \iff x = x^9 \iff x=0, y=0$ or $x=1, y=1$ or $x=-1, y=-1$.

Then,

$f_{xx}(x,y) = 12x^2$, $f_{yy}(x,y) = 12y^2$, $f_{yx}(x,y) = f_{xy}(x,y) = -4$.

① For $(0,0)$: $D = 16 - 0(0) > 0 \Rightarrow$ saddle point.

② For $(1,1)$: $D = 16 - 12(12) < 0$, and $f_{xx}(1,1) = 12 > 0$
$\Rightarrow f(1,1) = 0$ is a local min

③ For $(-1,-1)$: $D = 16 - 12(12) < 0$, and $f_{xx}(-1,-1) = 12 > 0$
$\Rightarrow f(-1,-1) = 0$ is a local min.

EX: ABS MAX/MIN OF $f(x,y) = 2x^3 + y^4$

Note that the "absolute max" and "absolute min" is defined by $\max f(K)$ & $\min f(K)$ respectively, which exist by EVT.

Q. Let $K = \overline{B_1(0,0)}$. Find the absolute max & min of $f: K \to \mathbb{R}$ by $f(x,y) = 2x^3 + y^4$.

Sol$^{\text{n}}$. Crit pts of $f: B_1(0,0) \to \mathbb{R}$:

$\nabla f(x,y) = (6x^2, 4y^3) = (0,0)$

$\iff (x,y) = (0,0)$.

Note that $f(0,0) = 0$.

Then,

$\partial(K) = \{(x,y) : x^2 + y^2 = 1\}$.

For $(x,y) \in \partial(K)$, we see that

$f(x,y) = 2x^3 + (1-x^2)^2$

$= x^4 + 2x^3 - 2x^2 + 1$. $(=g(x))$.

consider $g(x)$ on $[-1,1]$.

$\Rightarrow g'(x) = 4x^3 + 6x^2 - 4x$

$= 2x(2x^2 + 3x - 2)$

$= 2x(2x-1)(x+2)$

$\therefore g'(x) = 0 \iff x=0,\ x=\frac{1}{2},\ \cancel{x=-2}$

Then

$g(0) = 1$, $g(\frac{1}{2}) = \frac{13}{16}$,

$\underline{g(1) = 2}$, $\underline{g(-1) = -2}$.

$\therefore$ abs max: $f(1,0) = 2$; &

abs min: $f(-1,0) = -2$. ☐

Module 9.1: Inverse Function Theorem

JACOBIAN [OF f AT a]: $Jf(a) = \det(Df(a))$

Let $U \subseteq \mathbb{R}^n$, $f: U \to \mathbb{R}^n$, & let $a \in U$.
Then, the "Jacobian" of f at a, denoted by "$Jf(a)$", is equal to

$$Jf(a) := \det(Df(a)).$$

$\overline{B_r(a)} \subseteq U$; $f: \overline{B_r(a)} \to \mathbb{R}^n$ IS CTS & 1-1; 1ST ORDER PARTIALS OF f EXIST ON $B_r(a)$; $Jf \neq 0 \Rightarrow \exists \varepsilon > 0 \ni B_\varepsilon(f(a)) \subseteq f(B_r(a))$

Let $U \subseteq \mathbb{R}^n$ be open, and let $a \in U$ so there exists a $r > 0$ such that $\overline{B_r(a)} \subseteq U$.
Let $f: U \to \mathbb{R}^n$ be cts and 1-1 when restricted to $\overline{B_r(a)}$, and assume its first order partials exist on $B_r(a)$.
Suppose $Jf \neq 0$ on $B_r(a)$.
Then there exists a $\varepsilon > 0$ such that $B_\varepsilon(f(a)) \subseteq f(B_r(a))$.

Proof. Consider $g: \overline{B_r(a)} \to \mathbb{R}$ by $g(x) = \|f(x) - f(a)\|$.
As f is cts & injective on $\overline{B_r(a)}$, thus g is cts, & note $g(x) > 0$ $\forall x \neq a$.
Thus, by EVT,

$$m = \inf\{g(x) : \|x - a\| = r\} > 0.$$

Take $\varepsilon = \frac{m}{2}$. We claim $B_\varepsilon(f(a)) \subseteq f(B_r(a))$.
Indeed, let $y \in B_\varepsilon(f(a))$. By EVT, there exists a $b \in \overline{B_r(a)}$ such that

$$\|f(b) - y\| = \inf\{\|f(x) - y\| : x \in \overline{B_r(a)}\}.$$

Suppose $\|b - a\| = r$. Then

$$\begin{aligned} \varepsilon &> \|f(a) - y\| \\ &\geq \|f(b) - y\| \\ &\geq \|f(b) - f(a)\| - \|f(a) - y\| \\ &= g(b) - \|f(a) - y\| \\ &\geq m - \varepsilon \\ &= 2\varepsilon - \varepsilon \\ &= \varepsilon, \end{aligned}$$

a contⁿ. Thus $b \in B_r(a)$.
If we show $y = f(b)$ we're done.
Consider the cts fn $h: \overline{B_r(a)} \to \mathbb{R}$ by $h(x) = \|f(x) - y\|$.
By construction, $h(b)$ is the min value of h.
Moreover, $h^2(b)$ is the min value of h^2.
As $b \in B_r(a)$, which is open, we have $\nabla h^2(b) = 0$.
However,

$$h^2(x) = \sum_{i=1}^n (f_i(x) - y_i)^2,$$

and so

$$0 = \frac{\partial h^2}{\partial x_j} = \sum_{i=1}^n 2(f_i(b) - y_i) \frac{\partial f_i}{\partial x_j}(b) \quad \forall 1 \leq j \leq n.$$

Thus $Df(b)\, x = 0$, where $x = (2(f_1(b) - y_1), \ldots, 2(f_n(b) - y_n))^T$.
Since $Jf(b) \neq 0$, thus $Df(b)$ is invertible, and so $x = 0$.
Hence $f(b) = y$. as needed. ☒

$f: U \to \mathbb{R}^n$ IS CTS & 1-1, ALL 1ST ORDER PARTIALS EXIST ON U, $Jf \neq 0$ ON $U \Rightarrow f^{-1}$ IS CTS ON $f(U)$

Let $U \subseteq \mathbb{R}^n$ be open & non-empty.
Then, if $f: U \to \mathbb{R}^n$ is cts, 1-1, has all first-order partials existing on U & $Jf \neq 0$ on U, then necessarily f^{-1} is cts on $f(U)$.

Proof. To show $f^{-1}: f(U) \to \mathbb{R}^n$ is cts it suffices to show $f(W)$ is open if $W \subseteq U \subseteq \mathbb{R}^n$ is open.
Well, let W be such a set, and take $b \in f(W)$
$\ni b = f(a)$ for some $a \in W$.
As W is open, $\exists r > 0 \ni \overline{B_r(a)} \subseteq W$. By the prev lemma, $\exists \varepsilon > 0 \ni$

$$B_\varepsilon(b) \subseteq f(B_r(a)).$$

$\Rightarrow B_\varepsilon(b) \subseteq f(W)$, so $f(W)$ is open. ☒

$f \in C^1(U, \mathbb{R}^n)$, $Jf(a) \neq 0 \Rightarrow \exists r > 0 \ni B_r(a) \subseteq U$, f IS 1-1 ON $B_r(a)$, $Jf \neq 0$ ON $B_r(a)$ & $\det\left(\frac{\partial f_i}{\partial x_j}(c_i)\right) \neq 0$ $\forall c_1, \ldots, c_n \in B_r(a)$

Let $U \subseteq \mathbb{R}^n$ be open, and let $f \in C^1(U, \mathbb{R}^n)$.
Suppose $a \in U$ such that $Jf(a) \neq 0$.
Then there exists a $r > 0$ such that

1. $B_r(a) \subseteq U$;
2. f is injective on $B_r(a)$;
3. $Jf \neq 0$ on $B_r(a)$; &
4. $\det\left(\frac{\partial f_i}{\partial x_j}(c_i)\right) \neq 0$ $\forall c_1, \ldots, c_n \in B_r(a)$.

Proof. Let $W = U^n$. Consider $h: W \to \mathbb{R}$ by

$$h(x_1, \ldots, x_n) = \det\left(\left(\frac{\partial f_i}{\partial x_j}(x_i)\right)_{ij}\right).$$

As $f \in C^1(U, \mathbb{R}^n)$, and a determinant is a polynomial of its entries, we must have that h is cts.
Note $h(a, \ldots, a) = Jf(a) \neq 0$.
So $\exists$ open interval $h(a, \ldots, a) \in I \subseteq \mathbb{R} \ni 0 \notin I$.
Then $h^{-1}(I)$ is open (as W is open) and so $\exists R > 0$
$\ni B_R(a, \ldots, a) \subseteq h^{-1}(I)$.
But then $\exists r > 0 \ni$

$$B_r(a) \times \cdots \times B_r(a) \subseteq B_R(a, \ldots, a) \subseteq h^{-1}(I).$$

Thus $Jf \neq 0$ on $B_r(a)$ &

$$\det\left(\left(\frac{\partial f_i}{\partial x_j}(c_i)\right)\right) \neq 0 \quad \forall c_1, \ldots, c_n \in B_r(a).$$

So, we just need to show f is 1-1 on $B_r(a)$.
Suppose $\exists x \neq y \in B_r(a) \ni f(x) = f(y)$.
As f is diff on $B_r(a)$, every f_i is diff on $B_r(a)$.
Fix $1 \leq i \leq n$. By the MVT, $\exists c_i \in L(x, y) \ni$

$$0 = f_i(x) - f_i(y) = Df_i(c_i)(x - y).$$

Letting $A = \left(\frac{\partial f_i}{\partial x_j}(c_i)\right)_{ij}$ we see $A(x - y) = 0$.
As $x - y \neq 0$, A is not invertible and so $\det\left(\left(\frac{\partial f_i}{\partial x_j}(c_i)\right)_{ij}\right) = 0$,
a contⁿ. ☒

$A \in M_{n \times n}(\mathbb{R})$ IS INVERTIBLE $\Rightarrow Ax = b$ HAS A UNIQUE SOLⁿ $x = (x_1, \ldots, x_n)^T$ BY $x_i = \frac{\det(A^{(i)})}{\det(A)}$, $A^{(i)} :=$ A WITH ith COLUMN $= b$ ⟪CRAMER'S RULE⟫

Let $A \in M_{n \times n}(\mathbb{R})$ be invertible, and consider a system of equations $Ax = b$.
Then this system has a unique solution $x = (x_1, \ldots, x_n)^T \in \mathbb{R}^n$ by

$$x_i = \frac{\det(A^{(i)})}{\det(A)},$$

where $A^{(i)} :=$ the matrix obtained by replacing its ith column with b.

$f \in C^1(U, \mathbb{R}^n)$, $Jf(a) \neq 0 \Rightarrow \exists$ open $a \in W \subseteq U \ni$ f IS 1-1 ON W, $f^{-1} \in C^1(f(W), \mathbb{R}^n)$, & $Df^{-1}(y) = [Df(x)]^{-1}$ $\forall y \in f(W)$, $x = f^{-1}(y)$

<< THE INVERSE FUNCTION THEOREM >>

Let $U \subseteq \mathbb{R}^n$ be open, with $f \in C^1(U, \mathbb{R}^n)$.
Let $a \in U$ such that $Jf(a) \neq 0$.
Then there exists an open $a \in W \subseteq U$ such that

1. f is injective on W;
2. $f^{-1} \in C^1(f(W), \mathbb{R}^n)$; &
3. For any $y \in f(W)$,
$$D(f^{-1})(y) = [Df(x)]^{-1},$$
where $x = f^{-1}(y)$.

Proof.

① By the most recent lemma (L3), $\exists r > 0 \ni W := B_r(a) \subseteq U \ni f$ is injective on W, $Jf \neq 0$ on W, &
$$\det\left(\frac{\partial f_i}{\partial x_j}(c_i)\right)_{ij} \neq 0 \quad \forall c_1, \dots, c_n \in W.$$
By the previous lemma (L2), f^{-1} is cts on $f(W)$.

② We claim $f^{-1} \in C^1(f(W), \mathbb{R}^n)$. Fix $y_0 \in f(W)$ & $1 \leq i, j \leq n$.
Choose $0 \neq t \in \mathbb{R}$ sufficiently small so that $y_0 + te_j \in f(W)$.
We may then find $x_0, x_1 = x_1(t) \in W \ni f(x_0) = y_0$ & $f(x_1) = y_0 + te_j$.
By MVT, $\forall 1 \leq i \leq n$ $\exists c_i = c_i(t) \in L(x_0, x_1) \ni$
$$\nabla f_i(c_i)(x_1 - x_0) = f_i(x_1) - f_i(x_0) = \begin{cases} t, & i = j \\ 0, & \text{otherwise.} \end{cases}$$
Thus
$$\nabla f_i(c_i)\left(\frac{x_1 - x_0}{t}\right) = \frac{1}{t}(f_i(x_1) - f_i(x_0)) = \begin{cases} 1, & i = j \\ 0, & \text{otherwise.} \end{cases}$$
Now let $A \in M_{n \times n}(\mathbb{R})$ with i^{th} row $= \nabla f_i(c_i)$.
By assumption, $\det(A_j) \neq 0$, and moreover, $A_j \cdot \left(\frac{x_1 - x_0}{t}\right) = e_j$.
For $1 \leq k \leq n$, we see that
$$\frac{(f^{-1})_k(y_0 + te_j) - (f^{-1})_k(y_0)}{t} = \frac{x_{1,k} - x_{0,k}}{t},$$
where by Cramer's Rule, $Q_k(t) := \dfrac{x_{1,k} - x_{0,k}}{t}$ is a quotient of determinants of matrices whose entries are either 0, 1 or a 1^{st} order partial evaluated at a c_i.
Since $t \to 0$, thus $y_0 + te_j \to y_0$. By continuity of f^{-1}, hence $x_1 \to x_0$, and so $c_i \to x_0$.
As f is C^1, thus $Q_k(t) \to Q_k$, where Q_k is a quotient of determinants of matrices whose entries are either 0, 1 or a first-order partial of f evaluated at a $x_0 = f^{-1}(y_0)$.
As $f \in C^1$ & f^{-1} is cts at y_0, thus Q_k is cts at each $y_0 \in f(W)$.
Moreover,
$$\lim_{t \to 0} \frac{(f^{-1})_k(y_0 + te_j) - (f^{-1})_k(y_0)}{t} = \lim_{t \to 0} \frac{x_{1,k} - x_{0,k}}{t} = Q_k.$$
Hence all the partial derivatives of f^{-1} exist and are cts at y_0, ie $f^{-1} \in C^1(f(W), \mathbb{R}^n)$.

③ Therefore, by the chain rule, for $y \in f(W)$, we have
$$I = DI(y) = D(f \circ f^{-1})(y) = Df(f^{-1}(y))\, D(f^{-1})(y).$$
The result follows. ∎

Ex: $f(x,y) = (x+y, \sin x + \cos y)$

Let $f: \mathbb{R}^2 \to \mathbb{R}^2$ by $f(x,y) = (x+y, \sin(x) + \cos(y))$.
Note
$$f_x(x,y) = (1, \cos(x)); \text{ \&}$$
$$f_y(x,y) = (1, -\sin(y)),$$
so $f \in C^1(\mathbb{R}^2, \mathbb{R}^2)$.

Prove f^{-1} exists & is diff on some open set containing $(0,1)$, and compute $D(f^{-1})(0,1)$.

Sol$^{\underline{n}}$. Note
$$\begin{aligned} f(x,y) = (0,1) &\iff (x+y, \sin x + \cos y) = (0,1) \\ &\iff y = -x,\ \sin(x) + \cos(-x) = 1 \\ &\iff y = -x,\ \sin(x) + \cos(x) = 1 \\ &\iff (x,y) = (2k\pi, -2k\pi),\ k \in \mathbb{Z} \quad \text{— case ①} \\ &\qquad \text{or} \\ &\qquad (x,y) = \left(\tfrac{\pi}{2} + 2k\pi, -\tfrac{\pi}{2} - 2k\pi\right),\ k \in \mathbb{Z} \quad \text{— case ②.} \end{aligned}$$

Case ①: $a = (2k\pi, -2k\pi)$, $k \in \mathbb{Z}$.
$$\Rightarrow Jf(a) = \begin{vmatrix} 1 & 1 \\ 1 & 0 \end{vmatrix} = -1 \neq 0,$$
and so by the Inv Fn Thm $\exists$ open $a \in W \subseteq \mathbb{R}^2 \ni f$ is 1-1 on W & $f^{-1} \in C^1(f(W), \mathbb{R}^2)$.
Note $(0,1) \in f(W)$.
$$\Rightarrow D(f^{-1})(0,1) = [Df(a)]^{-1} = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}^{-1} = \begin{pmatrix} 0 & 1 \\ 1 & -1 \end{pmatrix}.$$

Case ②: $a = \left(\frac{\pi}{2} + 2k\pi, -\frac{\pi}{2} - 2k\pi\right)$
$$\Rightarrow Jf(a) = \begin{vmatrix} 1 & 1 \\ 0 & 1 \end{vmatrix} = 1 \neq 0.$$
So, $\exists$ open $a \in W \subseteq \mathbb{R}^2 \ni f^{-1} \in C^1(f(W), \mathbb{R}^2)$ with
$$D(f^{-1})(0,1) = Df(a)^{-1} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}^{-1} = \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}.$$

* Note that the way we restrict f to make it injective depends on our choice for $f^{-1}(y)$.

Module 9.2:

Implicit Function Theorem

$U \subseteq \mathbb{R}^{n+p}$; $f = (f_1, \dots, f_n) \in C'(U, \mathbb{R}^n)$; $x_0 \in \mathbb{R}^n$, $t_0 \in \mathbb{R}^p$ $\ni$ $f(x_0, t_0) = 0$; $\det\left[\frac{\partial f_i}{\partial x_j}(x_0, t_0)\right]_{n \times n} \neq 0 \Rightarrow \exists$ open $t_0 \in V \subseteq \mathbb{R}^p$ & A UNIQUE $g \in C'(V, \mathbb{R}^n) \ni g(t_0) = x_0$ & $f(g(t), t) = 0 \ \forall t \in V$

<< IMPLICIT FUNCTION THEOREM >>

Let $U \subseteq \mathbb{R}^{n+p}$ be open, and let $f = (f_1, \dots, f_n) \in C'(U, \mathbb{R}^n)$.

Let $x_0 \in \mathbb{R}^n$, $t_0 \in \mathbb{R}^p$ be such that $f(x_0, t_0) = 0$.

Suppose $\det\left(\frac{\partial f_i}{\partial x_j}(x_0, t_0)\right)_{\underline{\underline{n \times n}}} \neq 0$.

Then there exists an open $t_0 \in V \subseteq \mathbb{R}^p$ and a <u>unique</u> $g \in C'(V, \mathbb{R}^n)$ such that

① $g(t_0) = x_0$; &

② $f(g(t), t) = 0 \quad \forall t \in V$.

<u>Proof</u>. For every $(x, t) \in U$, let

$$F(x,t) := (f(x,t), t) = (f_1(x,t), \dots, f_n(x,t), t_1, \dots, t_p).$$

Note $F(x_0, t_0) = (0, t_0)$.

Then, F is a fn from U to $\mathbb{R}^{n+p}$ with

$$DF = \begin{pmatrix} \left(\frac{\partial f_i}{\partial x_j}\right)_{n \times n} & B \\ 0_{p \times n} & I_{p \times p} \end{pmatrix}$$

where B is a matrix whose entries are first order partials of the f_i's wrt the t_j's.

Taking the determinant of DF evaluated at (x_0, t_0) yields that

$$JF(x_0, t_0) = \det\left(\frac{\partial f_i}{\partial x_j}(x_0, t_0)\right)_{n \times n} \cdot \det I_{n \times p} \neq 0,$$

so by the Inv Fn Thm there exists an open set $(x_0, t_0) \in W \subseteq U \ni F$ is inj on W & $F^{-1} \in C'(F(W), \mathbb{R}^{n+p})$.

Let $G = F^{-1} = (G_1, \dots, G_n, G_{n+1}, \dots, G_{n+p})$. Consider $\varphi : F(W) \to \mathbb{R}^n$ by

$$\varphi = (G_1, \dots, G_n).$$

By construction,

$$\varphi(F(x,t)) = x \qquad \forall (x,t) \in W.$$

&

$$F(\varphi(x,t), t) = (x,t) \qquad \forall (x,t) \in F(W).$$

Consider $V = \{t \in \mathbb{R}^p : (0,t) \in F(W)\}$ & $g : V \to \mathbb{R}^n$ by $g(t) = \varphi(0,t)$.

As G is C', it follows φ is also C', so $g \in C'(V, \mathbb{R}^n)$.

Also note V is open since $F(W)$ is open.

Finally, see that

$$g(t_0) = \varphi(0, t_0) = \varphi(F(x_0, t_0)) = x_0,$$

and for $(x,t) \in F(W)$,

$$f(\varphi(x,t), t) = x.$$

In particular,

$$0 = f(\varphi(0,t), t) = f(g(t), t) = 0 \quad \forall t \in W.$$

Uniqueness follows from injectivity of F. ∎

eg¹: $xyz + \sin(x+y+z) = 0$

Consider

$$f(x,y,z) = xyz + \sin(x+y+z),$$

so $f \in C'(\mathbb{R}^3)$.

Note $f(0,0,0) = 0$.

Now,

$$f_z(x,y,z) = xy + \cos(x+y+z)$$

$\Rightarrow f_z(0,0,0) = 1 \neq 0$.

Hence

$$\det[1] = 1 \neq 0.$$

So, by the Implicit Function Theorem, there exists a open $V \subseteq \mathbb{R}^2$ with $(0,0) \in V$ and $g(x,y)$ in $C'(V)$ such that $g(0,0) = 0$ and

$$f(x, y, g(x,y)) = 0 \quad \forall (x,y) \in V.$$

ie $z = g(x,y)$ on V.

eg²: $u, v : \mathbb{R}^4 \to \mathbb{R}$

Prove there exist $u, v : \mathbb{R}^4 \to \mathbb{R}$ and $(2,-1,-1,2) \in U \subseteq \mathbb{R}^4$ open such that

① $u, v \in C'(U)$;

② $u(2,1,-1,-2) = 4$ & $v(2,1,-1,-2) = 3$; and

③ For all $(x,y,z,w) \in U$, we have

$$u^2 + v^2 + w^2 = 29; \ \& \qquad (u^2 := (u(x,y,w,z))^2)$$

$$\frac{u^2}{x^2} + \frac{v^2}{y^2} + \frac{w^2}{z^2} = 17.$$

Sol$^{\underline{n}}$. Let $f : \mathbb{R}^6 \to \mathbb{R}^2$ by

$$f(\underbrace{u, v}_{\text{replace}}, \underbrace{x, y, z, w}_{\text{keep}}) = \left(u^2 + v^2 + w^2 - 29, \ \frac{u^2}{x^2} + \frac{v^2}{y^2} + \frac{w^2}{z^2} - 17\right)$$

we want to replace u, v & fns of x, y, z, w,

ie keep u, v & replace x, y, z, w.

$\to n = 2$, $p = 4$.

See that

$$f(\underbrace{4, 3}_{x_0}, \underbrace{2, -1, -1, 2}_{t_0}) = 0.$$

and

$$\det\begin{pmatrix} \frac{\partial f_1}{\partial u} & \frac{\partial f_1}{\partial v} \\ \frac{\partial f_2}{\partial u} & \frac{\partial f_2}{\partial v} \end{pmatrix} = \begin{vmatrix} 2u & 2v \\ \frac{2u}{x^2} & \frac{2v}{y^2} \end{vmatrix}$$

$$= 4uv\left(\frac{1}{y^2} - \frac{1}{x^2}\right).$$

This is non-zero at

$$(4, 3, 2, -1, -1, 2).$$

So by the impl fn thm, $\exists$ open $(2,1,-1,-2) \in U$ & $g \in C'(U, \mathbb{R}^2) \ni g(2,1,-1,-2) = (4,3)$ &

$$\forall (x,y,z,w) \in U, \quad f(\underbrace{g(x,y,z,w)}_{g = (g_1, g_2)}, x, y, z, w) = 0.$$

Take $u(x,y,z,w) = g_1(x,y,z,w)$

& $v(x,y,z,w) = g_2(x,y,z,w)$.

Since $g \in C'(U)$, thus $u, v \in C'(U)$.

See that

$$u(2,1,-1,-2) = 4$$

& $v(2,1,-1,-2) = 3$,

and see that

$$f(g(x,y,z,w), x, y, z, w) = 0$$

$$\Rightarrow f(\underbrace{u(x,y,z,w)}_{u}, \underbrace{v(x,y,z,w)}_{v}, x, y, z, w) = 0$$

$$\Rightarrow u^2 + v^2 + w^2 = 29 \quad \& \quad \frac{u^2}{x^2} + \frac{v^2}{y^2} + \frac{w^2}{z^2} = 17. \ ∎$$

Module 9.3: Lagrange Multipliers

LOCAL MAXIMUM/MINIMUM [SUBJECT TO THE CONSTRAINTS $g_i : U \to \mathbb{R}$]

1. Let $U \subseteq \mathbb{R}^n$ be open, with $f : U \to \mathbb{R}$.
Let $a \in U$.
Then, we say $f(a)$ is a "local maximum" of f subject to the constraints
$$g_i : U \to \mathbb{R}, \quad 1 \le i \le m$$
if $g_i(a) = 0$ for each i and there exists a $r > 0$ such that whenever $x \in B_r(a)$ and $g_i(x) = 0$ $\forall i$, then $f(x) \le f(a)$.

2. Similarly, we say $f(a)$ is a "local minimum" of f subject to the constraints
$$g_i : U \to \mathbb{R}, \quad 1 \le i \le m$$
if $g_i(a) = 0$ for each i and there exists a $r > 0$ such that whenever $x \in B_r(a)$ and $g_i(x) = 0$ $\forall i$, then $f(x) \ge f(a)$.

$f, g_1, \dots, g_n \in C^1(U)$, $\det\left(\frac{\partial g_i}{\partial x_j}(a)\right)_{m \times m} \ne 0$, $f(a)$ IS A LOCAL EXTREMUM SUBJECT TO THE CONSTRAINTS g_i $\Rightarrow \exists \lambda_1, \dots, \lambda_m \in \mathbb{R} \ni \nabla f(a) + \sum_{i=1}^m \lambda_i \nabla g_i(a) = 0$

Let $U \subseteq \mathbb{R}^n$ be open, and let $m < n$.
Let $f, g_1, \dots, g_m \in C^1(U)$.
Suppose $a \in U$ such that
$$\det\left(\frac{\partial g_i}{\partial x_j}(a)\right)_{\underline{\underline{m \times m}}} \ne 0,$$
we only use $x_1, \dots, x_m$.

and let $f(a)$ be a local extremum of f subject to the constraints g_i.
Then there exists $\lambda_1, \dots, \lambda_m \in \mathbb{R}$ such that
$$\nabla f(a) + \sum_{i=1}^m \lambda_i \nabla g_i(a) = 0.$$

Idea. $m=2$, $n=3$.
We want to show $\exists \lambda, \mu \in \mathbb{R}$
$$\ni \lambda \frac{\partial g_1}{\partial x_j}(a) + \mu \frac{\partial g_2}{\partial x_j}(a) = -\frac{\partial f}{\partial x_j}(a) \quad \forall j = 1, 2, 3.$$
Let $A = \begin{pmatrix} \frac{\partial g_1}{\partial x_1}(a) & \frac{\partial g_1}{\partial x_2}(a) \\ \frac{\partial g_2}{\partial x_1}(a) & \frac{\partial g_2}{\partial x_2}(a) \end{pmatrix}$.
So $\det A \ne 0$.
In particular,
$$(\lambda \ \ \mu) A = \left(-\frac{\partial f}{\partial x_1}(a) \quad -\frac{\partial f}{\partial x_2}(a)\right)$$
has a unique sol$^{\underline{n}}$ $(\lambda \ \ \mu)$.
We need to show
$$\lambda \frac{\partial g_1}{\partial x_3}(a) + \mu \frac{\partial g_2}{\partial x_3}(a) = -\frac{\partial f}{\partial x_3}(a). \quad — (*)$$
We then "use" the Impl Fn Thm to replace x_3 w/ $h(x_1, x_2)$, and prove $(*)$ with what we know about x_1, x_2. ⧣

eg': $f(x,y,z) = x + 2y$

Maximize & minimize $f(x,y,z) = x + 2y$ subject to the constraints
① $x + y + z = 1$; &
② $y^2 + z^2 = 4$.

Sol$^{\underline{n}}$. Geometrically, such a max/min must exist (intersection is compact, so follows from EVT).
Let
$$f(x,y,z) = x + 2y$$
$$g(x,y,z) = x + y + z - 1$$
$$h(x,y,z) = y^2 + z^2 - 4.$$
Note
$$\begin{vmatrix} 1 & 1 \\ 0 & 2y \end{vmatrix} = 2y \ne 0 \quad \text{for } y \ne 0.$$
Then, if $g(x,0,z) = h(x,0,z) = 0$, then
$$z = 2, x = -1 \quad \text{or} \quad z = -2, x = 3.$$
So
$$f(-1,0,2) = -1 \quad \& \quad f(3,0,-2) = 3.$$
Otherwise, such a max/min is of the form
$$\nabla f = \lambda \nabla g + \mu \nabla h$$
$$\Rightarrow (1,2,0) = \lambda(1,1,1) + \mu(0, 2y, 2z).$$
$$\vdots$$
$$\Rightarrow y = \pm\sqrt{2}, \quad z = \mp\sqrt{2}, \quad x = 1.$$
Then
$$f(1, \sqrt{2}, -\sqrt{2}) = 1 + 2\sqrt{2} \quad (\text{max})$$
&
$$f(1, -\sqrt{2}, \sqrt{2}) = 1 - 2\sqrt{2} \quad (\text{min}).$$

Module 10.1-2: Riemann Integration

$\underline{\int}_a^b f(x)\,dx := \inf\{U(f,P)\}$; $\overline{\int}_a^b f(x)\,dx := \sup\{L(f,P)\}$

1. Let P be a partition, and let $f: [a,b] \to \mathbb{R}$ be bounded.
Recall that
$$U(f,P) = \sum_{i=1}^{n} M_i(x_i - x_{i-1}), \quad M_i = \sup\{f(x),\ x \in [x_{i-1}, x_i]\}; \ \&$$
$$L(f,P) = \sum_{i=1}^{n} m_i(x_i - x_{i-1}), \quad m_i = \inf\{f(x): x \in [x_{i-1}, x_i]\}.$$

2. Then, denote
$$\overline{\int}_a^b f(x)\,dx := \inf\{U(f,P): P \text{ is a partition}\}; \ \&$$
$$\underline{\int}_a^b f(x)\,dx := \sup\{L(f,P): P \text{ is a partition}\}.$$

3. Recall that f is integrable iff
$$\overline{\int}_a^b f(x)\,dx = \underline{\int}_a^b f(x)\,dx,$$
and in this case we set
$$\int_a^b f(x)\,dx = \overline{\int}_a^b f(x)\,dx = \underline{\int}_a^b f(x)\,dx.$$

RECTANGLE

A "rectangle" in $\mathbb{R}^n$ is a set of the form
$$R = [a_1,b_1] \times \cdots \times [a_n,b_n].$$

PARTITION [OF A RECTANGLE]

Let R be a rectangle, say
$$R = [a_1,b_1] \times \cdots \times [a_n,b_n].$$
Then, a "partition" of R is a grid of rectangles on R obtained by partitioning each $[a_i,b_i]$.

eg

this is a partition! this is not a partition.

VOLUME [OF A RECTANGLE]

Let $R \subseteq \mathbb{R}^n$ be a rectangle, say
$$R = [a_1,b_1] \times \cdots \times [a_n,b_n]$$
Then, the "volume" of R is
$$v(R) = (b_1 - a_1) \cdots (b_n - a_n).$$

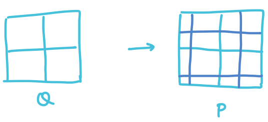
UPPER/LOWER SUM [OF f RELATIVE TO P]: $U(f,P)$ & $L(f,P)$

1. Let $R \subseteq \mathbb{R}^n$ be a rectangle, and let $f: R \to \mathbb{R}$ be bounded.
Let $P = \{R_1, \dots, R_n\}$ be a partition of R.
Then, the "upper sum" of f relative to P, denoted by $U(f,P)$, is defined by
$$U(f,P) = \sum_{i=1}^{n} M_i v(R_i),$$
where $M_i := \sup\{f(x): x \in R_i\}$.

2. Similarly, the "lower sum" of f relative to P, denoted by $U(f,P)$, is defined by
$$L(f,P) = \sum_{i=1}^{n} m_i v(R_i),$$
where $m_i := \inf\{f(x): x \in R_i\}$.

REFINEMENT [OF A PARTITION]: $P \leq Q$

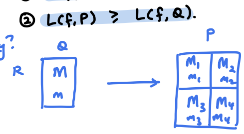
Let P, Q be partitions of $R \subseteq \mathbb{R}^n$.
Then, we say P is a "refinement" of Q, written "$P \leq Q$", if P is obtained from Q by partitioning the sides of R even further.



$P \leq Q \Rightarrow U(f,P) \leq U(f,Q),\ L(f,P) \geq L(f,Q)$

Let $P \leq Q$ on R, and let f be bounded.
Then necessarily
① $U(f,P) \leq U(f,Q)$; and
② $L(f,P) \geq L(f,Q)$.

why?



$$Mv(R) = Mv(R_1) + \cdots + Mv(R_4)$$
$$\geq M_1 v(R_1) + \cdots + M_4 v(R_4)$$
$$mv(R) \leq m_1 v(R_1) + \cdots + m_4 v(R_4).$$
Argument can be generalized. ☒

LOWER/UPPER INTEGRAL [OF f ON R]

Let $f: R \to \mathbb{R}$ be bd, where R is a rectangle.
Then, the "lower integral" of f, denoted by $\underline{\int}_R f$, is defined to be
$$\underline{\int}_R f = \sup\{L(f,P): P\}$$
and the "upper integral" of f, denoted by $\overline{\int}_R f$, is defined to be
$$\overline{\int}_R f = \inf\{U(f,P): P\}.$$

RIEMANN INTEGRABLE [OVER R]

Let $f: R \to \mathbb{R}$ be bd.
Then, we say f is "(Riemann) integrable" over R if
$$\int_R f := \underline{\int}_R f + \overline{\int}_R f.$$

SOME FUNCTIONS ARE NOT INTEGRABLE

Consider
$$f(x,y) = \begin{cases} 1, & x \in \mathbb{Q} \\ 0, & x \notin \mathbb{Q} \end{cases}$$
Then
$$\forall P,\ U(f,P) = 1$$
$$\& \ \forall P,\ L(f,P) = 0$$
So
$$\underline{\int}_R f = 0 \neq 1 = \overline{\int}_R f.$$

$f: R \to \mathbb{R}$ BD $\Rightarrow L(f,P) \leq U(f,Q)$

1. Let $f: R \to \mathbb{R}$ be bounded, and let P & Q be partitions of R.
Then necessarily $L(f,P) \leq U(f,Q)$.

Why? find a common refinement $S \leq P$, $S \leq Q$. (eg "overlap" P & Q).
Then
$$L(f,P) \leq L(f,S) \leq U(f,S) \leq U(f,Q).$$

2. Note: we just proved for any P, Q:
$$L(f,P) \leq U(f,Q).$$
Thus
$$L(f,P) \leq \overline{\int_R} f$$
and
$$\underline{\int_R} f \leq \overline{\int_R} f.$$

$f: R \to \mathbb{R}$ IS INTEGRABLE $\Leftrightarrow$ $\forall \varepsilon > 0$, $\exists P \ni$ $U(f,P) - L(f,P) < \varepsilon$

Let $f: R \to \mathbb{R}$ be bd.
Then f is integrable iff for any $\varepsilon > 0$, there exists a partition P such that
$$U(f,P) - L(f,P) < \varepsilon.$$

Proof. $(\Rightarrow)$ Assume f is integrable, so that
$$\underline{\int_R} f = \overline{\int_R} f.$$
Let $\varepsilon > 0$. We know we may find partition $P, Q \ni$
$$\underline{\int_R} f - \frac{\varepsilon}{2} < L(f,P)$$
&
$$U(f,Q) < \overline{\int_R} f + \frac{\varepsilon}{2}.$$
Thus
$$U(f,Q) \leq L(f,P) + \varepsilon.$$
Let S be a common refinement of P & Q, so $S \leq P, Q$.
Thus
$$U(f,S) < U(f,Q) < L(f,P) + \varepsilon < L(f,S) + \varepsilon.$$
$$\Rightarrow U(f,S) - L(f,S) < \varepsilon.$$

$(\Leftarrow)$ Let $\varepsilon > 0$. We may find $P \ni$
$$U(f,P) - L(f,P) < \varepsilon.$$
Hence
$$0 \leq \overline{\int_R} f - \underline{\int_R} f \leq U(f,P) - L(f,P) < \varepsilon$$
$\Rightarrow \overline{\int_R} f = \underline{\int_R} f$ (as ε was arbitrary).

Module 10.3: Content and Measure

LEBESGUE MEASURE ZERO

Let $A \subseteq \mathbb{R}^n$.
Then, we say A has "(Lebesgue) measure zero"
if for all $\varepsilon > 0$, there exists rectangles
R_i $(i \in \mathbb{N})$ such that
$$A \subseteq \bigcup_{i=1}^{\infty} R_i$$
and
$$\sum_{i=1}^{\infty} v(R_i) < \varepsilon.$$

JORDAN CONTENT ZERO

Let $A \subseteq \mathbb{R}^n$.
Then, we say A has "(Jordan) content zero"
if for all $\varepsilon > 0$, there exists rectangles $R_1, \ldots, R_m$
such that
$$A \subseteq \bigcup_{i=1}^{m} R_i$$
and
$$\sum_{i=1}^{m} v(R_i) < \varepsilon.$$

Note that if $A \subseteq \mathbb{R}^n$ has content zero, then it has measure zero.

Proof. Let $\varepsilon > 0$. Suppose $A \subseteq \mathbb{R}^n$ has content zero.
Then, $\exists$ rects $R_1, \ldots, R_m$ $\ni$
$$A \subseteq R_1 \cup \cdots \cup R_m \quad \& \quad \sum_{i=1}^{m} v(R_i) < \varepsilon.$$
Now, for $i > m$, let $R_i \subseteq \mathbb{R}^n$ be any rectangle w/ volume 0.
$$\therefore A \subseteq R_1 \cup \cdots \cup R_m \cup R_{m+1} \cup \cdots \quad \& \quad \sum_{i=1}^{\infty} v(R_i) < \varepsilon. \quad \blacksquare$$

The converse is not true!

eg $A = \mathbb{Q} \subseteq \mathbb{R}$.

① $\mathbb{Q}$ has measure zero.

Proof. We know $|\mathbb{Q}| = |\mathbb{N}|$, ie
$$\mathbb{Q} = \{q_1, q_2, \ldots\}$$
Let $\varepsilon > 0$. Let $R_i = [q_i - \frac{\varepsilon}{2^{i+2}}, q_i + \frac{\varepsilon}{2^{i+2}}]$.
Then
$$\mathbb{Q} \subseteq \bigcup_{i=1}^{\infty} R_i.$$
And
$$\sum_{i=1}^{\infty} v(R_i) = \sum_{i=1}^{\infty} \frac{\varepsilon}{2^{i+1}} = \frac{\varepsilon}{4} \sum_{i=1}^{\infty} \frac{1}{2^{i-1}} = \frac{\varepsilon}{4}\left(\frac{1}{1-\frac{1}{2}}\right) = \frac{\varepsilon}{2} < \varepsilon.$$

② $\mathbb{Q}$ does not have content zero.

Proof. Because it is unbounded. $\blacksquare$

Note that if A has Jordan content zero, it has to be bounded.

$A_i \subseteq \mathbb{R}^n$, $i \in \mathbb{N}$ HAVE MEASURE ZERO $\Rightarrow$ $A = \bigcup_{i=1}^{\infty} A_i$ HAS MEASURE ZERO

Let $A_1, A_2, \ldots \subseteq \mathbb{R}^n$ have measure zero.
Then necessarily $A = \bigcup_{i=1}^{\infty} A_i$ has measure zero.

Why? Let $\varepsilon > 0$. We know for each A_i,
$$A_i \subseteq \bigcup_{j=1}^{\infty} R_{i,j}.$$
Also
$$\sum_{j=1}^{\infty} v(R_{i,j}) < \frac{\varepsilon}{2^i}.$$
Then
$$A \subseteq \bigcup_{i,j} R_{i,j}.$$
Then
$$\sum_{i,j} v(R_{i,j}) = \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} v(R_{i,j}) < \sum_{i=1}^{\infty} \frac{\varepsilon}{2^i} = \frac{\varepsilon}{2} \sum_{i=1}^{\infty} \frac{1}{2^{i-1}} = \varepsilon. \quad \blacksquare$$

$A \subseteq \mathbb{R}^n$ IS COMPACT AND HAS MEASURE ZERO $\Rightarrow$ A HAS CONTENT ZERO

Let $A \subseteq \mathbb{R}^n$ be compact, and have measure zero.
Then necessarily A has content zero.

Proof. Let $\varepsilon > 0$. By asmt, $\exists$ open rects R_i $\ni$
$$A \subseteq \bigcup_{i=1}^{\infty} R_i \quad \& \quad \sum_{i=1}^{\infty} v(R_i) < \varepsilon.$$
Then, as A is compact,
$$A \subseteq \bigcup_{i=1}^{m} R_i$$
for some m.
Moreover,
$$\sum_{i=1}^{m} v(R_i) \leq \sum_{i=1}^{\infty} v(R_i) < \varepsilon. \quad \blacksquare$$

Module 10.4: Integrability and Measure

OSCILLATION [OF f AT a]: $\theta(f,a)$

Let $A \subseteq \mathbb{R}^n$, and let $f: A \to \mathbb{R}$ be bounded.
For $a \in A$ & $\delta > 0$, denote

$M(a,f,\delta) = \sup\{f(x) : x \in A,\ \|x-a\| < \delta\}$; &

$m(a,f,\delta) = \inf\{f(x) : x \in A,\ \|x-a\| < \delta\}$.

Then, the "oscillation of f at a", denoted by "$\theta(f,a)$", is defined to be

$$\theta(f,a) = \lim_{\delta \to 0} M(a,f,\delta) - m(a,f,\delta).$$

Note that
① $\theta(f,a)$ always exists; and
② f is cts at a iff $\theta(f,a) = 0$.

$A \subseteq \mathbb{R}^n$ IS CLOSED, $f: A \to \mathbb{R}$ IS BD $\Rightarrow \forall \varepsilon > 0$, $\{x \in A : \theta(f,x) \geq \varepsilon\}$ IS CLOSED

Let $A \subseteq \mathbb{R}^n$ be closed, and let $f: A \to \mathbb{R}$ be bounded.
Then for any $\varepsilon > 0$, necessarily $\{x \in A : \theta(f,x) \geq \varepsilon\}$ is closed.

Proof. Let $B = \{x \in A : \theta(f,x) \geq \varepsilon\}$.
Take $x \in A \setminus B$. We will show $A \setminus B$ is rel open in A.
$\Rightarrow \theta(f,x) < \varepsilon$, and so $\exists \delta > 0$ with
$M(x,f,\delta) - m(x,f,\delta) < \varepsilon$.
Consider $y \in B_{\frac{\delta}{2}}(x) \cap A$.
Then, for $z \in A$ w/ $|y-z| < \frac{\delta}{2}$, thus
$|z-x| \leq |z-y| + |y-x| < \delta$,
and
$m(x,f,\delta) \leq f(z) \leq M(x,f,\delta)$.
Thus
$M(y,f,\frac{\delta}{2}) - m(y,f,\frac{\delta}{2}) < \varepsilon$.
$\Rightarrow \theta(f,y) < \varepsilon$
$\Rightarrow B_{\frac{\delta}{2}}(x) \cap A \subseteq A \setminus B$
$\Rightarrow A \setminus B$ is rel open in A
$\Rightarrow B$ is rel closed in A
$\Rightarrow B$ is closed (since A is closed). ▨

$f: R \to \mathbb{R}$ BD, $\theta(f,x) < \varepsilon\ \forall x \in R \Rightarrow \exists P \ni U(f,P) - L(f,P) < \varepsilon \cdot v(R)$

Let $R \subseteq \mathbb{R}^n$ be a rectangle, and let $f: R \to \mathbb{R}$ be bounded.
Let $\varepsilon > 0$. Suppose for each $x \in R$, we have $\theta(f,x) < \varepsilon$.
Then necessarily there exists a partition P such that

$U(f,P) - L(f,P) < \varepsilon \cdot v(R)$.

Proof. For all $x \in R$, $\exists \delta_x > 0 \ni$
$M(x,f,\delta_x) - m(x,f,\delta_x) < \varepsilon$.
For all $x \in R$, let R_x be an open rect s.t.
$x \in R_x \subseteq B_{\frac{\delta_x}{2}}(x)$.
Let $U_x = R_x \cap R$. Then
$R = \bigcup_{x \in R} U_x$
is a rel open cover of R. Since R is compact,
$\exists x_1, \dots, x_m \in R \ni$
$R = U_{x_1} \cup \dots \cup U_{x_m}$.
Let P be a partition of R so fine $\ni$ each subrectangle in P is contained in some $\overline{U_{x_i}}$.
Note
$\overline{U_{x_i}} = \overline{R_{x_i}} \cap R$
$\subseteq \overline{B_{\frac{\delta_{x_i}}{2}}(x_i)} \cap R$ (by const^n of R_{x_i})
$\subseteq B_{\delta_{x_i}}(x_i) \cap R$.
$\therefore$ for every $R_i \in P$,
$M_i - m_i < \varepsilon$.
$\Rightarrow U(f,P) - L(f,P) = \sum_{R_i \in P} (M_i - m_i) v(R_i)$
$< \sum_{R_i \in P} \varepsilon v(R_i)$
$= \varepsilon v(R)$. ▨

$f: R \to \mathbb{R}$ BD, $A = \{x \in R : f$ IS NOT CTS AT $x\}$; f IS INTEGRABLE $\iff$ A HAS MEASURE ZERO

Let $R \subseteq \mathbb{R}^n$ be a rectangle, and let $f: R \to \mathbb{R}$ be bounded.
Let $A = \{x \in R : f$ is not cts at $x\}$.
Then necessarily f is integrable iff A has measure zero.

Proof. ($\Leftarrow$)
Let $B = \{x \in R : \theta(f,x) \geq \varepsilon\}$, so B is compact (R is bounded) from our first proposition.
Since $B \subseteq A$, & $\theta = 0$ at pts of continuity, then B also has measure zero.
Since B is compact, B also has content zero.
In particular, $\exists$ finitely many rectangles $U_1, \dots, U_m$ (by assmt) whose interiors cover B & $\sum v(U_i) < \varepsilon$.
Let X denote the set of subrects of R contained in ≥ 1 U_i.
Let Y denote the set of subrectangles of R contained in $R \setminus B$.
Now, since the U_i's cover B, we may find a partition $P = \{R_1, \dots, R_k\}$ fine enough so that the elements of P are either from X or Y.
Then, since f is bounded, $\exists M > 0 \ni |f(x)| \leq M\ \forall x \in R$.
In particular, $\forall R_i \in P$, $M_i - m_i \leq 2M$. By def^n of X:
$$\sum_{R_i \in X} (M_i - m_i) v(R_i) \leq 2M \sum_{R_i \in X} v(R_i) \leq 2M \sum_{i=1}^{m} v(U_i) < 2M\varepsilon.$$
Now, if $R_i \in Y$ & $x \in R_i$, we have $\theta(f,x) < \varepsilon$. By the 2nd prop in this page, we may find a partition $P_i = \{S_{i_1}, \dots, S_{i_{\alpha(i)}}\}$ of $R_i \ni$
$$\sum_{j=1}^{\alpha(i)} (M_j - m_j) v(S_{i_j}) < \varepsilon v(R_i).$$
By replacing each $R_i \in Y$ w/ $S_{i_1}, \dots, S_{i_{\alpha(i)}}$ (and leaving $R_i \in X$ alone), this creates a refinement $Q \subseteq P$. Finally,
$$U(f,Q) - L(f,Q) = \sum_{R_i \in X} (M_i - m_i) v(R_i) + \sum_{R_i \in Y} \sum_{j=1}^{\alpha(i)} (M_j - m_j) v(S_{i_j})$$
$$< 2M\varepsilon + \sum_{R_i \in Y} \varepsilon v(R_i)$$
$$\leq 2M\varepsilon + \varepsilon v(R),$$
which can be made arbitrarily small, and so f is integrable, as needed. #

($\Rightarrow$)
For every $n \in \mathbb{N}$, let $B_n = \{x \in R : \theta(f,x) \geq \frac{1}{n}\}$.
As $A = B_1 \cup B_2 \cup \cdots$, it suffices to show each B_n has measure zero.
Fix $n \in \mathbb{N}$. Since f is integrable, we may find a partition P of $R \ni$
$$U(f,P) - L(f,P) < \frac{\varepsilon}{n}.$$
Let X be the collection of rects in P that intersect B_n.
In particular, the elements of X cover B_n and are rectangles!
Now, if $R_i \in X$, then $M_i - m_i \geq \frac{1}{n}$ by def^n of θ. Then
$$\frac{1}{n} \sum_{R_i \in X} v(R_i) \leq \sum_{R_i \in X} (M_i - m_i) v(R_i)$$
$$\leq \sum_{R_i \in P} (M_i - m_i) v(R_i)$$
$$= U(f,P) - L(f,P) < \frac{\varepsilon}{n},$$
and so $\sum_{R_i \in X} v(R_i) < \varepsilon$, and so B_n has measure (content) zero.

Module 11.1: General Integrability

CHARACTERISTIC FUNCTION [OF A ON R]: $\chi_A(x)$

Let $A \subseteq \mathbb{R}^n$ be bd, and let R be a rectangle such that $A \subseteq R$.

Then, the "characteristic function" of A on R, denoted by χ_A, is

$$\chi_A : R \to \mathbb{R}$$

defined by

$$\chi_A(x) = \begin{cases} 1, & x \in A \\ 0, & x \notin A \end{cases}.$$

RIEMANN INTEGRABLE [OVER A]

(1) Let $A \subseteq \mathbb{R}^n$ be bd, and let $f: A \to \mathbb{R}$ be bd.

Let R be a rectangle with $A \subseteq R$.

Then, extend $f: R \to \mathbb{R}$ by setting $f(x) = 0 \ \forall x \in R \setminus A$.

We say $f: A \to \mathbb{R}$ is "integrable" iff $f \cdot \chi_A : R \to \mathbb{R}$ is integrable, in which case we define

$$\int_A f := \int_R f \cdot \chi_A .$$

(2) Note that this definition is independent of choice of $A \subseteq R$. (asmt).

(3) Note that if $f: R \to \mathbb{R}$ & $\chi_A : R \to \mathbb{R}$ are integrable, then $f \cdot \chi_A$ is also integrable, so that f is integrable on A.

$\chi_A : R \to \mathbb{R}$ IS INTEGRABLE $\Leftrightarrow$ $\partial(A)$ HAS MEASURE ZERO

Let $A \subseteq \mathbb{R}^n$ be bd, and let $A \subseteq R$ for some rect R.

Then, $\chi_A : R \to \mathbb{R}$ is integrable iff $\partial(A)$ has measure zero.

Proof. Let $a \in R$.

(1) $a \in \text{Int}(A)$.

$\Rightarrow \exists$ open ball $B_\delta(a) \subseteq A$.

Since $\chi_A = 1$ on $B_\delta(a)$, χ_A is clearly cts at a.

(2) $a \notin \overline{A}$.

$\Rightarrow a \in \text{Int}(\mathbb{R}^n \setminus A)$. $\Rightarrow \exists B_\delta(a) \subseteq \mathbb{R}^n \setminus A$.

Since $\chi_A = 0$ on $B_\delta(a) \cap R$, χ_A is clearly cts at a.

(3) $a \in \overline{A} \setminus \text{Int}(A) = \partial(A)$

$\Rightarrow$ then $a \in \overline{A}$ & $a \in \mathbb{R}^n \setminus \text{Int}(A)$
$= \overline{\mathbb{R}_n \setminus A}$.

$\Rightarrow \forall \delta > 0, \exists x \in A, y \in R \setminus A \ni$
$\|x - a\|, \|y - a\| < \delta$.

Thus

$\theta(\chi_A, a) \geq 1$

So f is not cts at a.

$\Rightarrow$ set of discontinuities of f is exactly $\partial(A)$.

$\Rightarrow$ proof follows from big theorem in previous thm.

JORDAN REGION

Let $A \subseteq \mathbb{R}^n$ be bd.

Then, we say A is a "Jordan region" iff $\partial(A)$ has measure zero

(ie iff χ_A is integrable on $R \supseteq A$).

VOLUME [OF A JORDAN REGION]: Vol(A)

Let A be a Jordan region, with $A \subseteq R$.

Then, the "volume" of A, denoted by "Vol(A)", is defined to be

$$\text{Vol}(A) = \int_R \chi_A = \int_A 1 .$$

A, B ARE JORDAN REGIONS $\Rightarrow$ A ∪ B IS A JORDAN REGION; $A \cap B = \emptyset$ & $f: A \cup B \to \mathbb{R}$ IS INTEGRABLE $\Rightarrow \int_{A \cup B} f = \int_A f + \int_B f$

Let $A, B \subseteq \mathbb{R}^n$ be Jordan regions.

Then necessarily

(1) $A \cup B$ is a Jordan region; and

(2) If $A \cap B = \emptyset$ & $f: A \cup B \to \mathbb{R}$ is integrable, then

$$\int_{A \cup B} f = \int_A f + \int_B f .$$

why? (1) $\partial(A \cup B) = (\overline{A \cup B}) \setminus \text{Int}(A \cup B)$

$= (\overline{A} \cup \overline{B}) \setminus (\text{Int}(A) \cup \text{Int}(B))$

$= (\overline{A} \setminus \text{Int}(A)) \setminus (\overline{B} \setminus \text{Int}(B))$

$= \partial(A) \cup \partial(B)$.

(2) Let $A \cup B \subseteq R$. Then

$$\begin{aligned} \int_{A \cup B} f &= \int_R f \cdot \chi_{A \cup B} \\ &= \int_R f(\chi_A + \chi_B) && \text{(since } A \cap B = \emptyset) \\ &= \int_R f \chi_A + \int_R f \chi_B && \text{(by asmt)} \\ &= \int_A f + \int_B f . \quad \blacksquare \end{aligned}$$

Module 11.2: Fubini's Theorem

$B \subseteq \mathbb{R}^2$ JR ; $\int_B f(v)\, dv \equiv \iint_B f(x,y)\, dA$;

$B \subseteq \mathbb{R}^3$ JR : $\int_B f(v)\, dv \equiv \iiint_B f(x,y,z)\, dV$

1. Let $B \subseteq \mathbb{R}^2$ be a JR, and let $f: B \to \mathbb{R}$ be integrable.
   Then, we denote
   $$\int_B f(v)\, dv \equiv \iint_B f(x,y)\, dA.$$

2. Similarly, if $B \subseteq \mathbb{R}^3$ is a JR & $f: B \to \mathbb{R}$ is integrable, then
   $$\int_B f(v)\, dv \equiv \iiint_B f(x,y,z)\, dV.$$

$R = [a,b] \times [c,d] \subseteq \mathbb{R}^2$, $f: R \to \mathbb{R}$ BD, $f(x,\cdot): [c,d] \to \mathbb{R}$ IS INTEGRABLE $\forall x \in [a,b]$ $\Rightarrow$

$$\underline{\iint}_R f(x,y)\, dA \le \underline{\int_a^b}\left(\int_c^d f(x,y)\, dy\right) dx \le \overline{\int_a^b}\left(\int_c^d f(x,y)\, dy\right) dx \le \overline{\iint}_R f(x,y)\, dA$$

Let $R = [a \times b] \times [c \times d] \subseteq \mathbb{R}^2$, and let $f: R \to \mathbb{R}$ be bd.
Let $f(x,\cdot): [c,d] \to \mathbb{R}$ by $f(x,\cdot)(y) = f(x,y)$ be integrable $\forall x \in [a,b]$.
Then necessarily
$$\underline{\iint}_R f(x,y)\, dA \le \underline{\int_a^b}\left(\int_c^d f(x,y)\, dy\right) dx$$
$$\le \overline{\int_a^b}\left(\int_c^d f(x,y)\, dy\right) dx$$
$$\le \overline{\iint}_R f(x,y)\, dA.$$

Proof. Middle ineq is trivial.
We prove the last ineq, and leave the first as an exercise.
Let $\varepsilon > 0$. Choose a partition P on R ∋
$$U(f,P) - \varepsilon \le \overline{\iint}_R f(x,y)\, dA,$$
say
$$P = \{R_{ij} : 1 \le i \le k,\ 1 \le j \le \ell\},$$
with $R_{ij} = [x_{i-1}, x_i] \times [y_{j-1}, y_j]$,
where $x_0 = a$, $x_k = b$, $y_0 = c$, $y_\ell = d$.
Set $M_{ij} = \sup\{f(v) : v \in R_{ij}\}$.
Then
$$\overline{\int_a^b}\left(\int_c^d f(x,y)\, dy\right) dx = \sum_{i=1}^{k} \overline{\int_{x_{i-1}}^{x_i}}\left(\sum_{j=1}^{\ell}\int_{y_{j-1}}^{y_j} f(x,y)\, dy\right) dx$$
$$\le \sum_{i=1}^{k}\sum_{j=1}^{k} \overline{\int_{x_{i-1}}^{x_i}}\left(\int_{y_{j-1}}^{y_j} f(x,y)\, dy\right) dx \quad \text{(addition of sups property)}$$
$$\le \sum_i \sum_j \int_{x_{i-1}}^{x_i}\left(\int_{y_{j-1}}^{y_j} M_{ij}\, dy\right) dx$$
$$= \sum_i \sum_j M_{ij}(x_i - x_{i-1})(y_j - y_{j-1})$$
$$= \sum_{i,j} M_{ij}\, v(R_{ij})$$
$$= U(f,P)$$
$$\le \overline{\iint}_R f(x,y)\, dA + \varepsilon.$$

$R = [a,b] \times [c,d] \subseteq \mathbb{R}^2$, $f: R \to \mathbb{R}$ INTEGRABLE, $f(x,\cdot)$, $f(\cdot,y)$ INTEGRABLE $\Rightarrow$ $\iint_R f(x,y)\, dA = \int_a^b\int_c^d f(x,y)\, dy\, dx = \int_c^d\int_a^b f(x,y)\, dx\, dy$ << FUBINI'S THEOREM >>

1. Let $R = [a,b] \times [c,d] \subseteq \mathbb{R}^2$, and let $f: R \to \mathbb{R}$ be integrable.
   Suppose $f(x,\cdot)$ and $f(\cdot,y)$ are integrable over $[c,d]$ & $[a,b]$, respectively, for all $x \in [a,b]$, $y \in [c,d]$.
   Then necessarily
   $$\iint_R f(x,y)\, dA = \int_a^b\int_c^d f(x,y)\, dy\, dx = \int_c^d\int_a^b f(x,y)\, dx\, dy.$$

   Proof. Since f is integrable,
   $$\underline{\iint}_R f(x,y)\, dA = \overline{\iint}_R f(x,y)\, dA.$$
   By the prev lemma,
   $$\underline{\int_a^b}\int_c^d f(x,y)\, dy\, dx = \overline{\int_a^b}\int_c^d f(x,y)\, dy\, dx.$$
   $$\Rightarrow \iint_R f(x,y)\, dA = \int_a^b\int_c^d f(x,y)\, dy\, dx.$$
   Reversing the roles of x & y proves the thm. ■

2. Note that this also applies when f is cts on R (because this satisfies the conditions needed.)

ITERATED INTEGRALS

An "iterated integral" is an integral of the form
$$\int_{a_n}^{b_n} \cdots \int_{a_1}^{b_1} f(x_1, \ldots, x_n)\, dx_1 \cdots dx_n.$$

EX: $R = [1,2] \times [0,\pi]$: $\iint_R y\sin(xy)\, dA$

Sol^n. Note that f, $f(x,\cdot)$, $f(\cdot,y)$ are all cts on closed JR's $\Rightarrow$ they are all integrable.
$$\iint_R y\sin(xy)\, dA \overset{FT}{=} \int_0^\pi \int_1^2 y\sin(xy)\, dx\, dy$$
$$= \int_0^\pi \left[-\cos(xy)\right]_{x=1}^{x=2} dy$$
$$= \int_0^\pi -\cos(2y) + \cos(y)\, dy$$
$$= \left[-\tfrac{1}{2}\sin(2y) + \sin(y)\right]_0^\pi$$
$$= (0 - 0)$$
$$= 0. \ \#$$

Module 11.3-4: Iterated Integrals

$R = [a_1,b_1] \times \cdots \times [a_n,b_n] \subseteq \mathbb{R}^n$, $f: R \to \mathbb{R}$ INTEGRABLE, $f(x,\cdot)$ INTEGRABLE $\forall x \in R_n \Rightarrow \int_{a_n}^{b_n} f(x,t)\,dt$ IS INTEGRABLE ON R_n & $\int_R f(v) = \int_{R_n}\int_{a_n}^{b_n} f(x,t)\,dt\,dx$

$①$ Let $R = [a_1,b_1] \times \cdots \times [a_n,b_n] \subseteq \mathbb{R}^n$, and let $f: R \to \mathbb{R}$ be integrable.

Let $R_n = [a_1,b_1] \times \cdots \times [a_{n-1},b_{n-1}]$.

Then, if $f(x,\cdot)$ is integrable for each $x \in R_n$, then

① $\int_{a_n}^{b_n} f(x,t)\,dt$ is integrable; and

② $\int_R f(v)\,dv = \int_{R_n}\int_{a_n}^{b_n} f(x,t)\,dt\,dx$.

$②$ Note once again that if f is cts, then

$$\int_R f(v)\,dv = \int_{a_1}^{b_1} \cdots \int_{a_n}^{b_n} f(x_1,\dots,x_n)\,dx_n \cdots dx_1.$$

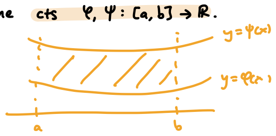
TYPE-1 & TYPE-2 [REGIONS IN $\mathbb{R}^2$]

$①$ We say $A \subseteq \mathbb{R}^2$ is "type 1" if

$A = \{(x,y) : x \in [a,b],\ \varphi(x) \le y \le \psi(x)\}$

for some cts $\varphi, \psi : [a,b] \to \mathbb{R}$.

eg

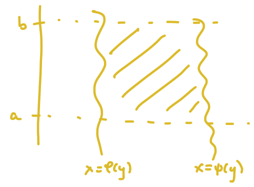


$②$ Similarly, we say $A \subseteq \mathbb{R}^2$ is "type 2" if

$A = \{(x,y) : y \in [a,b],\ \varphi(y) \le x \le \psi(y)\}$

for some cts $\varphi, \psi : [a,b] \to \mathbb{R}$.

eg



TYPE 1,2,3 [REGIONS IN $\mathbb{R}^3$]

$①$ Let $A \subseteq \mathbb{R}^3$. Then

① A is "type 1" if it is of the form

$A = \{(x,y,z) : (x,y) \in H,\ \varphi(x,y) \le z \le \psi(x,y)\}$;

② A is "type 2" if it is of the form

$A = \{(x,y,z) : (x,z) \in H,\ \varphi(x,z) \le y \le \psi(x,z)\}$; and

③ A is "type 3" if it is of the form

$A = \{(x,y,z) : (y,z) \in H,\ \varphi(y,z) \le x \le \psi(y,z)\}$,

where $H \subseteq \mathbb{R}^2$ is a closed Jordan region & $\varphi, \psi : H \to \mathbb{R}$ are cts.

TYPE 1,2,3 REGIONS ARE JORDAN REGIONS

$①$ Note that regions of type 1, 2 or 3 (in $\mathbb{R}^2$ or $\mathbb{R}^3$) are Jordan regions.

$A \subseteq \mathbb{R}^2$, $f: A \to \mathbb{R}$ CTS: A IS TYPE-1 $\Rightarrow$ $\int_A f(v)\,dv = \int_a^b \int_{\varphi(x)}^{\psi(x)} f(x,y)\,dy\,dx$; A IS TYPE-2 $\Rightarrow$ $\int_A f(v)\,dv = \int_a^b \int_{\varphi(y)}^{\psi(y)} f(x,y)\,dx\,dy$

$①$ Let $A \subseteq \mathbb{R}^2$, and let $f: A \to \mathbb{R}$ be cts.

Suppose A is type-1, so that

$A = \{(x,y) : x \in [a,b],\ \varphi(x) \le y \le \psi(x)\}$

for some cts $\varphi, \psi : [a,b] \to \mathbb{R}$.

Then

$$\int_A f(v)\,dv = \int_a^b \int_{\varphi(x)}^{\psi(x)} f(x,y)\,dy\,dx.$$

$②$ Similarly, suppose A is type-2, so that

$A = \{(x,y) : y \in [a,b],\ \varphi(y) \le x \le \psi(y)\}$

for some cts $\varphi, \psi : [a,b] \to \mathbb{R}$.

Then

$$\int_A f(v)\,dv = \int_a^b \int_{\varphi(y)}^{\psi(y)} f(x,y)\,dx\,dy.$$

Proof. We prove the first part; second part is similar.

Let $R = [a,b] \times [c,d]$ be a rect containing A.

Extend f to R by setting $f = 0$ on $R \setminus A$.

By Fubini,

$$\int_A f(v)\,dv = \int_R f(v)\,dv = \int_a^b \int_c^d f(x,y)\,dy\,dx.$$

However, $f(x,y) = 0$ if it is not the case that $\varphi(x) \le y \le \psi(x)$.

$$\therefore \int_A f(v)\,dv = \int_a^b \int_{\varphi(x)}^{\psi(x)} f(x,y)\,dy\,dx. \ \blacksquare$$

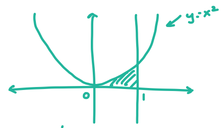
$③$ The analogous theorem exists for $\mathbb{R}^3$; eg if $A \subseteq \mathbb{R}^3$ is type-1 and $f: A \to \mathbb{R}$ is cts, then

$$\int_A f(v)\,dv = \int_H \int_{\varphi(x,y)}^{\psi(x,y)} f(u,z)\,dz\,du,$$

etc.

EX: $D \subseteq \mathbb{R}^2$ = REGION BD BY $y=0$, $y=x^2$, $x=1$: CALC $\iint_D x\cos(y)\,dA$.

Soln.



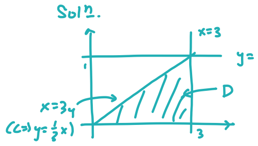
Region is Type-1.

So by thm.

$$\iint_D x\cos y\,dA = \int_0^1 \int_0^{x^2} x\cos(y)\,dy\,dx$$
$$= \int_0^1 [x\sin(y)]_{y=0}^{y=x^2}\,dx$$
$$= \int_0^1 x\sin(x^2)\,dx$$
$$= \left[-\tfrac{1}{2}\cos(x^2)\right]_0^1$$
$$= \tfrac{1}{2} - \tfrac{1}{2}\cos(1). \ \#$$

Ex: $\int_0^1 \int_{3y}^3 e^{x^2}\,dx\,dy$

Soln.

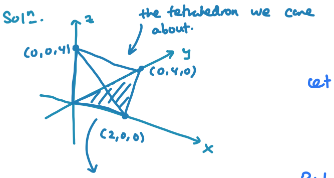


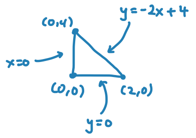
D is a region of type 1 & type 2.

$$\Rightarrow \int_0^1 \int_{3y}^3 e^{x^2}\,dx\,dy = \iint_D e^{x^2}\,dA$$
$$= \int_0^3 \int_0^{\frac{1}{3}x} e^{x^2}\,dy\,dx$$
$$= \int_0^3 \tfrac{1}{3}x e^{x^2}\,dx$$
$$= \left[\tfrac{1}{6}e^{x^2}\right]_0^3$$
$$= \tfrac{1}{6}(e^9 - 1). \ \#$$

EX: VOLUME OF TETRAHEDRON T ENCLOSED BY $x=0$, $y=0$, $z=0$, $2x+y+z=4$.

Sol$^{\underline{n}}$.





See that

$$T=\{(x,y,z):\ 0\le x\le 2,\ \ 0\le y\le -2x+4,\ \ 0\le z\le 4-2x-y\}.$$

Let

$$H=\{(x,y):\ 0\le x\le 2,\ \ 0\le y\le -2x+4\}.$$

$\Rightarrow$ H is type I in $\mathbb{R}^2$.

But also

$$T=\{(x,y,z):\ (x,y)\in H,\ \ 0\le z\le 4-2x-y\}.$$

$$\Rightarrow \iiint_T 1\,dV = \int_H \underbrace{\int_0^{4-2x-y}}_{\text{cts!}} 1\,dz\,dA$$

$$= \int_0^2\int_0^{-2x+4}\int_0^{4-2x-y} 1\,dz\,dy\,dx$$

$$= \int_0^2\int_0^{-2x+4} (4-2x-y)\,dy\,dx$$

$$= \int_0^2 \left[(4-2x)y-\tfrac{1}{2}y^2\right]_{y=0}^{y=4-2x} dx$$

$$= \int_0^2 \tfrac{1}{2}(4-2x)^2\,dx$$

$$= \cdots$$

$$= \frac{16}{3}. \quad \#$$

Module 12.1: Change of Variables

$U \subseteq \mathbb{R}^n$ OPEN, $A \subseteq U$ CLOSED JR, $f: A \to \mathbb{R}$ CTS, $\varphi \in C^1(U, \mathbb{R}^n)$, $\exists B \subseteq A \ni \mathrm{Vol}(B) = 0$, φ IS 1-1 ON $A \setminus B$, $J\varphi(a) \neq 0 \ \forall a \in A \setminus B$, $f: \varphi(A) \to \mathbb{R}$ IS CTS $\Rightarrow$ $\varphi(A)$ IS A JR, f IS INTEGRABLE ON $\varphi(A)$, $\int_{\varphi(A)} f(x)\,dx = \int_A f(\varphi(x))\,|Jf(x)|\,dx$

Let $U \subseteq \mathbb{R}^n$ be open, and let $A \subseteq U$ be a closed Jordan region.
Let $f: A \to \mathbb{R}$ be cts, and let $\varphi \in C^1(U, \mathbb{R}^n)$.
Suppose there exists some $B \subseteq A$ with
1. $\mathrm{Vol}(B) = 0$;
2. φ is injective on $A \setminus B$; and
3. $J\varphi(a) \neq 0 \quad \forall a \in A \setminus B$.

Also suppose $f: \varphi(A) \to \mathbb{R}$ is continuous.
Then necessarily
1. $\varphi(A)$ is a JR;
2. f is integrable on $\varphi(A)$; and
3. $\int_{\varphi(A)} f(x)\,dx = \int_A f(\varphi(x))\,|J\varphi(x)|\,dx$.

CARTESIAN → POLAR COORDINATES

Recall polar coordinates are coordinates of the form (r, θ), where "x" $= r\cos\theta$ & "y" $= r\sin\theta$.

Consider $\varphi \in C^1(\mathbb{R}^2, \mathbb{R}^2)$ by $\varphi(r, \theta) = (r\cos\theta, r\sin\theta)$.
Note φ is injective on $\mathbb{R}^2 \setminus \{(0, \theta) : 0 \le \theta < 2\pi\}$,
and see that $\{(0, \theta) : 0 \le \theta < 2\pi\}$ has volume zero.
Then,

$$|J\varphi(r, \theta)| = \left| \det \begin{pmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{pmatrix} \right|$$
$$= |r| = r \quad (\text{since } r > 0).$$

Therefore, given the right conditions,

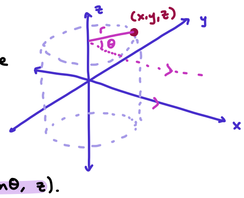
$$\iint_{\varphi(D)} f(x, y)\,dA = \iint_D f(r\cos\theta, r\sin\theta)\,r\,dA.$$

Ex: $\iint_D \cos(x^2 + y^2)\,dA$. D IS THE REGION BD BY $x^2 + y^2 = 9$ & ABOVE THE X-AXIS

Sol^n. See that
$D = \{(r, \theta) : 0 \le r \le 3, \ 0 \le \theta \le \pi\}$
$\Rightarrow \iint_D \cos(x^2 + y^2)\,dA = \iint_D \cos(r^2) \cdot r\,dA$
$= \int_0^\pi \int_0^3 \cos(r^2) \cdot r\,dr\,d\theta$
$= \int_0^\pi \left[\frac{1}{2}\sin(r^2)\right]_0^3 d\theta$
$= \int_0^\pi \frac{1}{2}\sin(9)\,d\theta$
$= \frac{\pi}{2}\sin(9)$.

CYLINDRICAL COORDINATES

Let $(x, y, z) \in \mathbb{R}^3$.
Then, we call the point (r, θ, z) (as shown to the right) the "cylindrical coordinates" of (x, y, z).



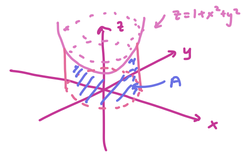
To convert, let
$\varphi(r, \theta, z) = (r\cos\theta, r\sin\theta, z)$.
Then

$$|J\varphi(r, \theta, z)| = \left| \det \begin{pmatrix} \cos\theta & -r\sin\theta & 0 \\ \sin\theta & r\cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix} \right|$$
$$= r,$$

so under the right conditions

$$\iiint_{\varphi(A)} f(x, y, z) = \iiint_A f(r\cos\theta, r\sin\theta, z)\,r\,dV.$$

Ex: $\iiint_A e^z\,dV$, A ENCLOSED BY 1) THE PARABOLOID $z = 1 + x^2 + y^2$, 2) $x^2 + y^2 = 5$, 3) THE XY-PLANE

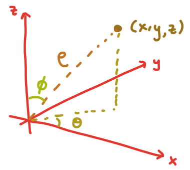


$A = \{(r, \theta, z) : 0 \le r \le \sqrt{5}, \ 0 \le \theta \le 2\pi, \ 0 \le z \le 1 + x^2 + y^2 = 1 + r^2\}$.

$\Rightarrow \iiint_A e^z\,dV = \int_0^{\sqrt{5}} \int_0^{2\pi} \int_0^{1+r^2} e^z \cdot r\,dz\,d\theta\,dr$
$= \int_0^{\sqrt{5}} \int_0^{2\pi} re^{1+r^2} - r\,d\theta\,dr$
$= 2\pi \int_0^{\sqrt{5}} re^{1+r^2} - r\,dr$
$= 2\pi \left[\frac{1}{2}e^{1+r^2} - \frac{1}{2}r^2\right]_0^{\sqrt{5}}$
$= 2\pi\left(\frac{1}{2}e^6 - \frac{5}{2} - \frac{1}{2}e\right)$
$= \pi(e^6 - 5 - e)$. ∎

SPHERICAL COORDINATES

Let $(x, y, z) \in \mathbb{R}^3$.
Then, the "spherical coordinates" of (x, y, z) is equal to (ρ, θ, ϕ), as shown in the right.



$0 \le \theta \le 2\pi$
$0 \le \phi \le \pi$

In particular,
1. $x = \rho\sin\phi\cos\theta$;
2. $y = \rho\sin\phi\sin\theta$;
3. $z = \rho\cos\phi$; and
4. $z^2 + y^2 + x^2 = \rho^2$.

To convert, consider
$\varphi(\rho, \theta, \phi) = (\rho\sin\phi\cos\theta, \rho\sin\phi\sin\theta, \rho\cos\phi)$,
and
$|J\varphi(\rho, \theta, \phi)| = \cdots = \rho^2\sin\phi$.
Hence

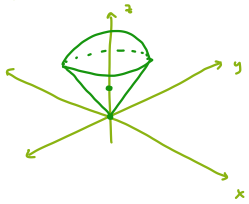
$$\iiint_{\varphi(A)} f(x, y, z)\,dV = \iiint_A f(\rho\sin\phi\cos\theta, \rho\sin\phi\sin\theta, \rho\cos\phi) \cdot \rho^2\sin\phi\,dV.$$

EX: VOLUME OF $x^2 + y^2 + z^2 = a^2$

Sol^n. Let S be the region (the sphere).
$\Rightarrow S = \{(\rho, \theta, \phi) : 0 \le \rho \le a, \ 0 \le \theta \le 2\pi, \ 0 \le \phi \le \pi\}$
& $\mathrm{Vol}(S) = \int_S 1\,dV$
$= \iiint_S 1\,dx\,dy\,dz$
$= \int_0^a \int_0^{2\pi} \int_0^\pi \rho^2\sin\phi\,d\phi\,d\theta\,d\rho$
$= \int_0^a \int_0^{2\pi} \left[-\rho^2\cos\phi\right]_0^\pi d\theta\,d\rho$
$= \int_0^a \int_0^{2\pi} 2\rho^2\,d\theta\,d\rho$
$= 2\pi \int_0^a 2\rho^2\,d\rho$
$= 2\pi\left[\frac{2}{3}\rho^3\right]_0^a$
$= \frac{4}{3}\pi a^3$.

EX: VOLUME OF SOLID WHICH 1) LIES ABOVE THE CONE $z=\sqrt{x^2+y^2}$, & 2) BELOW THE SPHERE $x^2+y^2+z^2=z$

Sol$^{\underline{n}}$. Note $x^2+y^2+z^2=z \iff x^2+y^2+(z-\frac{1}{2})^2=\frac{1}{4}$.



Cone:

$$\rho\cos\phi = \sqrt{\rho^2\sin^2\phi\cos^2\theta+\rho^2\sin^2\phi\sin^2\theta}$$
$$= \rho\sin\phi \qquad (\text{since } 0 \le \phi \le \pi)$$

$\Rightarrow$ the cone, $C=\{(\rho,\theta,\phi) : \rho=0 \text{ or } \phi=\frac{\pi}{4}\}$

Sphere:

$$\rho^2 = \rho\cos\phi$$

$\Rightarrow$ the sphere, $S=\{(\rho,\theta,\phi) : \rho=0 \text{ or } \rho=\cos\phi\}$.

letting D = above region,

volume of $D = \iiint_D 1\,dV = \int_0^{2\pi}\int_0^{\frac{\pi}{4}}\int_0^{\cos\phi}\rho^2\sin\phi\,d\rho\,d\phi\,d\theta$