MATH 249
Personal Notes

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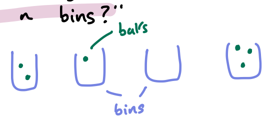
Section 1: Enumeration

THE BALLS & BINS EXAMPLE

Consider the following problem:
"How many ways are there to place k balls in n bins?"



This is an ill-formed question, since there is ambiguity.

RESTRICTIONS & DISTINGUISHABILITY

We can impose the following restriction on the balls:

① None

② At most 1 ball per bin; or

③ At least 1 ball per bin.

Similarly, we can impose the following distinguishability criterion for the bins & balls:

Ⓐ Balls and bins are all distinguishable;

Ⓑ Balls are indistinguishable, bins are not;

Ⓒ Bins are indistinguishable, balls are not;

Ⓓ Balls and bins are all indistinguishable.

Thus, combining the restriction & distinguishability criteria gives us 12 variations of the problem.

FORMALIZING THE PROBLEM

Each "way" of "placing the balls in bins" can be viewed as a function $f: K \to N$, where

① $|K| = k$ (ie the set of balls); &

② $|N| = n$ (ie the set of bins).

Then, each of the restrictions on the way corresponds to a respective restriction on the function:

① None;

② f is injective, &

③ f is surjective.

Similarly, each of the distinguishability criteria correspond to a respective equivalence relation on the functions:

Ⓐ $f \sim g \iff f = g$;

Ⓑ $f \sim g \iff \exists$ bijection $\alpha: K \to K$ s.t $f = g \circ \alpha$;

Ⓒ $f \sim g \iff \exists$ bijection $\beta: N \to N$ s.t. $f = \beta \circ g$; &

Ⓓ $f \sim g \iff \exists$ bijections $\alpha: K \to K$, $\beta: N \to N$ s.t. $f = \beta \circ g \circ \alpha$.

We can then re-state the problem:

"For each restriction / equivalence relation, determine the number of equivalence classes of functions from K to N, where $|K| = k$ & $|N| = n$."

FALLING/RISING FACTORIAL: $x^{\underline{k}}$, $x^{\overline{k}}$

The "rising factorial" notation is

$x^{\overline{k}} := x(x+1)\cdots(x+k-1)$

and the "falling factorial" notation is

$x^{\underline{k}} = x(x-1)\cdots(x-k+1)$.

* $x^{\overline{0}} = 1$;

$x^{\underline{0}} = 1$.

PROBLEM 1A

This is simply how many functions exist from K to N?

Answer: n^k.

Why? → n choices for each ball $b \in K$.

PROBLEM 2A

This is asking at most 1 ball per bin, and all the bins are distinguishable.

Answer: $n^{\underline{k}}$.

Why? Case #1: $k \leq n$.

Then n choices for 1st ball,

$n-1$ choices for 2nd ball,

⋮

$n-k+1$ choices for kth ball,

and as choices are independent, they

$\# = n(n-1)\cdots(n-k+1)$.

Case #2: $k > n$.

Then since no inj fns exist from K to N,

$\# = 0$,

and note that this is $n^{\underline{k}}$ since 0 is in the "product expansion" of $n^{\underline{k}}$.

We can write a more rigorous proof via the following:

Let $S =$ set of inj fns $f: K \to N$, $K = \{1, \ldots, k\}$ & $N = \{1, \ldots, n\}$.

Let $S_i = \{f \in S \mid f(k) = i\}$.

We then use induction to show that

$$|S_i| = (n-1)^{\underline{(k-1)}},$$

and so

$$\begin{aligned} |S| &= |S_1| + \cdots + |S_n| \\ &= n(n-1)^{\underline{(k-1)}} \\ &= n^{\underline{(k-1)}}. \end{aligned}$$

PROBLEM 2B

1. At most one ball per bin (2), and balls are indistinguishable (B).

2. This involves binomial coefficients;

   if $k \in \mathbb{Z}^+$, then $\binom{x}{k} = \frac{x^{\underline{k}}}{k!}$.

   If $0 \le k \le n$, then $\binom{n}{k} = \frac{n!}{k!(n-k)!}$ $\left(= \binom{n}{n-k}\right)$.

   But note that x can be anything! (real, complex, matrices, etc)

3. The answer is $\binom{n}{k}$.

Proof. Let $X = \{ f: K \to N \mid f \text{ is injective} \}$.

For $f, g \in X$, let $f \sim g \iff \exists$ bijection $\varphi: K \to K$ s.t. $f = g \circ \varphi$.

Let $Y = X/\sim$ be the set of equiv classes of the equiv relation $\sim$.

Then $|Y|$ is by defn the answer to the question.

eg $K = \{1,2\}$, $N = \{a,b,c\}$

Then $X = \{ \{1 \to a, 2 \to b\}, \{1 \to b, 2 \to a\}, \ldots \}$

& $Y = \{ \{ \{1\to a, 2\to b\}, \{1\to b, 2\to a\} \}, \{ \{1\to a, 2\to c\}, \{1\to c, 2\to a\} \}, \{ \{1\to b, 2\to c\}, \{1\to c, 2\to b\} \} \}$

Since if we swap 1 & 2, we get the other.

So $|Y| = 3$.

We saw $|X| = n^{\underline{k}}$ (Problem 2A).

But also $|X| = \sum_{C \in Y} |C|$ (since Y is a partition of X).

We claim $\forall C \in Y$, $|C| = k!$.

Proof. Let $C \in Y$, $g \in C$. Then

$$C = \{ f \in X \mid f = g \circ \varphi \text{ for some bij } \varphi: K \to K \}$$
$$= \{ g \circ \alpha \mid \alpha: K \to K \text{ is a bij} \}$$

As g is injective, $g \circ \varphi = g \circ \varphi' \iff \varphi = \varphi'$.

$\therefore |C| = \#$ of bijections $K \to K$
$= k!$. #

Finally, putting it together,

$$n^{\underline{k}} = |X| = \sum_{C \in Y} |C| = k! \cdot |Y|$$

$$\therefore |Y| = \frac{n^{\underline{k}}}{k!} = \binom{n}{k}. \quad \blacksquare$$

4. If $n, k \in \mathbb{Z}^+$, then $\binom{n}{k}$ is the # of k-element subsets of N, where $|N| = n$.

5. So, if $f, g \in X$, then $f \sim g \iff \mathrm{Image}(f) = \mathrm{Image}(g)$.

   Using this, we get a bijection

   $Y \leftrightarrow \{ k\text{-element subsets of } N \}$.

   Hence

   $C = \{ g \circ \varphi \mid \varphi \circ k \to k \text{ is bijective} \} \mapsto \mathrm{Image}(g)$.

   &

   $\{ f \in X \mid \mathrm{Image}(f) = S \} \leftarrow\!\shortmid S$.

n MULTICHOOSE k: $\left(\!\binom{x}{k}\!\right)$

We define

$$\left(\!\binom{x}{k}\!\right) = \frac{x^{\overline{k}}}{k!} = \binom{x+k-1}{k}.$$

PROBLEM 1B

1. No restriction on functions (1), and the balls are indistinguishable (B).

2. Interpretation: "k-element multisets" from an n-element set.

   eg $n=2$, $k=4$;

   There are 5 4-element multisets from $N = \{a,b\}$.

   $\Rightarrow \{a,a,a,a\}, \{a,a,a,b\}, \{a,a,b,b\}, \{a,b,b,b\}, \{b,b,b,b\}$.

   A multiset is a set where we can have an element more than once.

3. The answer is $\left(\!\binom{n}{k}\!\right)$.

Proof. Assume $N = \{1, \ldots, n\}$.

Let X = set of k-element multisets from N.

Let Y = set of k-element subsets of $\{1, 2, \ldots, n+k-1\}$.

Since we know $|Y| = \binom{n+k-1}{k}$, it suffices to show there's a bijection $f: X \to Y$.

Let's define f via an algorithm:

1. Let $A \in X$, and write $A = \{a_1, \ldots, a_n\}$, where $a_1 \le \cdots \le a_n$.
2. Define $f(A) = \{a_1, a_2+1, \ldots, a_k+k-1\}$.

To finish, verify f is well-defined and really is a bijection.

Details are exercise.

PROBLEM 3B

1. At least 1 ball per bin (3), and balls are indistinguishable (B).

   $\rightarrow$ this boils down to integer compositions.

PROBLEM 3D

1. At least 1 ball per bin (3), and balls & bins are indistinguishable (D).

   $\rightarrow$ this relates to integer partitions.

PROBLEM 3C

1. At least one ball per bin, but bins are indistinguishable.

2. We write $\left\{ {k \atop n} \right\}$ as the answer to this question.

3. These are called "Stirling numbers of the second kind".

PROBLEM 3A

1. This is counting the number of surjective functions from K to N.

2. <u>Answer</u>: $n! \cdot \left\{ {k \atop n} \right\}$.

ENUMERATION FRAMEWORK

1. We introduce sets, including the set of objects, we wish to count.
   ie we find sets that "represent" the objects we wish to count.
2. Then, we establish relationships between these sets.
   - may involve unions, bijections, cartesian products.
3. Next, we turn said relationships into formulas for cardinalities.

$[n] = \{1, \dots, n\}$

We denote $[n] = \{1, \dots, n\}$ for $n \in \mathbb{N}$.

$[x^k]\, A(x) = a_k$

Let $A(x) = \sum_{j \geq 0} a_j x^j$. Then we write

$$[x^k]A(x) = a_k.$$

$(1-x)^{-a} = \sum_{k \geq 0} \left(\!\binom{a}{k}\!\right) x^k$

<< NEGATIVE BINOMIAL THEOREM >>

Note that

$$(1-x)^{-a} = \sum_{k \geq 0} \left(\!\binom{a}{k}\!\right) x^k.$$

Proof. $(1-x)^{-a} = (1+(-x))^{-a}$

$$= \sum_{k \geq 0} \binom{-a}{k} (-x)^k.$$

Then check that

$$\binom{-a}{k}(-1)^k = \left(\!\binom{a}{k}\!\right).$$

COMBINATORIAL IDENTITIES

$\binom{n}{k} = \binom{n-1}{k} + \binom{n-1}{k-1}$ << PASCAL'S IDENTITY >>

* we say $\binom{x}{n} = 0$ if $n < 0$.

1. We can prove $\binom{n}{k} = \binom{n-1}{k} + \binom{n-1}{k-1}$.

Proof 1. (Combinatorial)

Let $X = \{k\text{-element subsets of } [n]\}$, &
$Y = \{k \text{ or } (k-1)\text{-element subsets of } [n-1]\}$.

Then $|X| = \binom{n}{k}$, $= \text{LHS}$, and $|Y| = \binom{n-1}{k} + \binom{n-1}{k-1}$ $= \text{RHS}$.

We prove $|X| = |Y|$ by giving a bijection between X & Y.

Define $f : X \to Y$ as follows:
for $A \in X$, let $f(A) = A \setminus \{n\}$.

Note $f(A) \subseteq [n-1]$, and $f(A)$ has either k or $k-1$ elements.

$\Rightarrow f(A) \in Y$.

Then, the inverse map $g : Y \to X$ is as follows:

for $B \in Y$, $g(B) = \begin{cases} B & \text{if } |B| = k; \\ B \cup \{n\} & \text{if } |B| = k-1. \end{cases}$

Exercise: check g is well-defined, and $g \circ f = id_X$, $f \circ g = id_Y$.

This suffices to show f is a bijection, as needed. ☐

Proof 2. (Algebraic)

We'll prove more generally that

$$\binom{a}{k} = \binom{a-1}{k} + \binom{a-1}{k-1} \quad \forall a \in \mathbb{R},\ k \in \mathbb{N}$$

Recall the binomial theorem:

$$(1+x)^a = \sum_{j \geq 0} \binom{a}{j} x^j \quad \forall |x| < 1$$

We can write this as

$$[x^k](1+x)^a = \binom{a}{k}.$$

Now, start with

$$(1+x)^{a-1} = \sum_{j \geq 0} \binom{a-1}{j} x^j$$

Multiply both sides by $(1+x)$:

$$\Rightarrow (1+x)^a = (1+x) \sum_{j \geq 0} \binom{a-1}{j} x^j$$

$$= \sum_{j \geq 0} \binom{a-1}{j} x^j + x \sum_{j \geq 0} \binom{a-1}{j} x^j$$

$$= \sum_{j \geq 0} \binom{a-1}{j} x^j + \sum_{j \geq 0} \binom{a-1}{j} x^{j+1}$$

$$= \left(1 + \sum_{j \geq 1} \binom{a-1}{j} x^j\right) + \sum_{j \geq 1} \binom{a-1}{j-1} x^j$$

$$= 1 + \sum_{j \geq 1} \left(\binom{a-1}{j} + \binom{a-1}{j-1}\right) x^j$$

$\therefore \binom{a}{0} = [x^0](1+x)^a = 1$, &

$[x^k](1+x)^a = \binom{a-1}{k} + \binom{a-1}{k-1} \quad \forall k \geq 1$

↳ but by bin thm this is $\binom{a}{k}$.

Proof follows. ☐

2. The idea behind this proof is we're trying to prove something about the seq $\binom{a-1}{0}, \binom{a-1}{1}, \binom{a-1}{2}, \dots$ which we can make into the coefficients of a power series

$$\boxed{A(x) = \binom{a-1}{0} + \binom{a-1}{1} x + \binom{a-1}{2} x^2 + \cdots}$$

$$(+)\quad xA(x) = \qquad \binom{a-1}{0} x + \binom{a-1}{1} x^2 + \cdots$$

Thus

$$\boxed{\begin{aligned} (1+x)A(x) &= \binom{a-1}{0} + \left[\binom{a-1}{1} + \binom{a-1}{0}\right] x + \left[\binom{a-1}{2} + \binom{a-1}{1}\right] x^2 + \cdots \\ \therefore (1+x)A(x) &= 1 + \binom{a}{1} x + \binom{a}{2} x^2 + \cdots \end{aligned}}$$

FIBONACCI SEQUENCE

1. The "Fibonacci sequence" is given by
$f_0 = 0,\quad f_1 = 1,\quad f_{n+2} = f_{n+1} + f_n \quad \forall n \geq 0.$

2. We can find an explicit formula for f_n.

let $F(x) = \sum_{n\geq 0} f_n x^n$, so

$$F(x) = f_0 + f_1 x + f_2 x^2 + f_3 x^3 + f_4 x^4 + \cdots$$
$$\Rightarrow (-)\ xF(x) = \quad f_0 x + f_1 x^2 + f_2 x^3 + f_3 x^4 + \cdots$$
$$\Rightarrow (-)\ x^2 F(x) = \quad f_0 x^2 + f_1 x^3 + f_2 x^4 + \cdots$$

$$(1-x-x^2)F(x) = f_0 + (f_1 - f_0)x + \cancel{(f_2 - f_1 - f_0)x^2} + \cdots$$
$$= f_0 + (f_1 - f_0)x$$
$$= x.$$
$$\therefore \quad F(x) = \frac{x}{1-x-x^2}$$

Then, we use partial fractions:
$$\frac{P(x)}{Q(x)} = \frac{P(x)}{(x-\alpha_1)(x-\alpha_2)\cdots(x-\alpha_k)}$$

Variation:
$$\frac{P(x)}{Q(x)} = \frac{P(x)}{(1-\beta_1 x)(1-\beta_2 x)\cdots(1-\beta_k x)} = \frac{A_1}{1-\beta_1 x} + \frac{A_2}{1-\beta_2 x} + \cdots$$
(assuming no repeated factors)

Then
$$F(x) = \frac{x}{\left(1-\left(\frac{1+\sqrt{5}}{2}\right)x\right)\left(1-\left(\frac{1-\sqrt{5}}{2}\right)x\right)}$$
$$= \frac{A}{1-\left(\frac{1+\sqrt{5}}{2}\right)x} + \frac{B}{1-\left(\frac{1-\sqrt{5}}{2}\right)x}$$

Solving for A & B: $A = \frac{1}{\sqrt{5}}$, $B = -\frac{1}{\sqrt{5}}$.

$$\therefore \quad F(x) = \frac{1}{\sqrt{5}}\left(\frac{1}{1-\left(\frac{1+\sqrt{5}}{2}\right)x}\right) - \frac{1}{\sqrt{5}}\left(\frac{1}{1-\left(\frac{1-\sqrt{5}}{2}\right)x}\right)$$
$$= \frac{1}{\sqrt{5}}\sum_{k\geq 0}\left(\frac{1+\sqrt{5}}{2}\right)^k x^k - \frac{1}{\sqrt{5}}\sum_{k\geq 0}\left(\frac{1-\sqrt{5}}{2}\right)^k x^k$$

$$\therefore \quad f_n = [x^n]F(x) = \frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^n - \frac{1}{\sqrt{5}}\left(\frac{1-\sqrt{5}}{2}\right)^n. \qquad \#$$

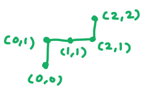
LATTICE PATH

1. A "lattice path" is a sequence $P_0, \ldots, P_r \subseteq \mathbb{Z}^2 \subseteq \mathbb{R}^2$ of points such that
$P_{i+1} - P_i \in \{(1,0), (0,1)\}.$

2. Note
$P_{i+1} - P_i = (1,0) \Rightarrow$ "east step"
$P_{i+1} - P_i = (0,1) \Rightarrow$ "north step"
and we call r the "length" of the path.

3. We represent a path starting at P_0 by a string of "N"s & "E"s.

eg



$\Longleftrightarrow$ NEEN starting at (0,0).

NUMBER OF LATTICE PATHS FROM $(0,0) \to (m,n)$ IS $\binom{m+n}{n}$

The number of lattice paths from $(0,0)$ to (m,n) is $\binom{m+n}{n}$.

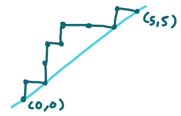
Why?
- each such path has $m+n$ steps, with n are "N" & m are "E".
- can be in any order
- if path is represented by the string $s_1 s_2 \cdots s_{m+n}$, let $\alpha = \{ i \in [m+n] \mid s_i = N \}$
- this gives a bijection b/w lattice paths from $(0,0) \to (m,n)$, & the n-element subsets of $[m+n]$.
- & the # of elements in the latter is $\binom{m+n}{n}$. ☒

DYCK PATH

A "Dyck path" is a lattice path $P_0, \ldots, P_{2n}$ such that
① $P_0 = (0,0)$;
② $P_{2n} = (n,n)$; &
③ $P_i = (x_i, y_i)$, where $x_i \leq y_i \quad \forall i$.

* these never cross the line $y = x$.

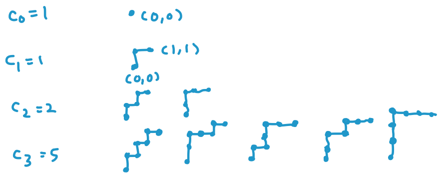
eg



CATALAN NUMBERS: C_n

1. For $n \in \mathbb{N}$, let C_n be the # of Dyck paths of length $2n$.

eg



2. The numbers C_n (for $n \geq 0$) are called the "Catalan numbers".

$$c_n = \frac{1}{n+1}\binom{2n}{n} \quad \forall n \in \mathbb{N}$$

Note that $c_n = \frac{1}{n+1}\binom{2n}{n}$.

ALGEBRAIC PROOF

Proof. **Step #1**: We'll show $\{c_n\}_{n\geq 0}$ satisfy the recurrence relation
$$c_0 = 1, \quad c_n = \sum_{k=0}^{n-1} c_k c_{n-k-1} \quad \forall n \geq 1.$$

Proof. Let $D_n = \{$Dyck paths of length $2n\}$.

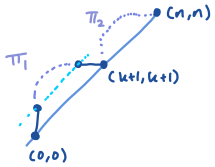
We claim that for $n \geq 1$, we have a bijection between
$$D_n \leftrightarrow \bigcup_{k=0}^{n-1} D_k \times D_{n-k-1}.$$
This would imply that
$$c_n = |D_n| = \sum_{k=0}^{n-1} |D_k \times D_{n-k-1}| = \sum_{k=0}^{n-1} c_k c_{n-k-1}$$
as needed.

So let's find the bijection.

$(\rightarrow)$ Let $\pi \in D_n$, $n \geq 1$.

- The first step must be N (otherwise we would go past "$y=x$").
- The path also ends at the line $y=x$.

So, consider the first time π returns to the line $y=x$, and suppose this occurs after $2k+2$ steps, where $k \geq 0$.



Then the $(2k+2)^{\text{th}}$ step must have been E, so we can write
$$\pi = N\pi_1 E \pi_2,$$
where
- π_1 has length $2k$, &
- π_2 has length $2(n-k-1)$.

Then π_2 is a Dyck path that has been shifted to start at $(k+1, k+1)$.

Moreover π_1 is a Dyck path, shifted to start at $(1,0)$.

- π_1 can never go below the line $y = x+1$.
- because if it did, there would be an "earlier" point which hits the line $y=x$.

Hence, we can define our map
$$\pi \mapsto (\pi_1, \pi_2).$$

$(\leftarrow)$ Given $(\pi_1, \pi_2) \in \bigcup_{k=0}^{n-1} D_k \times D_{n-k-1}$, we map
$$(\pi_1, \pi_2) \mapsto \pi$$
in a "symmetric" manner.

We check these maps are mutually inverse bijections.

Since the bijection exists, we have proven the recurrence. #

Step #2: Consider $C(x) = \sum_{n\geq 0} c_n x^n$. We claim $xC(x)^2 - C(x) + 1 = 0$.

Proof.
$$\begin{aligned} xC(x)^2 &= x\Big(\sum_{j\geq 0} c_j x^j\Big)\Big(\sum_{k\geq 0} c_k x^k\Big) \\ &= x\Big[\sum_{k\geq 0}\Big(\sum_{j\geq 0} c_j x^j\Big) c_k x^k\Big] \\ &= x\sum_{k\geq 0}\Big(\sum_{j\geq 0} c_j x^j \cdot c_k x^k\Big) \\ &= x\sum_{k\geq 0}\sum_{j\geq 0} c_j c_k x^{j+k} \\ &= \sum_{k\geq 0}\sum_{j\geq 0} c_j c_k x^{j+k+1} \end{aligned}$$

If we let $n = j+k+1$, we get that
$$\begin{aligned} xC(x)^2 &= \sum_{n\geq 1}\sum_{k=0}^{n-1} c_{n-k-1} c_k x^n \\ &= \sum_{n\geq 1} c_n x^n \qquad \text{(by Step \#1)} \\ &= C(x) - 1. \end{aligned}$$

Proof follows. ☒

Step #3: Solve for $C(x)$.

By Step #2, $xC(x)^2 - C(x) + 1 = 0$ is quadratic.

So, we can use the quadratic formula to find $C(x)$.
$$\therefore C(x) = \frac{1 \pm \sqrt{1-4x}}{2x} = \frac{1 \pm (1-4x)^{1/2}}{2x}.$$

Then see that by the bin thm:
$$\begin{aligned} (1-4x)^{1/2} &= \sum_{k\geq 0}\binom{1/2}{k}(-4x)^k \\ &= 1 + \sum_{k\geq 1}\binom{1/2}{k}(-4)^k x^k \end{aligned}$$

See that
$$\begin{aligned} \binom{1/2}{k}(-4)^k &= \frac{\frac{1}{\cancel{2}}\left(\frac{-1}{\cancel{2}}\right)\left(\frac{-3}{\cancel{2}}\right)\cdots\left(\frac{-2k+3}{\cancel{2}}\right)}{k!}(-1)^k \cancel{2^k} 2^k \\ &= -\frac{1\cdot 3 \cdot \ldots \cdot (2k-3)}{k!}\cdot 2^k \\ &= -\frac{1\cdot 3\cdot \ldots \cdot (2k-3)}{k!}\cdot \frac{2}{1}\cdot\frac{4}{2}\cdot\frac{6}{3}\cdot \ldots \cdot \frac{2k-2}{k-1}\cdot 2 \\ &= -2\cdot\frac{(2k-2)!}{k!(k-1)!} \\ &= -\frac{2}{k}\cdot\frac{(2k-2)!}{(k-1)!(k-1)!} \\ &= -\frac{2}{k}\binom{2k-2}{k-1}, \end{aligned}$$

and after back substituting, we get that
$$C(x) = \frac{1}{2x} \pm \Big(\frac{1}{2x} - \frac{1}{x}\sum_{k\geq 1}\frac{1}{k}\binom{2k-2}{k-1}x^k\Big)$$

But there's only one $C(x)$, so it cannot be that both + & − are correct. Suppose

Indeed, the "+" solution cannot be correct.
$$C(x) = \frac{1}{2x} + \Big(\frac{1}{2x} - \frac{1}{x}\sum_{k\geq 1}\frac{1}{k}\binom{2k-2}{k-1}x^k\Big) = \frac{1}{x} - 1 - x - \cdots.$$
Then
$$xC(x) = 1 - x - x^2 - \ldots.$$
but we already know that
$$xC(x) = 0 + x + x^2 + \ldots,$$
which cannot happen by uniqueness of PS.

Thus the "−" solution is correct; ie
$$\begin{aligned} C(x) &= \frac{1}{2x} - \frac{1}{2x} + \frac{1}{x}\sum_{k\geq 1}\frac{1}{k}\binom{2k-2}{k-1}x^k \\ &= \sum_{k\geq 1}\frac{1}{k}\binom{2k-2}{k-1}x^{k-1} \\ &= \sum_{n\geq 0}\frac{1}{n+1}\binom{2n}{n}x^n \qquad (n := k-1) \end{aligned}$$
$$\therefore c_n = \frac{1}{n+1}\binom{2n}{n},$$
and we're done. ☒

COMBINATORIAL PROOF 1

Proof. Let D_n = set of Dyck paths of length $2n$

& P_n = set of all lattice paths $(0,0)$ to (n,n).

We give a bijection

$$f : P_n \to D_n \times [n+1],$$

and as $|P_n| = \binom{2n}{n}$ & $|[n+1]| = n+1$, hence $|D_n| = \frac{1}{n+1}\binom{2n}{n}$.

Let $s_1 s_2 \cdots s_{2n} \in P_n$, and let $s_0 = N$. Consider the paths

$$\begin{cases} s_0 s_1 \cdots s_{2n-1} \\ s_1 s_2 \cdots s_{2n} \\ s_2 s_3 \cdots s_{2n} s_0 \\ \vdots \\ s_{2n} s_0 s_1 \cdots s_{2n-2}. \end{cases}$$

Cross out ones that are not in P_n to give us a list of $n+1$ paths.

We claim exactly one path on this list is a Dyck path, say π.

If π is the k^{th} path on the list, we define

$$f(s_1, \dots, s_{2n}) = (\pi, k).$$

eg NEENNE $\in P_3$.

The list would be

~~NNEENN~~ ← don't end up at (3,3).
1. NEENNE
2. EENNEN
~~ENNENE~~
~~NNENNE~~
3. NENNEE → this is the only Dyck path.
4. ENNEEN

$\therefore$ $f(\text{NEENNE}) = (\text{NENNEE}, 3)$

See course notes for explanation for why f is a bijection. ☒

COMBINATORIAL PROOF 2

For $0 \le a \le b$, $a, b \in \mathbb{N}$, the # of lattice paths from $(0,0)$ to (a,b) weakly above $y=x$ is

$$\binom{a+b}{a} - \binom{a+b}{a-1}.$$

If $(a,b) = (n,n)$, this # is

$$\binom{2n}{n} - \binom{2n}{n-1} = \frac{1}{n+1}\binom{2n}{n}.$$

Proof. Let X be the set of lattice paths from $(0,0)$ to (a,b) that go strictly below $y=x$ at some point.

If we can show $|X| = \binom{a+b}{a-1}$, we're done, since the total # of paths is $\binom{a+b}{a}$.

Let Y be the set of lattice paths from $(1,-1)$ to (a,b). (ie no restriction).

$\Rightarrow |Y| = \binom{a+b}{a-1}$.

$\therefore$ It suffices to give a bijection $f : X \to Y$.

So, let $s_1 s_2 \cdots s_{a+b} \in X$.

Since there is ≥ 1 point below $y=x$, there must be at least one point on the line $y = x-1$.

Suppose the first such point is after $2k-1$ steps.

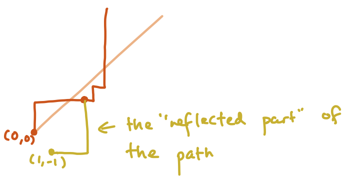
Define

$$f(s_1 \cdots s_{a+b}) = \overline{s_1}\,\overline{s_2} \cdots \overline{s_{2k-1}}\, s_{2k} \cdots s_{a+b},$$

where

$$\overline{s} = \begin{cases} N, & s = E \\ E, & s = N. \end{cases}$$

Exercise: f is a bijection.



GENERATING FUNCTIONS

GENERAL COMBINATORIAL FRAMEWORK

1. This is the general combinatorial framework.
2. Ingredients:
   ① a set $\mathcal{S}$ of "combinatorial objects"
   ↳ usually at most countable
   ② a function $w: \mathcal{S} \to \mathbb{N}$ called the "weight function"
   ↳ $w(\sigma)$ is called the "weight" of σ.
3. General counting problem:

> "For $n \in \mathbb{N}$, determine the # of objects in $\mathcal{S}$ that have weight n",
>
> ie $\#\{\sigma \in \mathcal{S} \mid w(\sigma) = n\}$.

4. We say a weight function is "good" if the answer to the above question is finite for all n.
5. Our basic strategy is to try to shove (almost) every problem we encounter into this framework.

EXAMPLE: BINARY STRINGS

1. Fix $n \in \mathbb{N}$, and let $\mathcal{S}_m = \{0,1\}^m$.
   * we call $\mathcal{S}_m$ the set of "binary strings" of length m.

   Let $w(\sigma) = \#$ of 1's in σ for any $\sigma \in \mathcal{S}_m$.

   eg $\mathcal{S}_3 = \{000, 001, 010, 011, 100, 101, 110, 111\}$

   $w(101) = 2$.
2. Then, the counting problem becomes

> "Determine the # of binary strings of length m with exactly n 1's."

which the answer to which is $\binom{n}{m}$.

EXAMPLE

1. Let $\mathcal{D}$ be the set of all Dyck paths,
   ie $\mathcal{D} = \bigcup_{n \geq 0} \mathcal{D}_n$.

   Define the weight function w to be:
   for $\pi \in \mathcal{D}$, let $w(\pi) = \#$ of N steps.
2. The counting problem becomes

> "Determine the # of Dyck paths of length $2n$",

for which the answer is $\frac{1}{n+1}\binom{2n}{n}$.

[ORDINARY] GENERATING FUNCTION/SERIES: $\Phi_{\mathcal{S}}(x)$

We define the "generating function/series" for $\mathcal{S}$ with respect to w to be the power series

$$\boxed{\Phi_{\mathcal{S}}(x) = \sum_{\sigma \in \mathcal{S}} x^{w(\sigma)}.}$$

$$\#\{\sigma \in \mathcal{S} \mid w(\sigma) = n\} = [x^n]\,\Phi_{\mathcal{S}}(x)$$

1. The answer to the counting problem is $[x^n]\,\Phi_{\mathcal{S}}(x)$;

   ie

   $\#\{\sigma \in \mathcal{S} \mid w(\sigma) = n\} = [x^n]\,\Phi_{\mathcal{S}}(x)$.

   Proof.
   $$\begin{aligned}[x^n]\,\Phi_{\mathcal{S}}(x) &= [x^n] \sum_{\sigma \in \mathcal{S}} x^{w(\sigma)} \\ &= [x^n] \sum_{m \geq 0} \sum_{\substack{\sigma \in \mathcal{S}, \\ w(\sigma) = m}} x^{w(\sigma)} \\ &= [x^n] \sum_{m \geq 0} \sum_{\substack{\sigma \in \mathcal{S}, \\ w(\sigma) = m}} x^m \cdot 1 \\ &= [x^n] \sum_{m \geq 0} x^m \Big(\sum_{\substack{\sigma \in \mathcal{S}, \\ w(\sigma) = m}} 1 \Big) \\ &= \sum_{\substack{\sigma \in \mathcal{S}, \\ w(\sigma) = n}} 1 \\ &= \#\{\sigma \in \mathcal{S} \mid w(\sigma) = n\}. \quad \blacksquare\end{aligned}$$

2. In particular, we want to do something similar to the following:
   ① We start with an easy counting problem;
   ② Turn that into a GF;
   ③ Convert that into a GF for a hard counting problem;
   * usually using the sum & product lemmas
   ④ And use that to solve the hard counting problem.

$S = A \cup B \Rightarrow \Phi_S(x) = \Phi_A(x) + \Phi_B(x) - \Phi_{A\cap B}(x)$

<< THE SUM LEMMA >>

1. Let S be a set with weight function w.
   If $S = A \cup B$, then
   $$\Phi_S(x) = \Phi_A(x) + \Phi_B(x) - \Phi_{A\cap B}(x).$$

Proof. For $A \subseteq S$, let $\chi_A : S \to \mathbb{Z}$ be the indicator fn s.t.
$$\chi_A(\sigma) = \begin{cases} 1, & \sigma \in A \\ 0, & \sigma \notin A. \end{cases}$$
Then since $S = A \cup B$, thus
$$\chi_A(\sigma) + \chi_B(\sigma) - \chi_{A\cap B}(\sigma) = 1 \quad \forall \sigma \in S.$$
Thus
$$\begin{aligned}
&\Phi_A(x) + \Phi_B(x) - \Phi_{A\cap B}(x) \\
&= \sum_{\sigma\in S} \chi_A(\sigma) x^{w(\sigma)} + \sum_{\sigma\in S} \chi_B(\sigma) x^{w(\sigma)} - \sum_{\sigma\in S} \chi_{A\cap B}(\sigma) x^{w(\sigma)} \\
&= \sum_{\sigma\in S} \left(\chi_A(\sigma) + \chi_B(\sigma) - \chi_{A\cap B}(\sigma)\right) x^{w(\sigma)} \\
&= \sum_{\sigma\in S} (1)\, x^{w(\sigma)} \\
&= \Phi_S(x). \quad \blacksquare
\end{aligned}$$

* most often used in the case $A \cap B = \emptyset$.

2. In general, if $S = \bigcup_i A_i$ with $A_i \cap A_j = \emptyset$ $\forall i \neq j$, then
   $$\boxed{\Phi_S(x) = \sum_i \Phi_{A_i}(x).}$$

* finite or countable union

* we can also write this as $\bigsqcup_i A_i$.

DISJOINT UNION: $A \sqcup B$

We write $A \sqcup B$ to mean $A \cup B$, where $A \cap B = \emptyset$.
This is called the "disjoint union" of A & B.

$w(a,b) = \alpha(a) + \beta(b) + \gamma \Rightarrow \Phi_{A\times B}(x) = x^{\gamma}\Phi_A(x)\Phi_B(x)$

<< THE PRODUCT LEMMA >>

1. Let A, B be sets, and let
   ① α be a weight function on A;
   ② β be a weight function on B; and
   ③ w be a weight function on $A \times B$.
   ④ γ is a constant (usually zero).

   Suppose that $w(a,b) = \alpha(a) + \beta(b) + \gamma$ $\forall a \in A,\ b \in B$.

   * this line is very important!

   Then
   $$\boxed{\Phi_{A\times B}(x) = x^{\gamma}\Phi_A(x)\Phi_B(x).}$$

   * if $\gamma = 0$, then $\Phi_{A\times B}(x) = \Phi_A(x)\Phi_B(x)$.

Proof.
$$\begin{aligned}
\Phi_{A\times B}(x) &= \sum_{(a,b)\in A\times B} x^{w(a,b)} \\
&= \sum_{(a,b)\in A\times B} x^{\gamma + \alpha(a) + \beta(b)} \\
&= x^{\gamma} \sum_{a\in A}\sum_{b\in B} x^{\alpha(a)} x^{\beta(b)} \\
&= x^{\gamma} \Big(\sum_{a\in A} x^{\alpha(a)}\Big)\Big(\sum_{b\in B} x^{\beta(b)}\Big) \\
&= x^{\gamma}\Phi_A(x)\Phi_B(x). \quad \blacksquare
\end{aligned}$$

2. Generalization: if $A_1, \dots, A_k$, $A_1 \times \dots \times A_k$ have weight functions $\alpha_1, \dots, \alpha_k, w$ respectively, and
   $$\boxed{w(a_1, \dots, a_k) = \gamma + \alpha_1(a_1) + \dots + \alpha_k(a_k)}$$
   for any $(a_1, \dots, a_k) \in A_1 \times \dots \times A_k$, then necessarily
   $$\boxed{\Phi_{A_1\times\dots\times A_k}(x) = x^{\gamma}\prod_{i=1}^{k}\Phi_{A_i}(x).}$$

3. Note that
   ① the sum lemma generalizes for infinite unions;
   but
   ② the product lemma requires a finite product.

   why? → infinite products are often uncountable.

EXAMPLE: LOONIES & TOONIES

1 Problem A:

> "You have 5 loonies ($1) & 3 toonies ($2).
> How many ways can you make $7?"

Sol$^{\underline{n}}$. We can make a table of all combinations of loonies & toonies:

toonies \ loonies	$0	$1	$2	$3	$4	$5
$0	$0	$1	$2	$3	$4	$5
$2	$2	$3	$4	$5	$6	$7
$4	$4	$5	$6	$7	$8	$9
$6	$6	$7	$8	$9	$10	$11

$7 appears 3 times, so there are 3 ways to make $7 dollars.

2 Problem B:

> "Compute $[x^7](1+x+x^2+x^3+x^4+x^5)(1+x^2+x^4+x^6)$".

Sol$^{\underline{n}}$. Multiply each pair of terms.

	x^0	x^1	x^2	x^3	x^4	x^5
x^0	x^0 +	x^1 +	x^2 +	x^3 +	x^4 +	x^5 +
x^2	x^2 +	x^3 +	x^4 +	x^5 +	x^6 +	x^7 +
x^4	x^4 +	x^5 +	x^6 +	x^7 +	x^8 +	x^9 +
x^6	x^6 +	x^7 +	x^8 +	x^9 +	x^{10} +	x^{11}

counting, thus coeff of x^7 is 3.

3 We can use the product lemma to solve problem A as well.

Let A = { set of loonies }, B = { set of toonies }.

...

EXAMPLE: WEAK COMPOSITIONS OF $k, n \in \mathbb{N}$

Let $k, n \in \mathbb{N}$.

> We want to determine the # of k-tuples $(c_1, \dots, c_k) \in \mathbb{N}^k$ such that $c_1 + \dots + c_k = n$.

Sol$^{\underline{n}}$. Consider $\mathbb{N}^k$ with weight function

$$w(c_1, \dots, c_k) = c_1 + \dots + c_k.$$

Then the answer to our question is $[x^n]\Phi_{\mathbb{N}^k}(x)$.

Let $\varphi: \mathbb{N} \to \mathbb{N}$ be the weight fn on $\mathbb{N}$ by $\varphi(i) = i$.

Then see that

$$\Phi_{\mathbb{N}}(x) = \sum_{i \in \mathbb{N}} x^{\varphi(i)} = \sum_{i=0}^{\infty} x^i = (1-x)^{-1}.$$

Since $w(c_1, \dots, c_k) = \sum_{i=1}^{k} \varphi(c_i)$, the presumptions to the Product Lemma are satisfied, and so we can use it:

$$\Phi_{\mathbb{N}^k}(x) = \left(\Phi_{\mathbb{N}}(x)\right)^k = [(1-x)^{-1}]^k = (1-x)^{-k}.$$

By the neg bin thm, thus the ans is

$$[x^n](1-x)^{-k} = \left(\!\!\binom{k}{n}\!\!\right) = \binom{k+n-1}{n}.$$

#

COMPOSITION

1 A "composition" of n with k parts is a k-tuple $(c_1, \dots, c_k) \in (\mathbb{N}_{\geq 1})^k$ with $c_1 + \dots + c_k = n$.

2 The numbers $c_1, \dots, c_k$ are called the "parts" of the composition.

3 Note 0 has a unique composition, and has 0 parts. (ie "()").

4 We say a composition is "weak" if $(c_1, \dots, c_k) \in \mathbb{N}^k$; ie parts of size zero can be included.

EXAMPLE: NON-WEAK COMPOSITIONS

Problem:

> "For $k, n \in \mathbb{N}$, determine the # of compositions of n with k parts".

Sol$^{\underline{n}}$. Similar to the previous example; but with the following changes:

Prev

- set of objs = $\mathbb{N}^k$
- $\Phi_{\mathbb{N}}(x) = 1 + x + x^2 + \dots = (1-x)^{-1}$
- answer: $[x^n](1-x)^{-k} = \left(\!\!\binom{n}{k}\!\!\right) = \binom{n+k-1}{n}$

Now

- set of objs is $(\mathbb{N}_{\geq 1})^k$
- $\Phi_{\mathbb{N}_{\geq 1}}(x) = x + x^2 + x^3 + \dots = x(1-x)^{-1}$
- answer: $[x^n]x^k(1-x)^{-k} = [x^{n-k}](1-x)^{-k} = \left(\!\!\binom{k}{n-k}\!\!\right) = \binom{n-1}{n-k}$.

EXAMPLE: COMPOSITIONS 2

Problem:

> "Determine the # of compositions of n with an even number of parts, and all parts are odd."

eg $(1,3,1,5)$ is a composition of 10 w/ these properties.

Sol$^{\underline{n}}$. Let $\mathcal{S}$ = set of all such compositions. (even # of parts, all parts are odd).

And define the weight of a composition to be the sum of the parts.

Let $\mathbb{N}_{odd} = \{1,3,5,7,\dots\}$ ← set of odd #. Then

$$\mathcal{S} = (\mathbb{N}_{odd})^0 \cup (\mathbb{N}_{odd})^2 \cup (\mathbb{N}_{odd})^4 \cup \cdots$$
$$= \bigcup_{k\geq 0} (\mathbb{N}_{odd})^{2k}.$$

Using the weight fn $\alpha(i) = i$ on $\mathbb{N}_{odd}$, we see that

$$\Phi_{\mathbb{N}_{odd}}(x) = x + x^3 + x^5 + x^7 + \cdots$$
$$= x(1-x^2)^{-1}.$$

By SL & PL,

$$\Phi_{\mathcal{S}}(x) = \sum_{k\geq 0} \Phi_{(\mathbb{N}_{odd})^{2k}}(x) \qquad \text{(Sum Lemma)}$$
$$= \sum_{k\geq 0} \left(\Phi_{(\mathbb{N}_{odd})}(x)\right)^{2k} \qquad \text{(Prod Lemma)}$$
$$= \sum_{k\geq 0} (x(1-x^2)^{-1})^{2k}$$

* must verify hyp of PL are satisfied.

$$= \frac{1}{1 - x^2(1-x^2)^{-2}}, \qquad \text{(by Geom Prog)}$$

and so

$$\therefore \text{ answer } = [x^n]\left(\frac{1}{1-x^2(1-x^2)^{-2}}\right).$$

Remark: the fn $(1-x^2(1-x^2)^{-2})^{-1}$ is an even function (ie $f(x) = f(-x)$).

(The power series expansion of any even function has only even powers of x, ie of the form

$$\sum_{n\geq 0} a_{2n}x^{2n}.\ )$$

For n odd, the answer should be 0, which is exactly what we see.

HOW TO SOLVE ENUMERATION PROBLEMS USING GEN FUNCS – A STEP-TO-STEP GUIDE

It's always the same for any problem:

1. Figure out the relevant set of objects and weight function.
2. Find a decomposition or bijection involving sets of objects.
   * involves thought
3. Step ② may introduce new sets — introduce weight functions for these.
   Verify that weight fns are "additive" (ie product lemma hypothesis applies)
4. Use the sum & product lemmas to convert bijections / decompositions into equations.
5. Solve said equations.
6. Extract the answer to the original question.

PLUGGING IN NUMBERS

1. If S is a finite set, then $\Phi_S(x)$ is a polynomial, and so we can plug in numbers into them.

2. In particular,
   ① $\Phi_S(1) = |S|$;
   ② $\Phi'_S(1) = \sum_{\sigma\in S} wt(\sigma)$;
   ③ $\dfrac{\Phi'_S(1)}{\Phi_S(1)}$ = avg of the weights.
   ④ we can also calculate the variance of the weights similarly.

3. If S is instead infinite, then S is a power series
   ① In some problems, we can evaluate $\Phi'_S(p)$ for some $0\leq p<1$; this can be interpreted as some kind of expected value calculation.
   ② In some problems, the radius of convergence is 0. Surprisingly, this is not a big deal, but we need to understand how to make sense of it.

FORMAL POWER SERIES

1. There are several variants on the concept of a "variable" x.
   1. "variables" → take values in a range.
   2. "constants" → take definite values.
   3. "parameter" → variable treated like a constant.
   4. "unknown" → constant treated like a variable
   5. "indeterminate" → follows the rules of algebra, but is not meant to have a value.
2. The idea of "formal power series" is that x is not a variable, but rather an indeterminate.
3. Consider the question:

$$\boxed{\text{"True or false: } \frac{1-x^2}{1-x} = 1+x\text{ ?"}}$$

Sol$^{\underline{n}}$. If x is a variable, this is not quite true ∵ $x=1$ is a problem.
If x is an indeterminate, then using the rule of algebra
$\frac{a}{b} = c \iff a = bc$,
so the formula is true ∵ $1-x^2 = (1+x)(1-x)$.

ANALYTIC VS FORMAL POWER SERIES

1. Analytic power series are of the form
$$\boxed{A(x) = \sum_{n\geq 0} a_n x^n},$$
where
   1. x is a variable;
   2. A(x) is a function; and
   3. $A: (-r, r) \to \mathbb{R}$.
2. Formal power series are of the form
$$\boxed{A(x) = \sum_{n\geq 0} a_n x^n},$$
where
   1. x is an indeterminate; and
   2. we perform manipulations, but never plug in values of x.
3. Advantage of FPS: We never have to worry about the radius of convergence.
4. So our calculations will be valid even if the radius of convergence is 0.
5. However, all the identities we know to be true for Analytic Power Series are also true for FPS.

SET OF ALL FPS WITH REAL COEFFICIENTS: $\mathbb{R}[[x]]$

1. We write "$\mathbb{R}[[x]]$" to denote the set of FPS in x with real coefficients.
2. We give $\mathbb{R}[[x]]$ the properties of a commutative ring:

   1. (Addition & Subtraction)
   If $A(x) = \sum_{n\geq 0} a_n x^n$ & $B(x) = \sum_{n\geq 0} b_n x^n$, then
   $$\boxed{\begin{aligned} A(x) + B(x) &= \sum_{n\geq 0} (a_n+b_n) x^n; \\ A(x) - B(x) &= \sum_{n\geq 0} (a_n - b_n) x^n. \end{aligned}}$$
   & * additive identity: $0 = \sum_{n\geq 0} 0x^n$

   2. (Scalar Multiplication)
   For $c \in \mathbb{R}$,
   $$\boxed{c \cdot A(x) = \sum_{n\geq 0} (ca_n) x^n.}$$

   3. Multiplication of FPS:
   $$\boxed{A(x) \cdot B(x) = \sum_{n\geq 0} \left(\sum_{k=0}^{n} a_k b_{n-k} \right) x^n.}$$
   * multiplicative identity: $1 = 1 + \sum_{n\geq 1} 0x^n$

   4. Partially-defined division:
   $$\boxed{\frac{A(x)}{B(x)} = C(x) \iff A(x) = B(x)C(x).}$$
   * if $B(x) \neq 0$.

   5. Coefficient extraction:
   $$\boxed{[x^n]\, A(x) = a_n.}$$

   6. Formal derivative:
   $$\boxed{\frac{d}{dx} A(x) = A'(x) = \sum_{n\geq 0} n a_n x^{n-1} = \sum_{n\geq 0} (n+1) a_{n+1} x^n}$$
3. These definitions satisfy the usual properties we would be familiar with.
   eg - $(A(x)B(x))C(x) = A(x)(B(x)C(x))$
   - distributive property
   - $\frac{d}{dx}(A(x)B(x)) = A'(x)B(x) + A(x)B'(x)$
   - etc

BINOMIAL THEOREM IN FPS

1. In FPS, the Binomial Theorem becomes a definition:
$$\boxed{(1+x)^a = \sum_{n\geq 0} \binom{a}{n} x^n.}$$
2. The resultant theorem is that this becomes like an exponential.
   eg $(1+x)^a (1+x)^b = (1+x)^{a+b}$.

SUBSTITUTION / COMPOSITION

We define $A(B(x))$ in two cases:

① If $A(x)$ is a polynomial, say $A(x) = \sum_{k=0}^{d} a_k x^k$,

then

$$\boxed{A(B(x)) = \sum_{k=0}^{d} a_k B(x)^k .}$$

- involves finitely many FPS operations
- so RHS is defined.

② If $[x^0]\,B(x) = 0$, then we define

$$\boxed{A(B(x)) = \sum_{k\geq 0} a_k B(x)^k .}$$

This makes sense.

Key point: if we want to know $[x^n]\,A(B(x))$, then summands involving $B(x)^m$, where $m > n$, don't contribute.

Therefore

$$\boxed{[x^n]\,A(B(x)) = [x^n] \sum_{k=0}^{n} a_k B(x)^k ,}$$

which only involves finitely many ops.

(We can "pretend" $A(x)$ is a polynomial.)

eg Consider

$A(x) = 1 + x + x^2 + \cdots = \sum x^n$,

$B(x) = x + x^2$.

Then

$$\begin{aligned} A(B(x)) &= 1 + (x+x^2) + (x+x^2)^2 + (x+x^2)^3 + \cdots \\ &= 1 + x + x^2 \\ &\quad + x^2 + 2x^3 + x^4 \\ &\quad + x^3 + 3x^4 + 3x^5 + x^6 \end{aligned}$$

When we collect like terms, there are only finitely many terms to collect for each coeff.

However, suppose

$B(x) = \frac{1}{2} + x$.

$$\begin{aligned} \Rightarrow A(B(x)) &= 1 + \left(\tfrac{1}{2} + x\right) + \left(\tfrac{1}{2} + x\right)^2 + \cdots \\ &= 1 \\ &\quad + \tfrac{1}{2} + x \\ &\quad + \tfrac{1}{4} + x + x^2 \\ &\quad + \tfrac{1}{8} + \tfrac{3}{8}x + \tfrac{3}{8}x^2 + \tfrac{1}{8}x^3 \\ &\quad + \cdots \end{aligned}$$

so each coefficient is an infinite sum, which we cannot do.

In all other cases, $A(B(x))$ is undefined.

This includes $A(b)$ for any $b \in \mathbb{R} \setminus \{0\}$.

Guiding rule:

> "FPS is valid if any coefficient of the result can be computed with finitely many arithmetic operations".

In particular, no limits or infinite sums.

$A(x)$ IS INVERTIBLE $\Leftrightarrow [x^0]\,A(x) \neq 0$

<< MULTIPLICATIVE INVERSES >>

$\frac{1}{A(x)}$ may or may not be defined.

However, we can prove $A(x)$ is invertible iff $[x^0]\,A(x) \neq 0$.

Proof. ($\Leftarrow$) Write $A(x) = \sum_{n\geq 0} a_n x^n$, and assume $a_0 \neq 0$.

Let $F(x) = a_0 - x$, $G(x) = \sum_{n\geq 0} a_0^{-n-1} x^n$.

Observe that $F(x)\,G(x) = 1$; indeed,

$G(x) = a_0^{-1} + a_0^{-2}x + a_0^{-3}x^2 + \cdots$

& so

$$\begin{aligned} F(x)\,G(x) &= a_0 G(x) - x\,G(x) \\ &= 1 + a_0^{-1}x + a_0^{-2}x^2 + \cdots \\ &\quad - a_0^{-1}x - a_0^{-2}x^2 - \cdots \\ &= 1. \end{aligned}$$

Let $B(x) = a_0 - A(x)$. Note $[x^0]\,B(x) = 0$.

$\therefore$ $B(x)$ can be substituted into other FPS.

Thus we can do

$F(B(x))\,G(B(x)) = 1$.

But $F(B(x)) = a_0 - B(x) = a_0 - (a_0 - A(x)) = A(x)$,

and so $G(B(x))$ is the mult. inverse of $A(x)$.

($\Rightarrow$) If $A(x)\,Z(x) = 1$, then necessarily

$A(0)\,Z(0) = 1$

$\Rightarrow a_0 Z(0) = 1$

$\Rightarrow a_0 \neq 0$. ☒

WEIGHT PRESERVING BIJECTION

Let S_1, S_2 be sets, & $w_1 : S_1 \to \mathbb{N}$ and $w_2 : S_2 \to \mathbb{N}$ be weight functions.

Suppose $f : S_1 \to S_2$ is bijective, such that $w_2 \circ f = w_1$.

Then we say f is a "weight-preserving bijection".

$f : S_1 \to S_2$ IS WEIGHT PRESERVING $\Rightarrow \Phi_{S_1}(x) = \Phi_{S_2}(x)$

Suppose $f : S_1 \to S_2$ is a weight preserving bijection.

Then necessarily $\Phi_{S_1}(x) = \Phi_{S_2}(x)$.

CATALAN NUMBERS REVISITED

We can use generating functions & FPS to prove Catalan numbers.

Sol^{n}. Let $\mathcal{D}$ be the set of all Dyck paths (of any length).

Define $w(\pi) = \#$ of N steps, for any $\mathcal{D} \in \pi$.

We have a bijection

$$f : \mathcal{D} \times \mathcal{D} \to \mathcal{D} \setminus \{\varepsilon\},$$

where "ε" is the path of length 0, defined by

$$f(\pi_1, \pi_2) = N\pi_1 E\pi_2 .$$

(previously we explained why this a bijection).

Then, consider the weight fn $w \circ f : \mathcal{D} \times \mathcal{D} \to \mathbb{N}$.

For any $(\pi_1, \pi_2) \in \mathcal{D} \times \mathcal{D}$, we have

$$(w\circ f)(\pi_1, \pi_2) = w(N\pi_1 E\pi_2) = w(\pi_1) + w(\pi_2) + 1 .$$

Hence, by the product lemma,

$$\underbrace{\Phi_{\mathcal{D}\times\mathcal{D}}}_{\text{wrt to } w\circ f}(x) = x\,\Phi_{\mathcal{D}}(x)\,\Phi_{\mathcal{D}}(x).$$

Moreover, as f is a weight-preserving bijection by construction, it follows that

$$\Phi_{\mathcal{D}\setminus\{\varepsilon\}}(x) = \Phi_{\mathcal{D}\times\mathcal{D}}(x).$$

But

$$\Phi_{\mathcal{D}\setminus\{\varepsilon\}}(x) = \Phi_{\mathcal{D}}(x) - \Phi_{\{\varepsilon\}}(x) = \Phi_{\mathcal{D}}(x) - 1.$$

and so by subst^{n} we get

$$x\,\Phi_{\mathcal{D}}(x)^2 - \Phi_{\mathcal{D}}(x) + 1 = 0 .$$

The rest of the solution proceeds as before, (ie solve this equation).

and we get

$$\Phi_{\mathcal{D}}(x) = \sum_{n\geq 0} \frac{1}{n+1}\binom{2n}{n} x^n . \quad \blacksquare$$

EXAMPLE: THE CRAZY DICE PROBLEM

Problem:

"Suppose we have 2 6-sided dice, & the probability of rolling any given total is

total	2	3	4	5	6	7	8	9	10	11	12
prob	$\frac{1}{36}$	$\frac{2}{36}$	$\frac{3}{36}$	$\frac{4}{36}$	$\frac{5}{36}$	$\frac{6}{36}$	$\frac{5}{36}$	$\frac{4}{36}$	$\frac{3}{36}$	$\frac{2}{36}$	$\frac{1}{36}$

Is it possible to replace the numbers on the dice with some different positive integers, without changing this probability table?

Sol^{n}. Let S = set of sides of an ordinary 6-sided die.

Let the weight of $\sigma \in S$ to be the number written on σ.

Then,

$$\Phi_S(x) = x + x^2 + x^3 + x^4 + x^5 + x^6 .$$

The set of ways to roll a pair of dice is $S\times S$.

Define the weight of a pair to be the total of the two numbers.

Then

$$\Phi_{S\times S}(x) = \Phi_S(x)^2 \quad \text{by PL.}$$

Hence the probability of rolling n is

$$p = \frac{1}{36}[x^n]\,\Phi_{S\times S}(x) = \frac{1}{36}[x^n]\,\Phi_S(x)^2$$

Now, suppose A, B are the sides of our "crazy dice", and the weight is the total of the numbers written on each side.

(eg if the first die had 1, 2, 4, 4, 4, 9, then $\Phi_A = x + x^2 + 3x^4 + x^9$.)

Then the set of ways to roll pair of dices is $A\times B$, and

$$\Phi_{A,B}(x) = \Phi_A(x)\,\Phi_B(x).$$

Thus

$$\text{prob.} = \frac{1}{36}[x^n]\,\Phi_A(x)\,\Phi_B(x)$$

We want

$$\Phi_A(x)\,\Phi_B(x) = \Phi_S(x)^2 .$$

Idea: factor $\Phi_S(x)^2$ & redistribute the factors to try and find a solution.

Other conditions:

1. $\Phi_A(1) = \Phi_B(1) = 6$;
2. Since #s on each side ≥ 1, for both Φ_A & Φ_B we need a factor of x.
3. Coeff of $\Phi_A(x)$, $\Phi_B(x)$ are ≥ 0.

In particular, see that

$$\Phi_S(x) = x\underbrace{(x+1)(x^2+x+1)}(x^2-x+1).$$

needed to ensure $\Phi_A(1) = \Phi_B(1) = 6$.

We need

$$\Phi_A(x) =$$
$$\Phi_B(x) =$$

Hence the only possibility (so that $A \neq S$ or $B \neq S$)

$$\Phi_A(x) = x(x+1)(x^2+x+1) = x + 2x^2 + 2x^3 + x^4$$

$$\& \quad \Phi_B(x) = x(x+1)(x^2+x+1)(x^2-x+1)^2 = x + x^3 + x^4 + x^5 + x^6 + x^8 .$$

These satisfy all the conditions, and so we conclude the crazy dice exist:

die 1 = 1, 2, 2, 3, 3, 4

& die 2 = 1, 3, 4, 5, 6, 8.

CYCLOTONIC POLYNOMIALS

In the Crazy Dice problem, how did we factor $\Phi_S(x)$?

See that

$$\Phi_S(x) = x+x^2+x^3+x^4+x^5+x^6$$
$$= x \cdot \frac{x^6-1}{x-1}$$

There exist polynomials $\phi_n(x) \in \mathbb{Z}[x]$ $\forall n \geq 1$, called the "cyclotonic polynomials", such that

① $x^n - 1 = \prod_{d \mid n} \phi_d(x)$; &

② $\phi_n(x) = \prod_{d \mid n} (x^d - 1)^{\mu(n/d)}$, where

$\mu(\cdot)$ is the "classical Mobius function", defined by

$$\mu(n) = \begin{cases} (-1)^k, & n \text{ is a product of } n \text{ distinct primes} \\ 0, & \text{otherwise.} \end{cases}$$

③ $\phi_n(x) = \prod_{\substack{0 \leq k < n, \\ \gcd(k,n)=1}} (x - e^{2\pi i \frac{k}{n}})$.

④ $\phi_n(x)$ is an irreducible polynomial.

Proof Sketch. Use ③ as the def^n of $\phi_n(x)$.

Then, we check ① holds.

In particular,

① $\Rightarrow$ ② by the "Mobius Inversion Theorem",

and

③ $\Rightarrow$ $\phi_n(x) \in \mathbb{C}[x]$,

but

② $\Rightarrow$ $\phi_n(x)$ is a ratio of integer polynomials.

Together, these tell us that $\phi_n(x) \in \mathbb{Z}[x]$.

④ is a non-trivial thm of Gauss, beyond the scope of the course. ⊠

So,

$$\boxed{x^6 - 1 = \phi_6(x)\phi_3(x)\phi_2(x)\phi_1(x),}$$

&

$$\boxed{\begin{aligned} \phi_6(x) &= x^2-x+1 &&= (x^6-1)(x^3-1)^{-1}(x^2-1)^{-1}(x-1) \\ \phi_3(x) &= x^2+x+1 &&= (x^3-1)(x-1)^{-1} \\ \phi_2(x) &= x+1 &&= (x^2-1)(x-1)^{-1} \\ \phi_1(x) &= x-1 &&= (x-1). \end{aligned}}$$

which we get via ②.

PARTITIONS OF AN INTEGER

An "integer partition" of n with k parts is a k-tuple $(\lambda_1, \dots, \lambda_k)$ of positive integers such that $\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_k$ and $\lambda_1 + \dots + \lambda_k = n$.

In particular,

① we simply say $\lambda = (\lambda_1, \dots, \lambda_k)$;

② n is called the "size" of λ, and we write $|\lambda| = n$ or $\lambda \vdash n$;

③ k is called the "length" of λ, and we write $\ell(\lambda) = k$;

④ $\lambda_1, \dots, \lambda_k$ are called the "parts" of λ.

OF PARTITIONS WHERE ALL PARTS ARE $\leq m$

<< PARTITIONS PROBLEM 1 >>

Problem:

> "Suppose $m, n \in \mathbb{N}$. Determine the # of partitions of n in which all parts are $\leq m$.

eg for $m=3$, $n=5$: $(1,1,1,1,1)$, $(2,1,1,1)$, $(2,2,1)$, $(3,1,1)$, $(3,2)$ — 5 such partitions.

Sol^n. Let P_m = set of partitions $\lambda = (\lambda_1, \dots, \lambda_k)$ such that $\lambda_i \leq m$ $\forall i$, where $k \in \mathbb{N}$.

Let the weight function $w: P_m \to \mathbb{N}$ by

$$w(\lambda) = |\lambda| = \lambda_1 + \dots + \lambda_k.$$

Let $S_m = \mathbb{N} \times 2\mathbb{N} \times 3\mathbb{N} \times \dots \times m\mathbb{N}$, w/ weight fn $\hat{\omega}: S_m \to \mathbb{N}$ by $\hat{\omega}(c_1, \dots, c_m) = c_1 + \dots + c_m$.

We claim we have a weight presering bijection $f: P_m \to S_m$.

Define $f(\lambda_1, \dots, \lambda_k) = (c_1, \dots, c_m)$, where

$$c_i = i \cdot \#\{j \mid \lambda_j = i\}.$$

eg $m=7$, $\lambda = (5,5,3,3,3,3,2,2,1)$, then

$$f(\lambda) = (1,2,4,0,2,0,0).$$

Thus

$$\Phi_{P_m}(x) = \Phi_{S_m}(x).$$

Now to figure out $\Phi_{S_m}(x)$, proceed as in composition examples.

Let $\varphi_i: i\mathbb{N} \to \mathbb{N}$ by $\varphi_i(n) = n$.

Since $\hat{\omega}(c_1, \dots, c_m) = \varphi_1(c_1) + \dots + \varphi_m(c_m)$, we can use the product lemma:

$$\Phi_{S_m}(x) = \prod_{i=1}^m \Phi_{i\mathbb{N}}(x)$$
$$= \prod_{i=1}^m \left(\sum_{k \geq 0} x^{ik}\right)$$
$$= \prod_{i=1}^m \frac{1}{1-x^i} \quad (= \Phi_{P_m}(x)),$$

and so

$$\text{answer} = [x^n] \prod_{i=1}^m \frac{1}{1-x^i}.$$

* There is no reasonable way to simplify this further.

\# OF PARTITIONS = $[x^n]\prod_{i=1}^{m}\frac{1}{1-x^i}$, $m \geq n$

<< PARTITIONS PROBLEM 2 >>

Problem:

$\boxed{\text{"Determine the \# of partitions of } n\text{".}}$

Sol^n. Let $m \geq n$. Then every partition of n has all parts $\leq m$.

By Problem 1, the answer is

$$[x^n]\prod_{i=1}^{m}\frac{1}{1-x^i}, \quad m \geq n.$$

In particular, since this is true for all sufficiently large m (ie $m \geq n$), we may write this answer as

$$[x^n]\prod_{i=1}^{\infty}\frac{1}{1-x^i}. \quad \text{(see below).}$$

In particular, the gen fn for the set of all partitions, wrt their size, is

$$\Phi_{\mathcal{P}}(x) = \prod_{i=1}^{\infty}\frac{1}{1-x^i}.$$

INFINITE PRODUCT: $\prod_{i=1}^{\infty} A_i(x)$

1. Suppose $\prod_{i=1}^{m} f_i(x)$ is the "correct answer" for any sufficiently large m.

Then, we may just write

$$\boxed{\prod_{i=1}^{\infty} f_i(x) \approx \prod_{i=1}^{m} f_i(x)}$$

for such sufficiently large n.

2. More formally, let $A_1(x), A_2(x), \ldots \in \mathbb{R}[[x]]$ be a sequence of FPS.

We say that

$$\boxed{a_n = [x^n]\prod_{i=1}^{\infty} A_i(x)}$$

if there exist a $N \in \mathbb{N}$ such that for any $m \geq N$ we have

$$\boxed{a_n = [x^n]\prod_{i=1}^{m} A_i(x).}$$

3. If a_n exists for all $n \in \mathbb{N}$, then we say

$$\boxed{\prod_{i=1}^{\infty} A_i(x) = \sum_{n \geq 0} a_n x^n.}$$

where $A_i(x)$ is a FPS for each i.

GENERATING FUNCTIONS WITH 2+ WEIGHT FUNCTIONS: $\Phi_S(x,y)$

1. Suppose S is a set of combinatorial objects, and we have two weight functions $w_1 : S \to \mathbb{N}$ & $w_2 : S \to \mathbb{N}$.

2. Then, we define the generating function with respect to both w_1 & w_2 is defined by

$$\boxed{\Phi_S(x,y) = \sum_{\sigma \in S} x^{w_1(\sigma)} y^{w_2(\sigma)}.}$$

* generalizes to more weight functions as well.

3. In particular, $[x^m y^n]\,\Phi_S(x,y)$ answers the question

$\boxed{\text{"How many } \sigma \in S \text{ with } w_1(\sigma) = m \ \& \ w_2(\sigma) = n\text{."}}$

SUM LEMMA FOR 2+ WEIGHT FNS

Let $S = \mathcal{A} \cup \mathcal{B}$, with weight functions $w_1 : S \to \mathbb{N}$ & $w_2 : S \to \mathbb{N}$.

Then

$$\boxed{\Phi_S(x,y) = \Phi_{\mathcal{A}}(x,y) + \Phi_{\mathcal{B}}(x,y) - \Phi_{\mathcal{A}\cap\mathcal{B}}(x,y).}$$

* generalizes to more sets & more weight fns.

PRODUCT LEMMA FOR 2+ WEIGHT FNS

Let $\mathcal{A}, \mathcal{B}, \mathcal{A}\times\mathcal{B}$ be sets with weight functions

1. $\alpha_1, \alpha_2 : \mathcal{A} \to \mathbb{N}$;
2. $\beta_1, \beta_2 : \mathcal{B} \to \mathbb{N}$;
3. $w_1, w_2 : \mathcal{A}\times\mathcal{B} \to \mathbb{N}$,

and let γ_1, γ_2 be constants.

Suppose

$$\boxed{w_i(x,y) = \alpha_i(x,y) + \beta_i(x,y) + \gamma_i, \quad i = 1,2.}$$

Then necessarily

$$\boxed{\Phi_{\mathcal{A}\times\mathcal{B}}(x,y) = x^{\gamma_1} y^{\gamma_2}\, \Phi_{\mathcal{A}}(x,y)\, \Phi_{\mathcal{B}}(x,y).}$$

* generalizes to more sets (finitely many) and more weight functions.

EXAMPLE: COMPOSITIONS, PART 1

Problem:

$\boxed{\text{"Determine the \# of compositions of } n.\text{"}}$

Sol^n. Let S = set of all compositions.
In particular,

$$S = \bigcup_{k \geq 0} \mathbb{N}_{\geq 1}^{k}.$$

Define two weight fns on S with

$$w_1(c_1, \dots, c_k) = c_1 + \dots + c_k,$$
$$w_2(c_1, \dots, c_k) = k.$$

Then the # of compositions of n with k parts is

$$[x^n y^k]\, \Phi_S(x,y).$$

Define two weight fns on $\mathbb{N}_{\geq 1}$ by

$$\varphi_1(c) = c$$

so that $w_1(c_1, \dots, c_k) = \varphi_1(c_1) + \dots + \varphi_1(c_k)$,

and

$$\varphi_2(c) = 1$$

so that $w_2(c_1, \dots, c_k) = k = k(1) = \varphi_2(c_1) + \dots + \varphi_2(c_k)$.

Thus, by SL & PL,

$$\Phi_S(x,y) = \sum_{k \geq 0} \Phi_{\mathbb{N}_{\geq 1}^k}(x,y) = \sum_{k \geq 0} \left(\Phi_{\mathbb{N}_{\geq 1}}(x,y)\right)^k$$

Then as

$$\Phi_{\mathbb{N}_{\geq 1}}(x,y) = xy + x^2y + x^3y + \cdots = \frac{xy}{1-x},$$

it follows that

$$\Phi_S(x,y) = \sum_{k \geq 0} \left(\frac{xy}{1-x}\right)^k = \frac{1}{1 - \frac{xy}{1-x}} = \frac{1-x}{1-x-xy}.$$

Therefore,

$$\#\text{ of compositions of } n \text{ with } k \text{ parts} = [x^n y^k]\, \frac{1-x}{1-x-xy}.$$

$$[x^n]A(x,y) = \sum_{m \geq 0} \left([x^n y^m] A(x,y)\right) y^m$$

<< A WORD OF CAUTION ABOUT COEFFICIENT NOTATION >>

1. Consider

$$F(x,y) = 1 + 3x + 5xy^2 - x^2y = 1 + (3+5y^2)x - yx^2.$$

Then

$$[xy^2]\, F(x,y) = 5.$$

2. However, note that

$$[x]\, F(x,y)$$

is ambiguous (it could be 3 or $3+5y^2$), but we define it to be

$$[x]\, F(x,y) = 3 + 5y^2.$$

3. In other words, we define

$$\boxed{[x^n]A(x,y) = \sum_{m \geq 0} \left([x^n y^m] A(x,y)\right) y^m,}$$

where the RHS is a FPS in y^m.

* this generalizes to more variables as well.

4. We will write

$$[x^n y^0]\, A(x,y)$$

to mean the "normal" coefficient of the original FPS.

ie in our example, $[xy^0]\, A(x,y) = 3$.

SPECIALIZATIONS OF MULTIVARIATE GEN FUNCTIONS

Note:

1. $\Phi_S(1,1) = |S|$ if $|S| < \infty$;
2. $\Phi_S(x,1)$ is the generating function wrt w_1;
   * if w_1 is a good weight function by itself; ie the answer to the problem "how many elements of S have $w_1(\sigma) = n$?" is finite for all n.
3. $\Phi_S(1,y)$ is the gen func for S wrt w_2;
   * need to check similar to above.
4. $[x^n]\, \Phi_S(x,y)$ is a power series in y.
   – the gen func for the set $\{\sigma \in S \mid w_1(\sigma) = n\}$ wrt w_2.
5. $[y^n]\, \Phi_S(x,y)$ is a power series in x.
   – the gen func for the set $\{\sigma \in S \mid w_2(\sigma) = n\}$ wrt w_1.
6. $\Phi_S(x,x)$ is the generating function for S wrt w, where $w(\sigma) = w_1(\sigma) + w_2(\sigma)$.

EXAMPLE: COMPOSITIONS, PART 2

Problem:

$\boxed{\text{"Determine the \# of compositions of } n, \text{ with any \# of parts."}}$

Sol^n. Let S = set of all compositions, as in prev example.

$$\Rightarrow \Phi_S(x,y) = \frac{1-x}{1-x-xy}$$

wrt $w_1(c_1, \dots, c_k) = c_1 + \dots + c_k$ & $w_2(c_1, \dots, c_k) = k$.

Thus, our answer is given by

$$[x^n]\, \Phi_S(x,1) = [x^n]\, \frac{1-x}{1-x-x} = [x^n]\, \frac{1-x}{1-2x}.$$

$$= [x^n]\, \frac{1}{1-2x} - [x^{n-1}]\, \frac{1}{1-2x}$$

$$= \begin{cases} 2^n - 2^{n-1}, & n \geq 1 \\ 1, & n = 0 \end{cases}$$

$$\boxed{\text{answer} = \begin{cases} 2^{n-1}, & n \geq 1 \\ 1, & n = 0 \end{cases}}$$

* However, consider $\Phi_S(1,y)$.
This should be the GF for S wrt # of parts; ie this answers "how many compositions w/ k parts?"
But this is a bad question (there are infinitely many such compositions!)
So we should expect the substitution $x=1$ to be nonsensical; however

$$\Phi_S(1,y) = \frac{1-1}{1-1-1\cdot y} = \frac{0}{-y} = 0,$$

which is a garbage calculation.

STRINGS

1. A "string" is a finite ordered list of symbols taken from an alphabet.
2. Initially, our alphabet will just be $\{0,1\}$; in this case our strings will be called "binary strings", "01-strings" or "$\{0,1\}$-strings".

   eg $a = 11010$ ← length = 5
   $b = 0001$ ← length = 4
3. The empty string "ε" has length 0.

CONCATENATION

1. We define the "product" of 2 strings to be their concatenation:

$$\boxed{\begin{aligned} ab &= (\text{digits of } a)(\text{digits of } b) \\ s\varepsilon &= \varepsilon s = s \end{aligned}}$$

2. Properties:
   1. It is associative;
   2. It has an identity element;

   → so the set of strings behaves like a monoid.

   3. But, it is non-commutative;
   4. And we have no inverses.

CONCATENATION & UNION [OF LANGUAGES]

If A & B are sets of strings (ie "languages"), then we define

$$\boxed{\begin{aligned} AB &= \{ab : a \in A,\ b \in B\} \\ A \cup B &= \{s : a \in A \text{ or } b \in B\} \end{aligned}}$$

KLEENE STAR [OF A LANGUAGE]

We define the "Kleene star" of A to be

$$\boxed{\begin{aligned} A^* &= \{\varepsilon\} \cup A \cup AA \cup AAA \cup \cdots \\ &= \bigcup_{i=0}^{\infty} A^i, \end{aligned}}$$

where $A^0 = \{\varepsilon\}$ & $A^k = \underbrace{A \cdots A}_{k}$.

eg $\{0,1\}^*$ is the set of all strings.

AMBIGUOUS / UNAMBIGUOUS [OPERATIONS]

1. For union:
   We say "$\underline{\underline{A \cup B}}$" is an "unambiguous" operation if $A \cap B = \emptyset$, and "ambiguous" otherwise.
2. For concatenation:
   Note that concatenation is a map $A \times B \to AB$.
   If this map is a bijection, we say AB is an "unambiguous" expression.
   (and "ambiguous" otherwise).
   - always surjective by defn of AB.
3. For the Kleene star:
   A^* is "unambiguous" if all unions are disjoint, and all concatenations are unambiguous.
4. For more complicated expressions, these are unambiguous if all the constituent operations are.

eg • $A = \{0, 00\}$, $B = \{1, 11\}$, $C = \{\varepsilon, 0\}$.

- ABC is unambiguous;
  ⇒ 001 must be $(00)(1)(\varepsilon)$
  ⇒ every other element of this set can be produced only in a single way.
- CAB is ambiguous.
  ⇒ because CA is ambiguous.
  ⇒ $00 = (\varepsilon)(00)$ or $(0)(0)$.
- $A \cup B$ is unambiguous; but
- $A \cup B \cup C$ is ambiguous
  ⇒ since $0 \in A \cup C$.
- A^* is ambiguous
  ⇒ $000 = 0(00) = (00)(0) = (0)(0)(0)$

ADDITIVE WEIGHT FUNCTION [ON A SET OF STRINGS]

A "weight function" w on a set of strings is "additive" if

$$\boxed{w(ab) = w(a) + w(b)}$$

for all strings $a, b \in \{0,1\}^*$.

eg - length of the string

DEFAULT WEIGHT FUNCTION ON SETS OF STRINGS = LENGTH

Unless otherwise specified, we assume the weight of a string is its length.

GENERATING FUNCTIONS OF COMBINATIONS OF SETS OF STRINGS

Let $A, B \in \{0,1\}^*$, and let w be an additive weight function.

Then

① If $A \cup B$ is unambiguous, then

$$\boxed{\Phi_{A\cup B}(x) = \Phi_A(x) + \Phi_B(x);}$$

② If AB is unambiguous, then

$$\boxed{\Phi_{AB}(x) = \Phi_A(x)\,\Phi_B(x);}$$

③ If A^* is unambiguous, then

$$\boxed{\Phi_{A^*}(x) = \frac{1}{1-\Phi_A(x)}.}$$

Proof. ① follows from the SL;

② : AB is unambiguous

$\Rightarrow$ we have a bijection $A\times B \to AB$.

This weight function is weight-preserving if we define

$$w(a,b) = w(a) + w(b).$$

$$\therefore \Phi_{AB}(x) = \Phi_{A\times B}(x) \overset{PL}{=} \Phi_A(x)\,\Phi_B(x).$$

③ :
$$\Phi_{A^*}(x) \overset{SL}{=} \sum_{i\geq 0}\Phi_{A^i}(x)$$
$$\overset{②}{=} \sum_{i\geq 0}\left(\Phi_A(x)\right)^i$$
$$= \frac{1}{1-\Phi_A(x)} \quad \text{(by geometric series).}$$

EXAMPLE : $\Phi_{\{0,1\}^*}(x)$

The set of binary strings is $\{0,1\}^*$.

$\rightarrow$ Then

$$\Phi_{\{0,1\}^*}(x) = \frac{1}{1-\Phi_{\{0,1\}}(x)}$$
$$= \frac{1}{1-2x}$$
$$= 1 + 2x + 4x^2 + 8x^3 + \cdots,$$

which is correct

(we have 2^i strings of length i.)

SUBSTRING

1. A "substring" is consecutive letters that form a word.
2. A "subsequence" / "subword" is a substring, but the letters might not be consecutive.

eg $\underline{1}0\underline{0}0\underline{1}0\underline{0}1\underline{0}1$

ZERO - DECOMPOSITION

A "zero-decomposition" is of the forms

①
$$\boxed{\begin{aligned}\{0,1\}^* &= (\{1\}^*\{0\})^*\{1\}^* \\ &= \{1\}^*(\{0\}\{1\}^*)^*\end{aligned}}$$

Proof. See that

$$\Phi_{(\{1\}^*\{0\})^*\{1\}^*}(x) = \frac{1}{1-\Phi_{\{1\}^*\{0\}}(x)}\,\Phi_{\{1\}^*}(x)$$
$$= \frac{1}{1-\frac{x}{1-x}}\cdot\frac{1}{1-x} = \frac{1}{1-2x}$$
$$= \Phi_{\{0,1\}^*}(x).$$

ONE - DECOMPOSITION

The "one-decomposition" is

$$\boxed{\begin{aligned}\{0,1\}^* &= (\{0\}^*\{1\})^*\{0\}^* \\ &= \{0\}^*(\{1\}\{0\}^*)\end{aligned}}$$

$p(A_1,\dots,A_k)$ IS UNAMBIGUOUS, $B_i \subseteq A_i \Rightarrow p(B_1,\dots,B_k)$ IS UNAMBIGUOUS

Let $A_1,\dots,A_k$ be sets of strings, $p(A_1,\dots,A_k)$ is an unambiguous expression, and $B_1\subseteq A_1,\dots,B_k\subseteq A_k$.

Then $p(B_1,\dots,B_k)$ is also unambiguous.

EXAMPLE : SET OF STRINGS THAT DON'T HAVE 111 AS A SUBSTRING

Problem :

> "Let S be the set of strings that do not have '111' as a substring.
> Find $\Phi_S(x)$."

Soln. We claim

$$S = \{0, 10, 110\}^*\{\varepsilon, 1, 11\}$$

is an unambiguous expression.

Why does this set generate S?

eg Consider $\underline{1101011001}$.

$\Rightarrow ((110)(10)(110)(0))\,(1)$.

From this, we get that

$$\Phi_S(x) = \Phi_{\{0,10,110\}^*}(x)\,\Phi_{\{\varepsilon,1,11\}}(x)$$
$$= \frac{1}{1-\Phi_{\{0,10,110\}}(x)}\,\Phi_{\{\varepsilon,1,11\}}(x)$$
$$= \frac{1}{1-(x+x^2+x^3)}(1+x+x^2)$$

$$\boxed{\Phi_S(x) = \frac{1+x+x^2}{1-x-x^2-x^3}.}$$

But why is the expression for S unambiguous?

In particular, $\{0,10,110\}^* \subseteq \{1\}^*\{0\}$.

Take

$p(A_1,A_2,A_3) = (A_1A_2)^*A_3$

w/ $A_1 = A_3 = \{1\}^*$, $A_2 = \{0\}$ (in below example)

we know

$(\{1\}^*\{0\})^*\{1\}^*$ is unambiguous.

Then, let

$B_1 = \{\varepsilon,1,11\} \subseteq A_1$, $B_2 = \{0\} \subseteq A_2$, $B_3 = \{\varepsilon,1,11\} \subseteq A_3$.

Using the theorem, $(B_1B_2)^*B_3$ is unambiguous, which is exactly our expression.

ZERO-BLOCK : $\{0\}\{0\}^*$

A "zero-block" is a maximal non-empty substring of zeroes.

Here, "maximal" means it cannot be extended.

→ on the other hand, "maximum" = biggest possible.

eg $1011\underline{0000}1\underline{0}1$

maximum ↑ ↑ maximal

ONE-BLOCK : $\{1\}\{1\}^*$

An "one-block" is a maximal non-empty substring of ones.

BLOCK-DECOMPOSITION

The "block-decomposition" is

$$\boxed{\begin{aligned}\{0,1\}^* &= \{0\}^*(\{1\}\{1\}^*\{0\}\{0\}^*)^*\{1\}^* \\ &= \{1\}^*(\{0\}\{0\}^*\{1\}\{1\}^*)\{0\}^*.\end{aligned}}$$

Idea $\{1\}\{1\}^*$ is a block of 1s; $\{0\}\{0\}^*$ is a block of 0s.

But,

$\{0\}^* = \{\varepsilon\} \cup \{0\}\{0\}^*$ (ie empty or 0-block)

$\{1\}^* = \{\varepsilon\} \cup \{1\}\{1\}^*$ (ie empty or 1-block).

EXAMPLE 1

Problem:

> "Determine the # of binary strings of length n such that all blocks are of odd length."

Sol$^{\underline{n}}$. Let S be the set of 01-strings where all blocks have odd length.

Then see that

$$S = (\{\varepsilon\} \cup \{1\}\{11\}^*)(\{0\}\{00\}^*\{1\}\{11\}^*)^*(\{\varepsilon\} \cup \{0\}\{00\}^*).$$

(Apply the rules to get $\Phi_S(x)$.)

REGULAR EXPRESSION

We say an expression for a set of strings which is built from

① finite sets of strings; &

② the operations of $\cup$, concat., $*$;

is a "regular expression".

*note regular expressions can still be ambiguous!

EXAMPLE 2

Problem:

> "Determine the number of strings of length n such that every block of 0's is followed by a longer block of 1's."

eg "111 0000 111111 00 1111"

Sol$^{\underline{n}}$. Let S be the set of such strings.

Then note that

$$S = \{1\}^*(\mathcal{A}\{1\}\{1\}^*)^*$$

where

$$\mathcal{A} = \{01, 0011, 000111, \dots \underbrace{0\dots0}_{n}\underbrace{1\dots1}_{n}, \dots\}$$

Then see that

$$\Phi_{\mathcal{A}}(x) = x^2 + x^4 + x^6 + \cdots = \frac{x^2}{1-x^2},$$

(and from here, we can get $\Phi_S(x)$). #

RELATING PREVIOUS CONCEPTS TO STRINGS

Notes about old examples:

① Dyck paths

We can think of Dyck paths as strings in the alphabet $\{N, E\}$.

If we let D = set of Dyck paths, then

$$D = \{\varepsilon\} \cup \{N\}D\{E\}D.$$

So we can get an expression for $\Phi_D(x)$ from here, and so on.

② Partitions.

We can also view partitions as strings in the alphabet $\{1, 2, 3, 4, \dots\} = \mathbb{N}_{\geq 1}$.

Let $\mathcal{P}$ be the set of partitions written backwards (ie smallest to largest.)

Then see that

$$\mathcal{P} = \{1\}^*\{2\}^*\{3\}^*\{4\}^*\cdots$$

INFINITE CONCATENATION PRODUCT: $\mathcal{A}_1\mathcal{A}_2\mathcal{A}_3\dots$

① Let $\mathcal{A}_1, \mathcal{A}_2, \dots$ be sets of strings, and suppose $\varepsilon \in \mathcal{A}_i \ \forall i$.

Then, we define the "infinite concatenation product" to be

$$\boxed{\mathcal{A}_1\mathcal{A}_2\mathcal{A}_3\cdots = \bigcup_{m=1}^{\infty} \mathcal{A}_1\cdots\mathcal{A}_m.}$$

② Since $\varepsilon \in \mathcal{A}_i$, the union is never disjoint, and so

$$\mathcal{A}_1 \subseteq \mathcal{A}_1\mathcal{A}_2 \subseteq \mathcal{A}_1\mathcal{A}_2\mathcal{A}_3 \subseteq \cdots$$

③ We say $\mathcal{A}_1\mathcal{A}_2\mathcal{A}_3\cdots$ is unambiguous if $\mathcal{A}_1\mathcal{A}_2\mathcal{A}_3\cdots\mathcal{A}_m$ is unambiguous for all m.

$\mathcal{A}_1\mathcal{A}_2\cdots$ IS UNAMBIGUOUS => $\Phi_{\mathcal{A}_1\mathcal{A}_2\cdots}(x) = \prod_{i=1}^{\infty}\Phi_{\mathcal{A}_i}(x)$

<< INFINITE PRODUCT LEMMA FOR STRINGS >>

Suppose $\mathcal{A}_1\mathcal{A}_2\cdots$ is unambiguous.

Then necessarily

$$\boxed{\Phi_{\mathcal{A}_1\mathcal{A}_2\cdots}(x) = \prod_{i=1}^{\infty}\Phi_{\mathcal{A}_i}(x)}$$

ADVANCED STRING TECHNIQUES

SUBSTITUTION : $a[q\to R]$, $\mathcal{A}[q\to R]$

1. Let Q be an alphabet, with $q \in Q$.
Let $R \subseteq Q^*$.
Suppose we have an $a \in Q^*$, and write

$$\boxed{a = a_0 q a_1 q a_2 \cdots q a_m}$$

Then, we define the "substitution of a of q by r" by

$$\boxed{a[q\to R] = \{a_0\}R\{a_1\}R \cdots R\{a_m\}.}$$

2. If $\mathcal{A} \subseteq Q^*$, then we similarly define

$$\boxed{\mathcal{A}[q\to R] = \bigcup_{a\in\mathcal{A}} a[q\to R].}$$

eg Take 01-strings, and let

$\mathcal{A} = \{0101, 001\}$

$R = \{111, 101\}$.

Consider $\mathcal{A}[1\to R]$:

$$\mathcal{A}[1\to R] = \left\{\begin{array}{cc} 0101 & 001 \\ \downarrow & \downarrow \\ 0\,111\,0\,111 & 00\,111 \\ 0\,111\,0\,101 & 00\,101 \\ 0\,101\,0\,111 & \\ 0\,101\,0\,101 & \end{array}\right\}$$

3. Then, we say $\mathcal{A}[q\to R]$ is unambiguous iff all concatenations & unions in the definition are unambiguous.

$$\Phi_{\mathcal{A}[q\to R]}(x) = \Phi_{\mathcal{A}}(x_0,\dots,x_{q-1},\Phi_R(x_0,\dots,x_k),x_{q+1},\dots,x_k)$$

<< SUBSTITUTION OF GFs >>

Suppose $Q = \{0,1,2,\dots,k\}$, and let $w_i : Q^* \to \mathbb{N}$ by $w_i(\sigma) = \#$ of i's in σ.

Let $\mathcal{A}, R \subseteq Q^*$.

Then necessarily

$$\boxed{\Phi_{\mathcal{A}[q\to R]}(x) = \Phi_{\mathcal{A}}(x_0,\dots,x_{q-1},\Phi_R(x_0,\dots,x_k),x_{q+1},\dots,x_k).}$$

Proof. By definition,

$$\mathcal{A}[q\to R] = \bigcup_{a\in\mathcal{A}} a[q\to R].$$

So by SL,

$$\Phi_{\mathcal{A}[q\to R]}(x_0,\dots,x_k) = \sum_{a\in\mathcal{A}} \Phi_{a[q\to R]}(x_0,\dots,x_k)$$

$$= \sum_{a\in\mathcal{A}} \Phi_{\{a_0\}R\{a_1\}\cdots R\{a_m\}}(x_0,\dots,x_k).$$

$$\overset{PL}{=} \sum_{a\in\mathcal{A}} \Phi_{\{a_0\}}(-)\,\Phi_R(-)\,\Phi_{\{a_1\}}(-)\cdots\Phi_R(-)\,\Phi_{\{a_m\}}(-)$$

$$= \sum_{a\in\mathcal{A}} x_{a_0}x_{a_1}x_{a_2}\cdots x_{a_m}\,\Phi_R(x_0,\dots,x_k)^{w_q(a)}$$

$$= \sum_{a\in\mathcal{A}} x_0^{w_0(a)}\,x_1^{w_1(a)}\cdots\Phi_R(x_0,\dots,x_k)^{w_q(a)}\cdots x_k^{w_k(a)}$$

$$= \Phi_{\mathcal{A}}(x_0,x_1,\dots,\Phi_R(x_0,\dots,x_k),\dots,x_k). \quad \blacksquare$$

EXAMPLE : SMIRNOV STRINGS

1. A "Smirnov string" is a string where no letter appears twice consecutively.

eg in $\{0,1,2,3\}^*$, 012130121301 is a Smirnov string.

2. Problem:

> "Find the GF of Smirnov strings in $\{0,1,\dots,k\}^*$, wrt to $w_0, w_1, \dots, w_k$ defined earlier".

Sol^n. Let S = set of Smirnov strings.

Then see that

$$S[0\to\{0\}\{0\}^*][1\to\{1\}\{1\}^*]\cdots[k\to\{k\}\{k\}^*]$$

$$= \{0,1,2,3,\dots,k\}^*.$$

So, by our theorem,

$$\Phi_S\Big(\underset{\substack{\uparrow\\\text{block of}\\0\text{'s}}}{\frac{x_0}{1-x_0}}, \underset{\substack{\uparrow\\\text{block of}\\1\text{'s}}}{\frac{x_1}{1-x_1}}, \dots, \underset{\substack{\uparrow\\\text{block of}\\k\text{'s}}}{\frac{x_k}{1-x_k}}\Big) = \frac{1}{1-(x_0+\dots+x_k)}.$$

Then, let $y_i = \frac{x_i}{1-x_i}$, so that $x_i = \frac{y_i}{1+y_i}$.

Thus

$$\boxed{\Phi_S(y_0,\dots,y_k) = \frac{1}{1-\left(\frac{y_0}{1+y_0}+\dots+\frac{y_k}{1+y_k}\right)}.}$$

MARKING TECHNIQUE

EXAMPLE: STRINGS WITH NO 011 SUBSTRINGS

Problem:

"Determine the # of 01-strings of length n that don't have 011 as a substring".

Sol^{n}. Consider the set $\mathcal{X}$ of 01-strings, where occurrences of 011 may or may not be marked.

eg
- 0101110110011110 $\in \mathcal{X}$ (no marks, has 011)
- 01(011)1011 0(011)110 (two marks, not all 011 marked)
- 01(011)(011)0(011)110 (all 011 marked)

all different elements of $\mathcal{X}$.

Let $\mathcal{Y} \subseteq \mathcal{X}$ be the subset of $\mathcal{X}$ of strings in which all occurrences of 011 are marked.

eg (011)(011) $\in \mathcal{Y}$

0101010 $\in \mathcal{Y}$

Let the weight fns w_0, w_1 be defined by

$w_0 : \mathcal{X} \to \mathbb{N}$ by w_0 = length of string

$w_1 : \mathcal{X} \to \mathbb{N}$ by w_1 = # of markings (ie circles).

Consider $\Phi_{\mathcal{X}}(x,y)$, $\Phi_{\mathcal{Y}}(x,y)$.

If we regard each circle as a separate "letter" in our alphabet, see that

$$\mathcal{X} = \mathcal{Y}[\bigcirc \to \{\underbrace{\bigcirc}_{\text{keep the circle}}, \underbrace{◌}_{\text{remove the circle}}\}]$$

Thus by our thm,

$$\Phi_{\mathcal{X}}(x,y) = \Phi_{\mathcal{Y}}(x, \Phi_{\{\bigcirc, ◌\}}(x,y))$$
$$= \Phi_{\mathcal{Y}}(x, y+1)$$

y with circle, 1 no circle

We want no occurrences of 011, which is given by

$$[y^0]\Phi_{\mathcal{Y}}(x,y) = \Phi_{\mathcal{Y}}(x,0) = \Phi_{\mathcal{X}}(x,-1)$$

Then

$$\mathcal{X} = \{0, 1, (011)\}^*,$$

and so

$$\Phi_{\mathcal{X}}(x,y) = \frac{1}{1-\Phi_{\{0,1,(011)\}}(x,y)} = \frac{1}{1-(x+x+x^3y)}.$$

Thus

$$[y^0]\Phi_{\mathcal{Y}}(x,y) = \Phi_{\mathcal{X}}(x,-1) = \frac{1}{1-(x+x+x^3(-1))} = \frac{1}{1-2x+x^3}.$$

Hence the # of strings of length n that don't have 011 as a substring is

$$[x^n]\Phi_{\mathcal{X}}(x,-1) = [x^n]\frac{1}{1-2x+x^3}.$$

What if we change '011' to '0101'?

Our previous strategy wouldn't work, since we need to allow things like

01 0011(01(01)01)01111

So, to solve this problem, we just introduce overlapping circles into $\mathcal{X}$:

$$\mathcal{X} = \{0, 1, (0101), (01(01)01), (01(01)(01)01), \dots\}$$

EXAMPLE: BALLS & BINS REVISITED

Problem:

"How many surjective functions exist from $[k]$ to $[n]$?"

(k balls, n bins, everything distinguishable, at least one ball per bin)

Sol^{n}. Idea: note

surjective <=> nothing not in range.

Then, consider the set $\mathcal{X}$ of marked functions

$$f : [k] \to [n]$$

where elements of $[n]$ which are not in range(f) may are marked.

eg $[k]$: 1 2 3 4 5; $[n]$: 1 2 (3) 4 $\in \mathcal{X}$

($3 \in [n]$ is marked.)

Define $\mathcal{Y} \subseteq \mathcal{X}$ be the subset of marked functions where all elements not in range are marked.

eg $[k]$: 1 2 3 4 5; $[n]$: 1 (2) (3) 4 $\in \mathcal{Y}$.

Then, let the weight function w be defined as the # of markings.

See that

$$\mathcal{X} = \mathcal{Y}[\bigcirc \to \{\bigcirc \to ◌\}],$$

and so

$$\Phi_{\mathcal{X}}(y) = \Phi_{\mathcal{Y}}(y+1).$$

We want elements of $\mathcal{Y}$ with no markings (which are surjective functions); ie we want

$$[y^0]\Phi_{\mathcal{Y}}(y) = \Phi_{\mathcal{Y}}(0) = \Phi_{\mathcal{X}}(-1).$$

Finally, to determine $\Phi_{\mathcal{X}}(x)$:

① we first pick the elements we want to circle; and then

② construct a function $f : [k] \to$ (uncircled).

Then

$$\Phi_{\mathcal{X}}(x) = \sum_{j=0}^{n} \underbrace{\binom{n}{j}(n-j)^k}_{\text{\# of ways to create object of } \mathcal{X} \text{ w/ } j \text{ circles}}$$

and so

$$\boxed{\text{answer} = \sum_{j=0}^{n}\binom{n}{j}(n-j)^k(-1)^j.}$$

Section 2: Graph Theory

GRAPH

A "graph" G consists of a finite set $V(G)$ and another set $E(G)$, where

① $V(G)$ is called the "set of vertices"; &

- Singular of 'vertices' is 'vertex'

② The elements of $E(G)$ are unordered pairs of distinct elements of $V(G)$, called "edges".

* usually, we write $p = |V(G)|$; & $q = |E(G)|$.

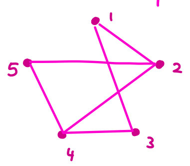
eg Let G be s.t.

$V(G) = \{1,2,3,4,5\}$

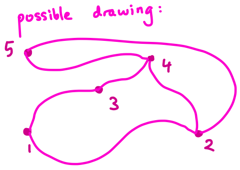
$E(G) = \{\{1,2\}, \{1,3\}, \{2,4\}, \{3,4\}, \{4,5\}, \{2,5\}\}$.

We often drop the $\{\ \}$ for edges, & just write 'uv' instead of $\{u,v\}$.

We can also represent G by a drawing:

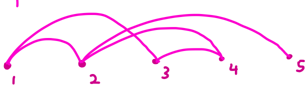


Another possible drawing:



* generally, we prefer straight lines over curly lines (since straight lines are easier to visualize and study)

Another possible drawing:



* we also don't want edges to cross themselves.

ALTERNATIVE VARIANTS OF GRAPHS

Variants (not studied in this course):

① Infinite graphs;
- $V(G)$ is an infinite set

② Directed graphs;
- Edges in $E(G)$ are ordered pairs of vertices

③ Graphs with loops;
- Two vertices in an edge can be the same

④ Graphs with multiple edges;
- $E(G)$ is a multiset.

etc.

ADJACENT, NEIGHBORS, INCIDENT, JOINS

If $e = xy$ (ie $e = \{x,y\}$) is an edge of G, we say:

① x is "adjacent" to y;

② x and y are "neighbors";

③ x is "incident" with e, or e is incident with x;

④ e "joins" x & y.

SET OF ALL NEIGHBORS: $N(v)$, DEGREE: $\deg(v)$

[1] If $v \in V(G)$, we define $N(v)$ to be the set of all neighbors of v.

[2] The "degree" of v is defined to be

$$\boxed{\deg(v) = |N(v)|.}$$

[3] $\deg(v)$ is also the number of edges incident with v.

ISOMORPHISM, ISOMORPHIC [GRAPHS]

[1] Let G, H be graphs. Then, we say a function $f: V(G) \to V(H)$ is an "isomorphism" if

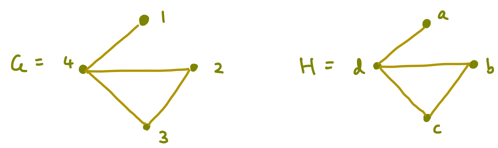
① it is bijective; &

② it "preserves adjacencies"; ie

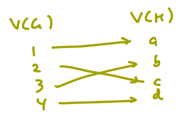
$$\boxed{uv \in E(G) \iff f(u)f(v) \in E(H).}$$

[2] If such an isomorphism exists, we say G & H are "isomorphic".

eg Consider the following two graphs:



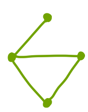
These graphs are different since $V(G) \neq V(H)$, but



is an isomorphism.

[3] Isomorphic graphs have the same structural properties; if we only care about these, we sometimes omit the vertex labels from drawings.

eg



* this is not a drawing of a graph, since the vertices are un-named.

This is a drawing of an isomorphism type.

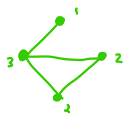
$$\sum_{v \in V(G)} \deg(v) = 2|E(G)|$$

<< THE HANDSHAKE THEOREM >>

1. Let G be a graph. Then

$$\boxed{\sum_{v \in V(G)} \deg(v) = 2|E(G)|.}$$

eg



LHS $= 1+2+2+3$
$= 8$

RHS $= 2(4) = 8$.

Proof. Consider the set

$P = \{(v,e) \mid v \in V(G),\ e \in E(G),\ v \text{ is incident to } e\}$.

For each vertex v, there are $\deg(v)$ edges incident with v, and so

$|P| = \sum_{v \in V(G)} \deg(v)$.

For each edge $e \in E(G)$, there are two vertices incident with e, and so

$|P| = 2|E(G)|$.

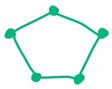
Proof follows. ☒

2. In particular, it follows from the thm that the number of vertices of odd degree is even.

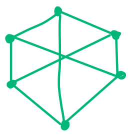
K-REGULAR [GRAPH]

1. A graph G is "k-regular" if all vertices of G have degree k.

eg



This is 2-regular.



This is 3-regular.

EXAMPLE 1: 7-REGULAR WITH 103 VERTICES

Problem:

> "Is there a 7-regular graph with 103 vertices?"

Sol^n. No, because such a graph we would have

$2|E(G)| = 7 \cdot 103$

and the RHS is odd. #

CUBE GRAPHS: Q_n

1. "Q_n" is the graph with

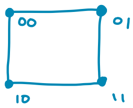
$V(Q_n) = \{$01-strings of length $n\}$

and two vertices σ, σ' are adjacent iff they differ in exactly one position.

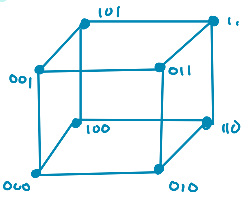
*Cube graphs are regular.

eg Q_1:

Q_2:



Q_3:



2. In particular,

$$\boxed{\begin{aligned} |V(Q_n)| &= 2^n \ ; \quad \& \\ |E(Q_n)| &= n \cdot 2^{n-1}. \end{aligned}}$$

why? $\sum \deg(v) = n2^n$.

KNESER GRAPHS

1. Fix n, m, k.

Then,

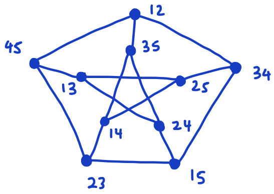
vertices $=$ m-element subsets of $[n]$

edges $=$ A & B are adjacent $\Leftrightarrow$ $|A \cap B| = k$

* kneser graphs are regular.

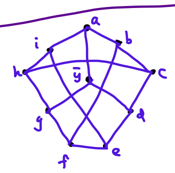
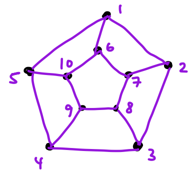
2. Special case: $n=5,\ m=2,\ k=0$.

(The Petersen graph).



EXAMPLE: ARE TWO GRAPHS ISOMORPHIC TO PETERSEN

Problem:

> " Let A =
>
> 
>
>
> & B =
>
> 
>
>
> Is A isomorphic to the Petersen graph? Is B? "

Sol^n. To prove 2 graphs are isomorphic, write down an isomorphism (and verify it).

To prove 2 graphs are non-isomorphic, we identify a structural property that is different.

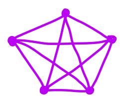
eg See that B has a "cycle" of length 4, but the petersen graph does not.

But A is isomorphic. → can check in finite time.

COMPLETE GRAPH: K_n

1. A "complete graph", denoted as K_n, has n vertices & all pairs of vertices are adjacent.

eg K_5



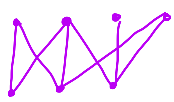
BIPARTITE GRAPHS & BIPARTITIONS

1. A graph is "bipartite" if there is a partition (A, B) of $V(G)$ such that every edge joins a vertex in A to a vertex in B.

2. In particular, (A, B) is called a "bipartition".

eg A

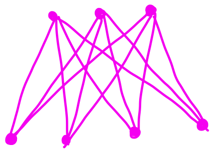
B



COMPLETE BIPARTITE GRAPH: $K_{m,n}$

The "complete bipartite graph" $K_{m,n}$ has bipartition (A,B), where
① $|A| = m$;
② $|B| = n$; &
③ Every $(a,b) \in A \times B$ is an edge ab.

eg $K_{3,4}$



SUBGRAPH

Let G be a graph.
A "subgraph" H of G is a graph such that
① $V(H) \subseteq V(G)$; &
② $E(H) \subseteq E(G)$.

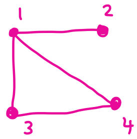
SPANNING SUBGRAPH

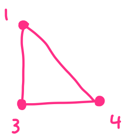
H is a "spanning subgraph" if $V(H) = V(G)$.

INDUCED SUBGRAPH

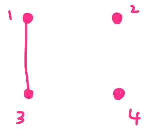
H is an "induced subgraph" if
$E(H) = \{ xy \in E(G) \mid x \in V(H),\ y \in V(H) \}$.

eg $G =$

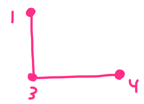




induced



spanning



neither

WALK, LENGTH, REVERSE WALK

1. A "walk" in a graph G is a sequence
$v_0 e_1 v_1 e_2 v_2 \cdots e_n v_n$
in which $v_i \in V(G)$ $\forall i = 0, \dots, n$ & $e_i = \{v_{i-1}, v_i\} \in E(G)$.

2. We sometimes just write $v_0 \cdots v_n$ (and omit the edges).

3. In particular, this is a walk "from v_0 to v_n".

4. The "length" of the walk is n.

5. The "reverse walk" is
$v_n e_n v_{n-1} \cdots e_1 v_0$
from v_n to v_0.

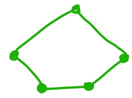
PATH

A "path" is a walk with no vertices repeated

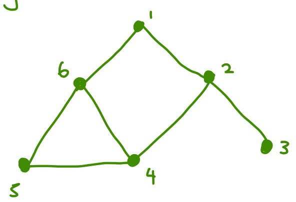
CYCLE

A "cycle" is a walk such that
① It is from a vertex to itself;
② No edges are repeated;
③ No vertices are repeated except first = last; &
④ It has at least one edge.

* We also use "path" & "cycle" to refer to subgraphs defined by the vertices/edges in the walk.



eg



1232456 is a walk
6123 is a path
124561 is a cycle.

SHORTEST WALK FROM u TO v IS A PATH

If there is a walk from u to v in a graph G, then any shortest length walk from u to v must be a path.

Proof. Consider a shortest walk
$w_0 w_1 \cdots w_n$, $u = w_0$, $v = w_n$
If this is not a path, then there is a repeated vertex; ie $w_i = w_j$ for $i < j$.
But then
$w_0 \cdots w_{i-1} w_i w_{j+1} \cdots w_n$
would be a shorter walk from $u \to v$, a contradiction.
Proof follows. ☒

THERE EXISTS A PATH: $\overset{G}{\rightsquigarrow}$

1. We define a relation $\overset{G}{\rightsquigarrow}$ on $V(G) \times V(G)$ by

$$\boxed{u \overset{G}{\rightsquigarrow} v \iff \text{there exists a path from } u \text{ to } v.}$$

2. By the previous lemma,

$$\boxed{u \overset{G}{\rightsquigarrow} v \iff \text{there exists a walk from } u \text{ to } v.}$$

$\overset{G}{\rightsquigarrow}$ IS AN EQUIVALENCE RELATION

Note $\overset{G}{\rightsquigarrow}$ is an equivalence relation.

Proof. Reflexive: For any $v \in V(G)$, $v \overset{G}{\rightsquigarrow} v$ since there is a path of length 0 from v to itself.

Symmetric: If $u \overset{G}{\rightsquigarrow} v$, given any path from $u \to v$, the reverse path exists, and is a path from $v \to u$.

Transitive: Suppose $u \overset{G}{\rightsquigarrow} v$, $v \overset{G}{\rightsquigarrow} w$. Let
$u = \gamma_0 \cdots \gamma_n = v$, $v = \omega_0 \cdots \omega_m = w$
be the paths from $u \to v$ & $v \to w$ resp.
Then
$u = \gamma_0 \cdots \gamma_n \omega_1 \cdots \omega_m = w$
is a walk from u to w.
Thus $u \overset{G}{\rightsquigarrow} w$. (by lemma). ☒

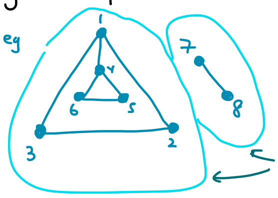
CONNECTED [GRAPH]

We say a graph G is "connected" if $\overset{G}{\sim}$ has exactly one equivalence class.

COMPONENT [OF A GRAPH]

A "component" of G is the subgraph induced by an equivalence class of $\overset{G}{\sim}$.

eg



Equivalence classes:
- $\{1, \dots, 6\}$
- $\{7, 8\}$

components

$_2$ Alternatively, a component is a "maximal connected subgraph".

$_3$ Exercise: If H is a connected subgraph of G, and $N_H(v) = N_G(v)$ $\forall v \in V(H)$, then H is a component.

G MINUS e: $G - e$, $e \in E(G)$

If e is an edge of G, we define "$G-e$" to be the spanning subgraph such that

$$\boxed{E(G-e) = E(G) \setminus \{e\}.}$$

BRIDGE / CUT-EDGE [OF A CONNECTED GRAPH]

If G is connected, we say e is a "bridge" if $G-e$ is not connected.

BRIDGE / CUT-EDGE [OF A GENERAL GRAPH]

We say $e \in E(G)$ is a bridge if it is a bridge in its component.

G IS CONNECTED, e IS A BRIDGE ⇒ G−e HAS 2 COMPONENTS, ENDPOINTS OF e IN EACH COMPONENT

If G is connected, and $e = xy$ is a bridge of G, then

1. $G-e$ has exactly two components; &
2. One component contains x, & the other contains y.

Proof. Let $v \in V(G)$ & let $v \dots x$ be a path in G from v to x.

If e does not appear in this path, then this is a path in $G-e$.

Thus
$$v \overset{G-e}{\sim} x.$$

Otherwise, since x can only appear once, e has to be the end; thus, the path is of the form
$$v \dots yex.$$

Since e can only appear once, thus
$$v \dots y$$
is a path from v to y in $G-e$.

Hence
$$v \overset{G-e}{\sim} y.$$

Therefore, there are at most two components in $G-e$, mainly the component containing x, & the component containing y.

Since $G-e$ is not connected, (as e is a bridge), these components must be distinct.

Proof follows. ⊠

EXAMPLE: 4-REGULAR GRAPHS HAVE NO BRIDGES

Problem:

> "Prove that a 4-regular graph has no bridges."

Sol^{n}. If $e = xy$ is a bridge:

Let G_x be the component of x in $G-e$.

Every vertex in G_x has degree 4, except for x, which has degree 3.

But this is impossible, since the Handshake Theorem says we must have an even # of nodes with odd degree ⊠

G MINUS v: $G-v$, $v \in V(G)$

If $v \in V(G)$, then we define "$G-v$" to be the induced subgraph with

$$\boxed{V(G-v) = V(G) \setminus \{v\}.}$$

* analagous definition for "$G+v$".

CUT-VERTEX

A "cut-vertex" is a vertex such that its deletion disconnects the graph.

$e \in E(G)$ ⇒ e IS A BRIDGE ⇔ e IS NOT IN ANY CYCLE

Let G be a graph, and let $e \in E(G)$.

Then e is a bridge iff e is not contained in any cycle.

∃u,v s.t. 2 DIFF PATHS FROM $u \to v$ ⇔ G HAS A CYCLE

Let G be a graph. Then the following are equivalent:

1. There exists $u, v \in V(G)$ such that there are two different paths from u to v; &
2. There exists a cycle in G.

Proof. ② ⇒ ①: easy.

① ⇒ ②: Let

$$P_1 = (u = u_0 \;\dots\; u_m = v)$$
$$P_2 = (u = v_0 \;\dots\; v_n = v)$$

be two different paths from u to v.

If $P_1 \neq P_2$, then there exists an edge e that appears in one but not the other. (exercise)

WLOG, suppose $e = u_{i-1}u_i$ is an edge of P_1.

But then

$$u_{i-1} \;\dots\; \underset{u = v_0}{u_0} v_1 \;\dots\; \underset{u}{v_n} \;\dots\; u_i$$

is a walk in $G-e$ from u_{i-1} to u_i.

Therefore e is not a bridge.

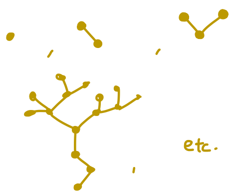
By the previous thm, ∃ a cycle containing e. ⊠

TREE

A "tree" is a connected graph with no cycles.

* the empty graph is not a tree!

eg



etc.

FOREST

(1) A "forest" is a graph in which every component is a tree.

(2) More concisely, a forest is a graph with no cycles.

EQUIVALENT DEFINITIONS OF BEING A TREE

Let T be a connected graph with p vertices and q edges.

Then the following are equivalent:

1. T is a tree (ie T has no cycles);
2. Every edge of T is a bridge;
3. There is exactly one path joining each pair of vertices; &
4. $q = p-1$.

★ "connected" is essential for ④⇒①.

eg △ • satisfies $q=p-1$ but isn't a tree.

Proof. ① ⇔ ② ⇔ ③ is solved by our previous theorems.

①⇒④: proceed by induction on p.

True for $p=1$ naively.

Let $p>1$ & the result is true for smaller p (ie for graphs w/ fewer vertices).

Let $e \in E(T)$. Since ①⇒②, e is a bridge.

So, $T-e$ has 2 components, say T_1, T_2.

T_1 & T_2 are connected, and have no cycles (since they are components of $T-e$, & are subgraphs of a graph with no cycles).

∴ T_1 & T_2 are trees.

By IH, $|E(T_1)| = |V(T_1)|-1$ & $|E(T_2)| = |V(T_2)|-1$.

Thus

$$\begin{aligned}|E(T-e)| &= |E(T_1)| + |E(T_2)| \\ &= |V(T_1)| + |V(T_2)| - 2 \\ &= |V(T)| - 2 =\end{aligned}$$

Hence $\underset{q}{|E(T)|} = \underset{p}{|V(T)|} - 1$

and we're done.

④⇒②: Suppose, for a contradiction, that $q=p-1$ & T has an edge that is not a bridge.

Delete non-bridges until none remain.

In the end, we're left with a connected graph (we only deleted non-bridges) with p vertices, and $< p-1$ edges (since we deleted ≥ 1 edge).

The result must be a tree since it's connected & has no non-bridges, which doesn't satisfy $q=p-1$.

This contradicts ①⇒④.

Proof follows. ▨

MT up to

F IS A FOREST WITH c COMPONENTS ⇒ $q=p-c$

Let F be a forest with p vertices, q edges and c components.

Then necessarily $q=p-c$.

SPANNING TREE

A subgraph T of G is called a "spanning tree" if T is a spanning subgraph & T is a tree.

G IS CONNECTED ⇔ IT HAS A SPANNING TREE

A graph G is connected iff it has a spanning tree.

Proof. (⇐) If G has a spanning tree, then we can use T to find a path joining any pair of vertices. Proof follows. #

(⇒) If G is connected, let

$S = \{$ connected spanning subgraphs of $G \}$.

$S \neq \emptyset$ since $G \in S$.

Consider an element of S with minimum amount of edges.

Exercise: show this is a tree.

STRUCTURE OF TREES

(1) Suppose T is a tree with p vertices, where $p \geq 2$.

Let n_i = # of vertices with degree i $\forall i = 0, 1, \ldots$

Then necessarily

1. $n_0 = 0$;
2. $\sum_{i \geq 1} n_i = p$; &
3. $\sum_{i \geq 1} i n_i = 2(p-1)$.

↳ handshake thm.

(2) We can combine these equations to get

④ $\boxed{n_1 = 2 + \sum_{i\geq 3} (i-2) n_i}$.

(from 2·② − ③)

LEAF [IN A TREE]

A "leaf" is a vertex of degree 1 in a tree.

EVERY TREE WITH ≥2 VERTICES HAS ≥2 LEAVES

Every tree with at least 2 vertices has at least 2 leaves.

Proof. Corollary from previous thm.

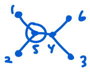
ENUMERATION OF TREES

ROOTED TREE : (T, r)

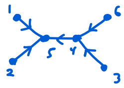
💡₁ A "rooted tree" is a pair (T, r) where
① T is a tree; &
② $r \in V(T)$.

💡₂ We call r the "root".

💡₃ We represent a rooted tree by



① circling the root; or
② by putting arrows on all edges so that the unique path from any vertex to r follows the arrows.



p^{p-2} TREES WITH VERTEX SET $[p]$

<<CAYLEY>>

💡₁ Problem:

> "Given a set of vertices, say $[p]$, how many trees are there with that vertex set?"

eg $p = 3$

💡₂ There are exactly p^{p-2} trees with vertex set $[p]$.

eg

4 trees

+

12 trees

$= 16$ trees $= 4^2$

Proof. We count sequences $(e_1, \dots, e_{p-1})$ of <u>directed</u> edges such that these edges collectively form a rooted tree on vertex set $[p]$.

We do this in 2 ways:

<u>Method #1</u>: Let $\mathcal{T}$ = set of all trees w/ vertex set $[p]$.

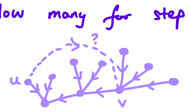
For each $T \in \mathcal{T}$, there are p ways to pick a root, and $(p-1)!$ ways to order the edges in a sequence.

$\therefore$ # of sequences $= \underbrace{|\mathcal{T}|}_{\text{pick a tree}} \cdot \underbrace{p}_{\text{pick a root}} \cdot \underbrace{(p-1)!}_{\text{pick an order}}$

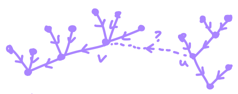
<u>Method #2</u>: Start with the vertex set $[p]$ & no edges. Add directed edges one at a time as follows:

① Pick any vertex $v \in [p]$;
② Pick another vertex u s.t. adding a directed edge from u to v creates a graph in which every component is a rooted tree.

There are p choices for step ①.
How many for step ②?



this is not a tree. ie u cannot be on the same component as v.



this is not a tree. (u has two different 'to' paths). ie u canot be on the different component as v.

But this works iff u is the root of its component.

So, the # of possibilities for u (ie step ②) is the # of components $- 1$.

By a previous theorem, # of components $= p - q$.

So # of possibilities $= p - q - 1$.

$\therefore$ # of choices at k^{th} iteration $= p(p-k)$.

Thus

$$\begin{aligned} \text{\# of sequences} &= p(p-1) \cdot p(p-2) \cdot \dots \cdot p(1) \\ &= p^{p-1} \cdot (p-1)!. \end{aligned}$$

Combining the two methods, we get

$$|\mathcal{T}| \cdot p \cdot (p-1)! = p^{p-1} \cdot (p-1)!$$

or in other words $|\mathcal{T}| = p^{p-2}$, as needed. ∎

BREADTH-FIRST SEARCH TREES / BFSTs

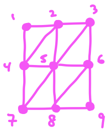
A "breadth-first search tree" of a graph G is a rooted tree (T, r), where T is a subgraph of G, which is the output by the "BFST algorithm".

BFST ALGORITHM

$_1$ Input: a graph G & vertex $r \in V(G)$.
Output: a BFST with root r.

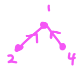
$_2$ Example:

G =

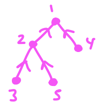


Let $r = 1$.
Form a queue, and push 1 to the queue.
$q = \underline{1}$.
Get q.first, and see its neighbors <u>we haven't "seen yet"</u>.
Draw "T", a rooted tree where the neighbors point.
Pop the queue, & add these neighbors to the queue.
$q = \underline{2}, 4$
Repeat the same process repetitively

T:



T:



$q = \underline{4}, 3, 5$

T:

And so on, until the queue is empty.
The final tree is our result.

$_3$ The general algorithm follows very similarly.

UNEXHAUSTED VERTICES

The vertices currently in the queue above are called "unexhausted".

ACTIVE VERTEX

The vertex at the head of the queue is called "active".

PARENT [OF A VERTEX]

The "parent" of x is the active vertex such that x is joined in the tree.
– the root has no parent.

LEVEL [OF A VERTEX]

The "level" of x is defined by
① $level(r) = 0$; &
② $level(x) = level(pr(x)) + 1$.

BFST IS A TREE, & IT IS SPANNING <=> IT IS CONNECTED

A BFST is a tree, and it is spanning iff it is connected.

Proof. To show it is a tree, we show by induction at each stage of the algorithm that T is connected & $|E(T)| = |V(T)| - 1$.

For the second point:
($\Rightarrow$) Clear.
($\Leftarrow$) Suppose G is connected, & (T, r) is a BFST.
Given $v \in V(G)$, let $v = v_0 \cdots v_k = r$ be a path from v to r.
(Such a path exists $\because G$ is connected).
If $v \notin V(T)$, then $\exists$ some index k such that $v_{k-1} \notin V(T)$ & $v_k \in V(T)$.
But this is impossible, because if $v_k \in V(T)$, then at some point in the BFST algorithm v_k is active.
At this point, all of its neighbors (including v_{k-1}) are added to T.
Thus $v_{k-1} \in V(T)$, a contradiction.
Proof follows. ▨

$x \in V(G)$ ACTIVE, $level(x) = k \Rightarrow$ ALL VERTICES IN THE QUEUE HAVE LEVEL k OR $k+1$

When $x \in V(G)$ is active, if $level(x) = k$, then all vertices in the queue have level k or $k+1$.

Proof. Exercise.

T IS A BFST, $e \in E(G) \Rightarrow |level(x) - level(y)| \leq 1$

«FUNDAMENTAL PROPERTY OF BFSTs»

Let G be a connected graph, and let T be a BFST.
Let $e = xy \in E(G)$. Then necessarily

$$\boxed{|level(x) - level(y)| \leq 1}$$

Proof. WLOG, suppose x is active before y.
Let $k = level(x)$. If y is not in the queue when x is active, then y is currently not in $V(T)$.
Then x becomes the parent of y.
If y is in the queue at this stage, then $level(y) = \{k, k+1\}$ by the lemma.
Proof follows. ▨

DISTANCE [BETWEEN TWO VERTICES]: $dist(x,y)$

Let G be a connected graph, and let $x, y \in V(G)$.
The "distance" between x & y is the length of a shortest path from x to y.

T IS A BFST WITH ROOT $x \Rightarrow x \to y$ IS THE SHORTEST PATH, $dist(x,y) = level(y)$

$_1$ Let G be a connected graph, and T is a BFST rooted at x.
Then the unique path from x to y in T is the shortest path.
In particular,

$$\boxed{dist(x,y) = level(y).}$$

Proof. Clearly the path
$$y \cdot pr(y) \cdot pr(pr(y)) \cdots x$$
is a path from y to x of length $level(y)$.
We must show that any other path has length $\geq level(y)$.
Let
$$y = v_0 v_1 \cdots v_k = x$$
be a path.
See that

$$\begin{aligned}
level(y) &= |level(y) - level(x)| \\
&= \left|\sum_{i=1}^{k} (level(v_{i-1}) - level(v_i))\right| && \text{(telescoping sum)} \\
&\leq \sum_{i=1}^{k} |level(v_{i-1}) - level(v_i)| && \text{(by } \Delta \text{ ineq)} \\
&\leq \sum_{i=1}^{k} (1) && \text{(by Fund Prop. of BFSTs)} \\
&= k
\end{aligned}$$

as needed. ▨

$_2$ Note this fails if neither x nor y is the root.

GIRTH [OF A GRAPH]: girth(G)

The "girth" of a graph G is the length of the shortest cycle.

ALGORITHM TO FIND THE GIRTH

Let G be a connected graph.

For each vertex $r \in V(G)$, let (T_r, r) be a BFST rooted at r.

Let

$$\boxed{m_r = \min_{xy \in E(G) \setminus E(T_r)} \{ \text{level}_{T_r}(x) + \text{level}_{T_r}(y) + 1 \}.}$$

Then

$$\boxed{\text{girth}(G) = \min \{ m_r \mid r \in V(G) \}.}$$

Proof Sketch. Let $r \in V(G)$ be a vertex & $xy \in E(G) \setminus E(T_r)$.

First, we need to show

$$\text{level}_{T_r}(x) + \text{level}_{T_r}(y) + 1 \geq \text{girth}(G).$$

Consider the walk

$$\underbrace{r, \dots, pr(y), y}_{\text{level}_{T_r}(y)}, \underbrace{x, pr(x), \dots, r}_{\text{level}_{T_r}(x)}.$$

This has length $\text{level}_{T_r}(x) + \text{level}_{T_r}(y) + 1$.

Now, consider the subgraph defined by the vertices & edges in this walk.

This has at most $\text{level}_{T_r}(x) + \text{level}_{T_r}(y) + 1$ edges, and it has a cycle (there are two distinct paths from r to x).

Thus, the subgraph has cycle of length $\leq \text{level}_{T_r}(x) + \text{level}_{T_r}(y) + 1$.

Hence, the girth (ie the shortest cycle) must be $\leq \text{level}_{T_r}(x) + \text{level}_{T_r}(y) + 1$.

Considering all vertices r, thus

$$\text{girth}(r) \leq \min \{ m_r : r \in V(G) \}.$$

To show equality, we prove

"If r is in a shortest cycle, then $m_r = \text{girth}(G)$".

(left as exercise.) ☒

Note the # of cycles in G is "exponential" in the # of edges for a fixed # of vertices.

So looking at every single cycle does not give a polynomial time algorithm.

However, this algorithm does run in polynomial time.

EQUIVALENT DEFINITIONS OF BIPARTITE GRAPHS

Let G be a connected graph, and T a BFST.

Then the following are equivalent:

1. G is bipartite;
2. G has no cycles of odd length; &
3. For every $xy \in E(G)$, $|\text{level}(x) - \text{level}(y)| = 1$.

Proof. ① ⇒ ②: Trivial — left as an exercise.

② ⇒ ③: Suppose ③ is false. Then $\exists\, xy \in E(G)$ such that $\text{level}(x) = \text{level}(y)$.

(since by the Fund. Prop. of BFSTs states $|\text{level}(x) - \text{level}(y)| \leq 1$.

Now consider the subgraph from the girth algorithm (to the left).

The cycle in this subgraph has odd length, which contradicts ②.

eg

③ ⇒ ①: If ③ holds, then let

$$A = \{ x \in V(G) \mid \text{level}(x) \text{ even} \}$$
$$B = \{ y \in V(G) \mid \text{level}(x) \text{ odd} \}.$$

Then ③ ⇒ (A, B) is a bipartition, which is sufficient for what we need. ☒

THE ADJACENCY MATRIX

💡1 Let G be a graph.
For convenience, assume $V(G) = [n]$.
The "adjacency matrix" of G is the matrix A s.t.

$$\boxed{A_{ij} = \begin{cases} 1, & ij \in E(G) \\ 0, & \text{otherwise} \end{cases}}$$

eg

Then

$$A = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}.$$

💡2 Notes:

① a vertex can never be adjacent to itself, so the diagonal will always be 0's.

② this matrix will also be symmetric.

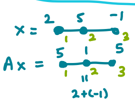
💡3 Thus, they are orthogonally diagonalizable, and so have real eigenvalues.

💡4 In particular,

① If $x = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$ & $y = Ax = \begin{pmatrix} y_1 \\ \vdots \\ y_n \end{pmatrix}$, then

$$\boxed{y_i = \sum_{j \in N(i)} x_j.}$$

eg



• 2, 5, −1 are the entries of the vector x.

OF WALKS OF LENGTH n FROM $i \to j$ IS $(A^n)_{ij}$

💡 Let $i, j \in V(G)$.
Then, the # of walks of length n from i to j is equal to $(A^n)_{ij}$.

Proof. Exercise.
Use induction on n & def$^{\underline{n}}$ of matrix multiplication:

$$(A^n)_{ij} = \sum_k (A^{n-1})_{ik} A_{kj}.$$

EIGENVALUES [OF A GRAPH]

💡 If G is a graph, the "eigenvalues" of G is the eigenvalues of its adjacency matrix.

G IS k-REGULAR $\Rightarrow$ k IS AN EIGENVALUE OF G, g_k = # OF COMPONENTS

💡 Let G be a k-regular graph.
Then k is an eigenvalue of G, and its geometric multiplicity is the # of components.

* geometric multiplicity $= \dim\{x \in \mathbb{R}^n \mid Ax = kx\}$

Proof. Let A be the adjacency matrix.
Let $x = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} \in \mathbb{R}^n$. We will prove
$Ax = kx \iff x_i = x_j \quad \forall ij \in E(G)$.
Rest of proof is exercise.

($\Leftarrow$) Let $Ax = \begin{pmatrix} y_1 \\ \vdots \\ y_n \end{pmatrix}$. If $x_i = x_j \ \forall ij \in E(G)$, then

$$y_i = \sum_{j \in N(i)} x_j = \sum_{j \in N(i)} x_i = kx_i \quad (\text{since } \deg(j) = k).$$

$\therefore Ax = kx$. #

($\Rightarrow$) Let $Ax = kx$. Then

$$kx_i = \sum_{j \in N(i)} x_j. \quad \text{— } (*)$$

Let $S = \{i \in V(G) \mid \exists j \in N(i) \text{ s.t. } x_i \neq x_j\}$. We want to show $S = \emptyset$.
If $S \neq \emptyset$, let $i \in S$ s.t. x_i is maximal.
Claim: $x_i \geq x_j \quad \forall j \in N(i)$.
Why? Case #1: $j \in S$. Then x_i is maximal $\Rightarrow x_i \geq x_j$.
Case #2: $j \notin S$. Then by def$^{\underline{n}}$ of S, $x_j = x_h$ for all $jh \in E(G)$.
In particular, $x_j = x_i \Rightarrow x_i \geq x_j$. #

Thus

$$\sum_{j \in N(i)} x_j \leq kx_i,$$

with equality iff $x_i = x_j \ \forall j \in N(i)$.
But since $i \in S$, this inequality is strict; ie $\sum_{j \in N(i)} x_j < kx_i$,
which contradicts $(*)$.
Hence $S = \emptyset$, as required. ☒

EXAMPLE: PETERSEN GRAPH

1. Let T be the Petersen graph.
Let A be T's adjacency matrix.

2. Problem 1:

> "What are A's eigenvalues & their respective multiplicities?"

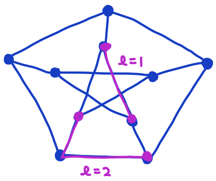
Sol^{n}. 3 is an eigenvalue with multiplicity 1.

We claim

$$\boxed{A^2 + A = 2I + J, \quad J = \text{all ones matrix}.}$$

Proof. Notice

$$(A^2+A)_{ij} = \underbrace{A^2_{ij}}_{\substack{\#\text{ of walks}\\ \text{of length } 2\\ \text{from } i\to j}} + \underbrace{A_{ij}}_{\substack{\#\text{ of walks}\\ \text{of length } 1\\ \text{from } i\to j}}.$$



$$(2I+J) = \begin{cases} 3, & i=j \\ 1, & i\neq j \end{cases}$$

Check: for any 2 distinct vertices i, j, there is exactly 1 walk of length 1 or of length 2.

Now, let λ be an eigenvalue. Assume $\lambda \neq 3$.
Let $Ax = \lambda x$, where $x \neq 0$.
Then

$$A^2x + Ax = 2Ix + Jx$$
$$\Leftrightarrow \quad \lambda^2 x + \lambda x = 2x + 0.$$

Since A is orthogonally diagonalizable
$\Rightarrow$ vectors from distinct eigenspaces are orthogonal.
Thus $(1 \cdots 1)x = 0$ (since $\begin{pmatrix}1\\ \vdots\\ 1\end{pmatrix}$ is an eigenvector of 3).
Hence

$$\lambda^2 + \lambda = 2.$$
$$\Rightarrow \lambda = -2, \ \lambda = 1.$$

It follows the Petersen graph has eigenvalues $3, 1, \dots, 1, -2, \dots, -2$

The sum of the eigenvalues $= \operatorname{trace}(A) = 0$.
$\therefore$ there are 5 1's & 4 -2's.

$$\boxed{\therefore \text{ The eigenvalues of } A \text{ are } 3, \underbrace{1, \dots, 1}_{5}, \underbrace{-2, \dots, -2}_{4}.}$$

HAMILTONIAN CYCLE

A "Hamiltonian cycle" in a graph G with p vertices is a cycle of length p.

OF CYCLES $\approx p!\left(\frac{q}{m}\right)^p$

Note the approximate number of cycles in a graph with p vertices & q edges, where $q \leq m = \binom{p}{2}$, is $\approx p!\left(\frac{q}{m}\right)^p$.

*very approximate value!

E^A_λ [FOR A SYMMETRIC MATRIX A]

If A is symmetric, & λ is an eigenvalue, write

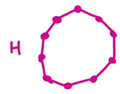
$$E^A_\lambda = \{ x \in \mathbb{R}^n \mid Ax = \lambda x \}.$$

THE PETERSEN GRAPH DOES NOT HAVE A HAMILTONIAN CYCLE

Problem:

> "Prove the Petersen Graph cannot have a Hamiltonian cycle".

Proof. Suppose P has a Hamiltonian cycle H.
Let A be the adjacency matrix of P.
Then H is a 2-regular subgraph that is spanning & connected.

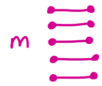


Let C be the adjacency matrix of H.
Then the eigenvalues of C are

$$\underbrace{2, \cos \pi/5, \cos 2\pi/5, \dots}_{\neq 2}$$

(by the previous theorem).

If we delete the edges of H, we get a 1-regular spanning subgraph, say M.



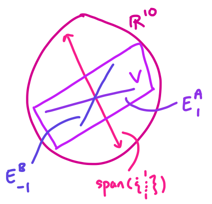
Let's say its adjacency matrix is B.
Then the eigenvalues of B are

$$\underbrace{1, 1, 1, 1, 1}_{5 \text{ components}}, \underbrace{-1, -1, -1, -1, -1}_{\substack{\text{have to be the same, \&}\\ \text{the sum of the } \lambda \text{ have}\\ \text{to be } 0}}$$

In particular, $\begin{pmatrix}1\\ \vdots\\ 1\end{pmatrix}$ is an eigenvector of

- 3 in A;
- 1 in B;
- 2 in C.

Additionally, the other eigenvectors are orthogonal to this vector.
Then, by construction, we have $A = B + C$.
Consider E^A_1 & E^B_{-1}.



$V = \left\{\begin{pmatrix}1\\ \vdots\\ 1\end{pmatrix}\right\}^{\perp}$.
(note $\dim(V) = 9$).
Also $\dim E^A_1 = \dim E^B_{-1} = 5$.
Since $\underset{\dim 5}{E^A_1}, \underset{\dim 5}{E^B_{-1}} \subseteq \underset{\dim 9}{V}$,
thus

$$\dim(E^A_1 \cap E^B_{-1}) \geq 5 + 5 - 9 = 1.$$

So, there is a non-zero vector

$$x \in E^A_1 \cap E^B_{-1}.$$

Thus

$$Ax = x \quad \& \quad Bx = -x,$$

and so

$$Cx = (A - B)x = 2x.$$

So we've found another eigenvector in E^C_2, and so

$$\dim E^C_2 \geq 2.$$

Hence H has 2 components, a contradiction. ▪

PLANAR & NON-PLANAR GRAPHS

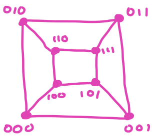
PLANAR EMBEDDING

A drawing of a graph in the plane with no edges crossing is called a "planar embedding".

PLANAR GRAPH

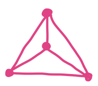
A graph is "planar" if it has at least one planar embedding.

eg Q_3:



So Q_3 is planar.

K_4:



So K_4 is planar.

$K_{3,2}$:

But the following are non-planar:

K_5, $K_{3,3}$, Q_4, the Petersen graph.

FACES [OF A PLANAR EMBEDDING]

A planar embedding divides the plane into regions, called "faces".

ADJACENT / INCIDENT [FACES]

We say two faces are "adjacent" iff they are "incident" on a common edge.

BOUNDARY [OF A FACE]

The "boundary" of a face is the subgraph defined by the vertices & edges incident with it.

Note: the above three are concepts associated with a planar embedding, not a planar graph! (graphs don't have faces.)

SET OF FACES: F(P)

let P be a planar embedding.

We write "F(P)" for the set of faces of P.

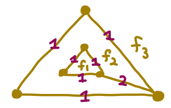
DEGREE [OF A FACE]: deg(f)

The "degree" of a face is

(# of non-bridges) + 2 × (# of bridges)

in the boundary.

eg



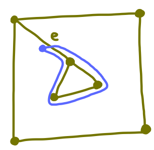
p = 6 vertices
q = 9 edges
s = 3 faces

$deg(f_1) = 3$
$deg(f_2) = 8$
$deg(f_3) = 3$.

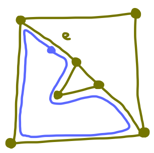
BRIDGES ARE INCIDENT W/ 1 FACE, NON-BRIDGES ARE INCIDENT W/ 2 FACES

Why are bridges special?

Consider the following two graphs:



e is a bridge



e is not a bridge

let e be an edge.

Starting on one side of e, trace along the boundary of a face. (in blue)

① In the first graph, we got to the opposite side.
- The two faces on either side of e are the same
- So e is incident with one face.
- Moreover, e is a bridge, because the line we drawn separates the two parts of G-e.

② In the second graph, we got to the same side.
- Thus e is not a bridge (since the boundary of f contains a cycle).
- e is also incident with two different faces.

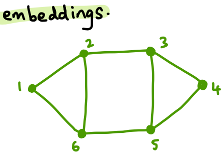
Key idea: bridges are incident with 1 face, whilst non-bridges are incident with 2 faces.

(Rigorous proof needs the Jordan Curve Theorem.)

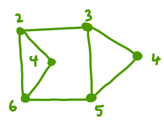
"DIFFERENT" PLANAR EMBEDDINGS

Note: a planar graph can have "different" planar embeddings.

eg



face sequence: 6, 4, 3, 3



face sequence, 5, 5, 3, 3

$\sum_{f \in F(P)} \deg(f) = 2q$

<< HANDSHAKE THM FOR FACES >>

For a planar embedding P, let $F(P)$ be the set of faces.

If P has q edges, then

$$\boxed{\sum_{f\in F(P)} \deg(f) = 2q.}$$

Proof. Let

$A = \{(f,e) : f \in F(P),\ e \in E(P),\ f \text{ incident with } e,\ e \text{ bridge}\}$

$B = \{(f,e) : f \in F(P),\ e \in E(P),\ f \text{ incident with } e,\ e \text{ non-bridge}\}$

Then

$$|A| + 2|B| = \underbrace{2q}_{\uparrow \text{ counting by edges}} = \underbrace{\sum_{f\in F(P)} \deg(f)}_{\uparrow \text{ counting by faces}}.$$

Proof follows. ∎

$p - q + s = c + 1$

<< EULER'S FORMULA >>

For any planar embedding with

① p vertices; (0-dimensional)

② q edges; (1-dimensional)

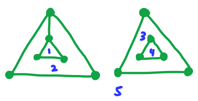
③ s faces; & (2-dimensional)

④ c components,

we have

$$\boxed{p - q + s = c + 1.}$$

eg



$p = 12$
$q = 13$
$s = 5$
$c = 3$

Then $p - q + s = 4 = 3 + 1 = c + 1$.

Proof. Assume p is fixed, and use induction on q.

Let P be our planar embedding.

Base case: $q = 0$. $\Rightarrow$ so $s = 1$, $c = p$.

$\Rightarrow p - q + s = p + 1 = p + 1$.

(So formula works out.)

Inductive step: Suppose $q > 0$ and the result holds for planar embeddings with p vertices & $q-1$ edges.

Let P be a planar embedding with p vertices & q edges, c components & s faces.

Let $e \in E(P)$, and consider $P - e$.

Case 1: e is not a bridge.

Then $P-e$ has p vertices, $q-1$ edges, $s-1$ faces (since two faces on either side of e become one) & c components (since e is not a bridge).

Thus $p - (q-1) + (s-1) = c + 1$

$\Rightarrow p - q + s = c + 1$. #

Case 2: e is a bridge.

Then $P-e$ has p vertices, $q-1$ edges, s faces (two faces on either side of e are already the same), & $c+1$ components (since e is a bridge).

Thus $p - (q-1) + s = (c+1) + 1$

$\Rightarrow p - q + s = c + 1$. #

Proof follows. ∎

P HAS A CYCLE ⇒ EVERY FACE'S BOUNDARY HAS A CYCLE, $\deg(f) \geq \text{girth}(P)$ $\forall f \in F(P)$

Let P be a planar embedding that has a cycle.

Then the following are true:

① The boundary of every face contains a cycle;

② The degree of every face $\geq \text{girth}(P)$.

Proof. ① Let $f \in F(P)$. Recall in a forest, the # of vertices, edges & components satisfy $q = p - c$.

P is not a forest, so

$q \geq p - c + 1$.

By Euler's formula,

$s = q - p + c - 1 + 2 \geq 2$ faces.

Thus f is not the whole plane.

Let Q be the boundary of f (a subgraph of P embedded in the plane.)

Then f is also a face of Q, and it's not the whole plane.

Thus Q has at least 2 faces.

So Q is not a forest, and so has a cycle.

② $\deg(f) \geq |E(Q)| \geq$ length of a cycle in $Q \geq \text{girth}(Q)$

$\geq \text{girth}(P)$. ∎

EXAMPLE 1: K_5 IS NOT PLANAR

We can show K_5 is non-planar.

Proof. Suppose P is a planar embedding of K_5.

Then

$p = 5$, $q = 10$, $c = 1$

and so by Euler's formula

$s = q - p + c + 1 = 7$.

For every $f \in F(P)$, $\deg(f) \geq \text{girth}(P) = 3$, & and by the HT

$2q = \sum_{f\in F(P)} \deg(f) \geq 21$,

but $q = 10$ so $2q = 20$, contradiction!

Proof follows. ∎

G IS PLANAR ⇒ $q \leq 3p - 6$

Let G be a graph with $p \geq 3$ vertices & q edges.

If G is planar, then $q \leq 3p - 6$.

Proof. If G has no cycles, then G is a forest

so $q = p - 1 \leq 3p - 6$. (since $p \geq 3$).

If G has a cycle, then $\text{girth}(G) \geq 3$.

Let P be a planar embedding of G. Then

$$2q = \sum_{f\in F(P)} \deg(f) \geq 3\underset{\substack{\uparrow \\ \#\text{ of faces}}}{s}. \quad \text{(by the Handshake Thm for faces)}$$

Hence

$s \leq \frac{2}{3} q$.

Since $c \geq 1$, by Euler's Theorem

$p - q + s = c + 1 \geq 2$

&

$p - q + s \leq p - q + \frac{2}{3} q = p - \frac{1}{3} q$.

So

$p - \frac{1}{3} q \geq 2 \Rightarrow q \leq 3p - 6$

as desired. ∎

VERTICES HAVE DEG d, FACES HAVE DEG d^* ⇒ GRAPH IS A CYCLE OR A PLATONIC SOLID

💡1 Consider connected planar embeddings such that

① Every vertex has degree d;
② Every face has degree d^*.

What do these look like?

Sol$^{\underline{n}}$. By HT for vertices, $dp = 2q$.
(dp: sum of deg of vertices)

By HT for faces, $d^*s = 2q$
(d^*s: sum of deg of faces.)

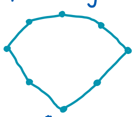
By Euler's Theorem, $p - q + s = 1 + 1 = 2$
(since the graph is connected).

Via elimination, it follows that
$$p - q + \frac{2q}{d^*} = 2.$$
Hence
$$q = \frac{d^*(p-2)}{d^*-2}$$
Similarly,
$$\frac{dp}{2} = \frac{d^*(p-2)}{d^*-2}.$$
This can be rewritten as
$$\frac{d(d^*-2)}{2d^*} = \frac{p-2}{p} < 1.$$
In particular, $p \geq 3$. So
$$\frac{d(d^*-2)}{2d^*} < 1.$$
Hence
$$dd^* - 2d - 2d^* < 0.$$
Thus
$$dd^* - 2d - 2d^* + 4 < 4.$$
$$\Rightarrow (d-2)(d^*-2) < 4.$$

Options:

① $d = 2$, $d^* =$ any

eg

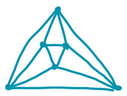


– cycle.

② $d = 3$, $d^* = 3$. (since $d^* \geq 3$).
This is K_4:

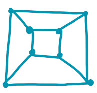
- tetrahedron

③ $d = 4$, $d^* = 3$.
This is



– octahedron.

④ $d = 5$, $d^* = 3$.
⑤ $d = 3$, $d^* = 4$.
This is Q_3:



– cube

⑥ $d = 3$, $d^* = 5$.

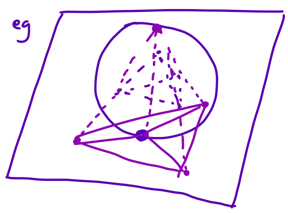
💡2 These graphs correspond to the cycle, & the five platonic solids.

GRAPH IS PLANAR ON A SPHERE ⇒ GRAPH IS PLANAR ON THE PLANE

💡 Any graph that can be drawn on a sphere without crossing can be drawn on a plane without crossing.

Proof. We can use "stereographic projection".

eg



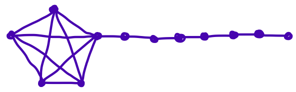
∎

girth(G) = k ⇒ $q \leq \frac{k(p-2)}{k-2}$

💡1 Let G be a planar graph with p vertices, q edges, & girth(G) = k.
Then necessarily
$$\boxed{q \leq \frac{k(p-2)}{k-2}.}$$

💡2 The converse is not true!

eg



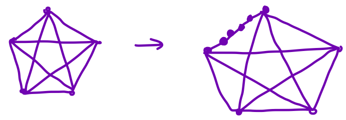
The inequality holds despite the fact we know the graph is not planar.

ANY SUBGRAPH OF A PLANAR GRAPH IS PLANAR

💡 Any subgraph of a planar graph is planar.

EDGE-SUBDIVISION

💡 An "edge-subdivision" of a graph involves taking an edge and replace it by a path repeatedly.



G IS NON-PLANAR ⇔ IT HAS A SUBGRAPH THAT IS AN EDGE SUBDIVISION OF K_5 OR $K_{3,3}$

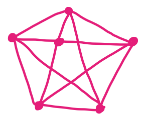
<< KURATOWSKI'S THEOREM >>

💡1 A graph is non-planar iff either

① It has a subgraph which is an edge subdivision of K_5; or
② It has a subgraph which is an edge subdivision of $K_{3,3}$.

💡2 Make sure we do not "repeat" any vertices!

eg



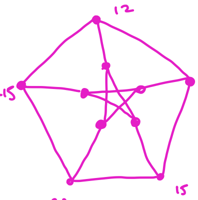
This is planar! 2 "lines" share a common vertex.

EXAMPLE: THE PETERSEN GRAPH

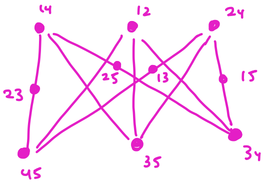
💡 Problem:

"Use Kuratowski's Theorem to show the Petersen graph is not planar".

Sol$^{\underline{n}}$. See that



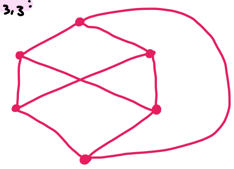
⇔



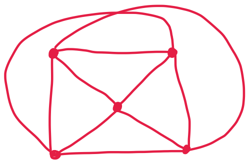
So the Petersen graph is an edge subdivision of $K_{3,3}$.
Thus, it is not planar.

BETTER WAYS TO DRAW $K_{3,3}$ & K_5

💡₁ $K_{3,3}$:



💡₂ K_5:



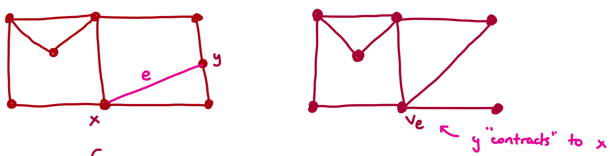
CONTRACTING AN EDGE: G/e

💡 Let G be a graph, and let $e = xy \in E(G)$.
Define a new graph "G/e" by

$$\boxed{\begin{aligned} V(G/e) &= V(G) \setminus \{x,y\} \cup \{v_e\}; \quad \& \\ E(G/e) &= \{ab \in E(G) \mid \{a,b\} \cap \{x,y\} = \emptyset\} \\ &\quad \cup \{av_e \mid ax \in E(G) \text{ or } ay \in E(G),\ a \notin \{x,y\}\} \end{aligned}}$$

where v_e = the vertex that "corresponds" to e

eg

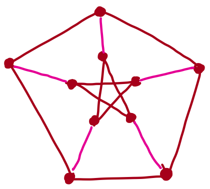


G $\qquad$ G/e

MINOR [OF A GRAPH]

💡 A graph obtained from G by deleting edges & vertices and/or contracting edges is called a "minor" of G.

eg K_5 is a minor of the Petersen graph.



Contracting the pink edges gives us K_5.

G IS NON-PLANAR $\iff$ EITHER $K_{3,3}$ OR K_5 IS A <u>MINOR</u> OF G

<< WAGNER'S THEOREM >>

💡 A graph is non-planar iff either
① K_5 is a minor; or
② $K_{3,3}$ is a minor.

* <u>Note</u>: this is also referred to as "Kuratowski's Theorem" (as well as the previous definition).

GRAPH COLORINGS

K-COLORING

1. A "k-coloring" of a graph G is an assignment of k or fewer colors to the vertices such that adjacent vertices have different colors.
2. Formally, this is a function $f: V(G) \to [k]$ s.t. (colors)

$$\boxed{xy \in E(G) \Rightarrow f(x) \neq f(y).}$$

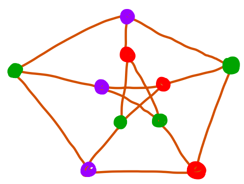
K-COLORABLE

1. If a k-coloring exists on G, we say G is k-colorable.
2. Note there is no efficient algorithm to determine if a graph is 3-colorable.

EXAMPLE: THE PETERSEN-GRAPH IS 3-COLORABLE

1. Show the Petersen graph is 3-colorable.

Sol^{n}.



$\Rightarrow$ The Petersen graph is 3-colorable.

2-COLORABLE $\Leftrightarrow$ BIPARTITE

1. Note a graph is 2-colorable iff it is bipartite.
2. This can be decided efficiently.

EVERY PLANAR GRAPH HAS A VERTEX WITH $\deg(v) \le 5$

1. Every planar graph has a vertex with degree at most 5.

Proof. Suppose not. Then $\exists$ a planar graph G s.t.

$$\deg(v) \geq 6 \quad \forall v \in V(G).$$

Since $|V(G)| \geq 3$ & G is planar, thus $q \le 3p-6$.

By the HT,

$$2q = \sum_{v \in V(G)} \deg(v) \geq 6p.$$

ie

$$q \geq 3p \quad \& \quad q \le 3p-6.$$

The two inequalities obtained are inconsistent — thus contradiction! Proof follows. ☒

ANY PLANAR GRAPH IS 6-COLORABLE
<< SIX-COLOR THEOREM >>

1. Any planar graph is 6-colorable.

"a warmup theorem".

Proof. We proceed by induction on p, the # of vertices.

Base case: $p=1$. (This is trivial.)

Inductive step: Suppose G has p vertices & the result is true for graphs with $p-1$ vertices.

By the above lemma, G has a vertex v s.t. $\deg(v) \le 5$.

By the inductive hypothesis, $G-v$ is 6-colorable.

We can extend this coloring of $G-v$ to a coloring of G, by giving v a color that is different from its neighbors.

Proof follows. ☒

ANY PLANAR GRAPH IS 5-COLORABLE
<< FIVE-COLOR THEOREM >>

1. Any planar graph is 5-colorable.

Proof. Argument of 6-color theorem still works if $\deg(v) \le 4$.

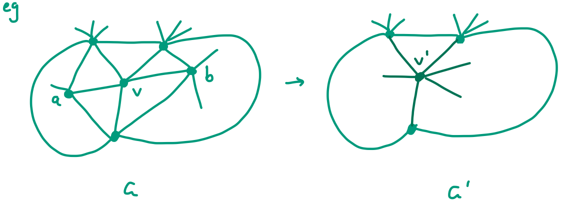
If $\deg(v) = 5$, v has 2 neighbors $a, b \in N(v)$ which are non-adjacent.

(Otherwise, the subgraph induced by $N(v)$ is K_5. But G is planar, so it can't have K_5 as a subgraph).

Let G' be the graph obtained by contracting va & vb.

Call the new vertex v'.

eg



G' is still planar, and has fewer vertices than G.

By our inductive hypothesis, G' is 5-colorable.

Use this coloring to get a coloring of $G-v$.

Every vertex $x \in V \setminus \{a,b\}$ gets the same color as in coloring of G'.

But a & b both get the color of v'.

(Because a & b are non-adjacent in G).

This uses at most 4 colors amongst the neighbors of v,

So we can extend this to a 5-coloring of G. ☒

ANY PLANAR GRAPH IS 4-COLORABLE
<< FOUR-COLOR THEOREM >>

1. Any planar graph is 4-colorable.

Proof. Details beyond the scope of the course.

2. It was first conjectured in ~1852.
3. First actual proof: ~1976 (Appel & Hacken)
4. This was turned into an efficient algorithm in ~1996 (Roberson, Sanders, Seymour, Thomas).
5. Idea: same as the 5-color theorem, except that we look for configurations that are more complicated than

$\mid \quad \vee \quad Y \quad + \quad \ast$.

6. In particular, we identify a list of unavoidable configurations using "discharging", such that colorings can be extended.

- but this a long list!

THE CHROMATIC POLYNOMIAL [OF A GRAPH]: $\chi_G(t)$

Let G be a graph with p vertices.
There exists a polynomial $\chi_G(t)$ with integer coefficients and $\deg(\chi_G(t)) \leq p$ such that

$$\boxed{\#\text{ of colorings of } G = \chi_G(k)}$$

eg For K_p,

$$\chi_{K_p}(t) = t(t-1)\cdots(t-p+1) = t^{\underline{p}}.$$

Why? - there are k ways to color the first vertex;
- there are $k-1$ ways to color the second vertex
- etc.

$\therefore$ # of k-colorings is $k(k-1)(k-2)\cdots(k-p+1) = \chi_G(k)$.

Proof. We proceed by induction on the # of edges.
If G has no edges, then every function $V(G) \to [k]$ is a k-coloring.
$\therefore \chi_G(t) = t^p$ works.

Now, fix G and assume the result is true for graphs with fewer edges.
Let $e = xy$ be an edge.

- every coloring of G is also a coloring of G.
- A k-coloring of $G-e$ is a k-coloring of G iff x and y have different colors.
- A k-coloring of $G-e$ in which x & y have the same colors is equivalent to a coloring of G/e.

Thus,

$$\#\text{ of } k\text{-colorings of } G = (\#\text{ of } k\text{-colorings of } G-e) - (\#\text{ of } k\text{-colorings of } G/e).$$

By our inductive hypothesis, this is

$$\#\text{ of } k\text{-colorings of } G = \chi_{G-e}(k) - \chi_{G/e}(k),$$

where $\chi_{G-e}(t)$ & $\chi_{G/e}(t)$ are polynomials of degree $\leq p$ with integer coefficients.

$\therefore$ We define

$$\boxed{\chi_G(t) = \chi_{G-e}(t) - \chi_{G/e}(t).}$$

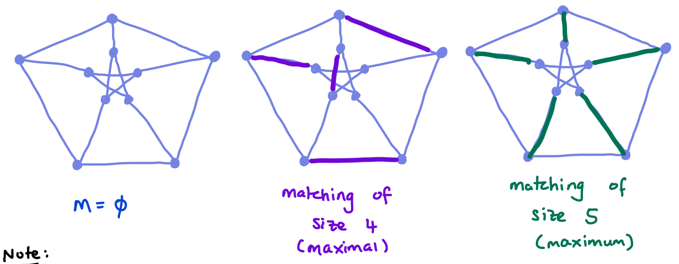
and this works. ☒

Note that calculating $\chi_G(t)$ is exponential.

MATCHINGS

1. A "matching" M in a graph G is a subset of $\underline{\underline{E(G)}}$ s.t. every vertex of G is incident with at most 1 edge of M.

eg



2. Note: The empty set ϕ is always a matching of G.

MAXIMUM / MAX MATCHING

1. A "maximum" matching is a matching with the maximum possible # of edges amongst all matchings of G.

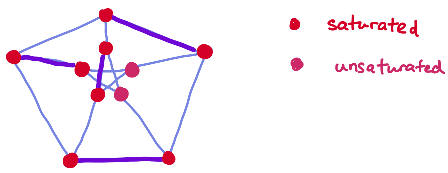
(Compared to maximal: matchings s.t we "cannot add any more edges" and let it still be a matching.

* max = maximum.

SATURATED [VERTEX]

1. A vertex $v \in V(G)$ is "saturated" by a matching M if v is incident with some edge in M.

eg



2. See that
   ① # of saturated vertices $= 2|M|$; &
   ② # of unsaturated vertices $: p - 2|M|$.

PERFECT MATCHING

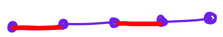
1. A "perfect matching" is a matching with $\frac{p}{2}$ vertices, where $p = |V(G)|$.
2. In a perfect matching, every vertex is saturated.
3. A perfect matching is automatically a maximum matching.
4. However, some graphs have no perfect matching.

eg

ALTERNATING PATH

1. Given a graph G and matching M, an "alternating path" is a path in which the edges are alternately in M / not in M.

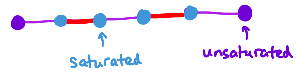
eg



AUGMENTING PATH

1. An "augmenting path" is an alternating path from u to v, where u & v are both unsaturated.

eg



M IS NOT A MAX MATCHING $\Leftrightarrow$ ∃ AN AUGMENTING PATH

1. Given an augmenting path P, we can replace

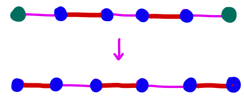
$$\boxed{M \leftarrow M \,\Delta\, E(P),}$$

where

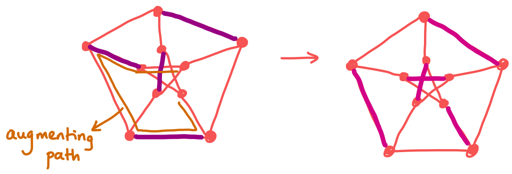
$$\boxed{A \,\Delta\, B = (A \setminus B) \cup (B \setminus A).}$$

2. This will be a strictly larger matching.

eg



eg The Petersen Graph:



3. If M is not a max matching, then an augmenting path exists.

FINDING AN AUGMENTING PATH

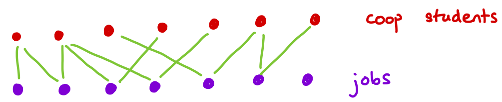
1. In this course, we focus on the bipartite case.

why?
① Easier;
② More important for applications; &
③ Stronger results.

EXAMPLE : JOB APPLICATIONS

1. Problem:

" Say we have a graph



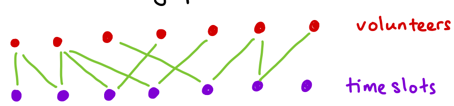
▨ = coop student is qualified.

Then matching the most students to jobs $\Leftrightarrow$ finding a maximum matching.

EXAMPLE : SCHEDULING

1. Problem:

" Say we have a graph



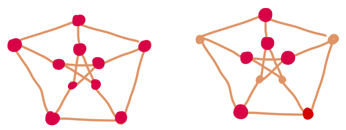
▨ = volunteer is available

Then filling the most timeslots $\Leftrightarrow$ finding a maximum matching.

COVER [OF A GRAPH]

A "cover" C in a graph G is a subset of $\underline{\underline{V(G)}}$ s.t. every edge of G is incident with at least one vertex in C.

eg



$C = V(G)$

Note: $C = V(G)$ is always a cover of G.

We are interested in minimum covers.

(but this is NP-hard in general.)

M IS A MATCHING, C IS A COVER IN G ⇒ $|M| \leq |C|$

If M is a matching and C is a cover in the same graph, then $|M| \leq |C|$.

(where the graph is bipartite.)

Proof. Let $M = \{e_1, \ldots, e_k\}$, $e_i = u_i v_j$.

For each i, we must have $u_i \in C$ or $v_i \in C$ (or both).

$\therefore \{u_1, \ldots, u_k\} \subseteq C$.

But since M is a matching, $u_i \neq u_j$ for $i \neq j$

$\therefore |C| \geq k = |M|$. $\boxed{}$

$|M| = |C|$ ⇒ M IS MAXIMUM, C IS MINIMUM

In particular, if $|M| = |C|$, then M is a maximal matching and C is a minimum cover.

Proof. Let M' be some matching, and C' be some cover.

Then

$|M'| \leq |C| = |M|$ & $|C'| \geq |M| = |C|$. $\boxed{}$

G IS BIPARTITE ⇒ ∃ MATCHING, COVER OF THE SAME SIZE

<< KÖNIG'S THEOREM >>

In a bipartite graph, there exists a matching M and a cover C of the same size.

Idea. Start with a matching M.
We will search for an augmenting path, and in the process of doing so, build a cover.
Either the augmenting path exists (so matching is not maximal, so start again), or $|C| = |M|$.

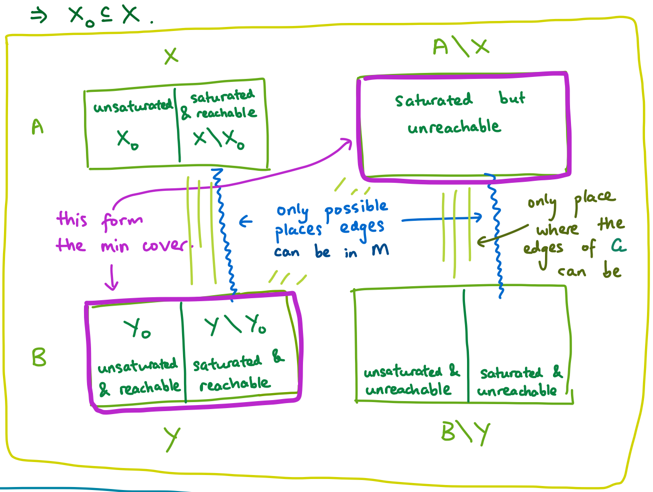
Proof. Part #1: Searching for augmenting paths.

Let M be a matching in G, and (A, B) be a bipartition.
Note that an augmenting path has one end in A and the other in B (since it has odd length).
WLOG, we can start searching from an unsaturated vertex in A.
Let

X_0 = set of unsaturated vertices in A;
X = set of reachable vertices in A;
Y = set of reachable vertices in B; &
Y_0 = set of unsaturated vertices in Y.

Define a vertex u to be "reachable" if there exists an alternating path from some vertex in X_0 to u.
Denote this path $P(u)$.

Note: an alternating path exists $\iff Y_0 \neq \emptyset$.
(If $u \in Y_0$, then $P(u)$ is an augmenting path.
If P is an augmenting path, one end is in X_0 & the other is in Y_0.)
Also note every vertex in X_0 is reachable since we can just consider paths of length 0.
$\Rightarrow X_0 \subseteq X$.



Lemma #1: If $u \in X$ & $e = uv \in E(G)$, then $v \in Y$.

Proof. Case #1: $e \in M$.
Then e must be the last edge in $P(u)$.
Why? → because $P(u)$ has even length (as $a \in A$), and the first step is not in M.
Thus, the last edge is in M (by alternation).
Since M is a matching, there is only one edge in M incident with u.
e is such an edge, so this edge is e.
$\therefore P(u) = P(v) e u$.
In particular v is reachable.

Case #2: $e \notin M$.
Consider $P(u)$. If $v \in P(u)$, then v is reachable, so we're done.
Otherwise, $P(u) e v$ is an alternating path so that v is reachable, and again we're done. ∎

Lemma #2: If $v \in Y$, & $e = xy \in M$, then $u \in X$.

Proof. Similar to Lemma #1.
Consider $P(v)$. Either $u \in P(v)$, so u is reachable,
or $u \notin P(v)$, so $P(v) e u$ is alternating, so u is reachable. ∎

Part #2: Building a cover.

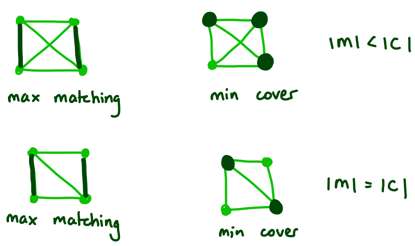
Lemma #3: Let $C = Y \cup (A \setminus X)$. Then C is a cover.
Moreover, $|C| = |M| + |Y_0|$.

Proof. By Lemma #1, every edge in G has one end in $A \setminus X$ or one end in Y.
So C is a cover.
By Lemma #2, every edge in M is incident with a vertex in $A \setminus X$ or a vertex in $Y \setminus Y_0$, but not both.
$\therefore |M| = |A \setminus X| + |Y \setminus Y_0|$
$= |C| - |Y_0|$,
which suffices to prove the claim. ∎

Main proof.
Suppose M is a maximum matching.
Construct X, Y, etc. as before.
Since M is a max matching, no augmenting path exists.
$\therefore Y_0 = \emptyset$, and so
$|C| = |M|$,
as needed. ∎

This may or may not be the case for non-bipartite graphs.

eg



BIPARTITE MATCHING ALGORITHM

1. To turn our proof of Konig's Theorem into an algorithm for max matchings/min covers, we need a systematic way to compute the sets X & Y.

2. **Idea**: Modify the BFST algorithm.

① Instead of starting with a single root, every vertex in X_0 is a root.

② If v is the active vertex:
- if $v \in A$ (even level), only add edges not in M.
- if $v \in B$ (odd level), only add edges in M.

③ In the end,

$$\boxed{\begin{array}{l} - X = \text{vertices of even level;} \\ - Y = \text{vertices of odd level;} \\ - X_0 = \text{vertices of level } 0; \\ - Y_0 = \text{vertices of odd level with no children (ie leaves)} \end{array}}$$

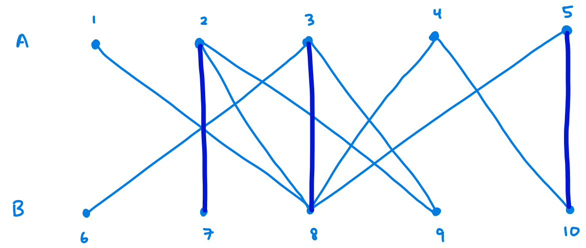
④ If $Y_0 = \emptyset$, we have a maximum matching. &

$C = Y \cup (A \setminus X)$

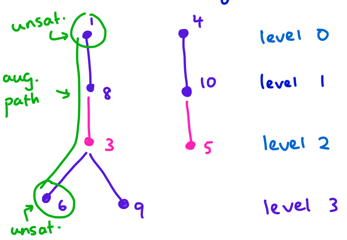
is a min cover; &

⑤ If $Y \neq \emptyset$, then a path from a vertex in Y_0 to its root is an augmenting path.

eg



Start with the matching shown.
WLOG, let A be the set of the top nodes, & B the bottom nodes.
$X_0 = \{1, 4\}$.
Run the BFST algorithm on X_0:



→ can only move along edges $\notin M$

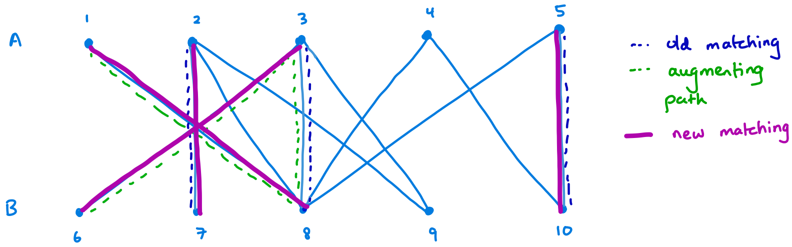
→ can only move along edges $\in M$

⋮

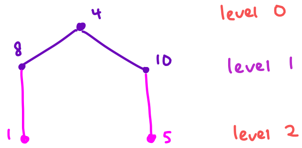
Queue: ~~1~~ ~~4~~ ~~8~~ ~~10~~ ~~3~~ ~~5~~ ~~6~~ ~~9~~

Then
① $X = \{1, 3, 4, 5\}$;
② $Y = \{6, 8, 9, 10\}$;
③ $Y_0 = \{6, 9\}$.

Then $\{1, 8, 3, 6\}$ is an augmenting path. We can then use this to obtain the new matching



We run our algorithm again.
$X_0 = \{4\}$.



$\Rightarrow X = \{1, 4, 5\}$
$Y = \{8, 10\}$
$Y_0 = \emptyset$.

Queue: 4 8 10 1 5

Since $Y_0 = \emptyset$, this is a maximum matching.
So

$$C = Y \cup (A \setminus X)$$
$$= \{8, 10, 2, 3\}$$

is a min cover.

G IS BIPARTITE; ∃ MATCHING OF SIZE |A| ⟺ $|N(D)| \geq |D|$ $\forall D \subseteq A$

<< HALL'S [MARRIAGE] THEOREM >>

Let G be a bipartite graph with bipartition (A, B).
For $D \subseteq A$, define

$$\boxed{N(D) = \bigcup_{v \in D} N(v).}$$

Then there exists a matching of size $|A|$ iff $|N(D)| \geq |D|$ for all subsets $D \subseteq A$.

Note: if a matching of size $|A|$ exists, then it is a maximum matching.

Also, if $|B| < |A|$, then taking $D = A$, we see that

$N(D) = N(A) \subseteq B$,

and so $|N(D)| < |D|$.

Thus, the theorem states no matching of size $|A|$ exists.

ie we must have $|A| \leq |B|$ for such a matching to exist.

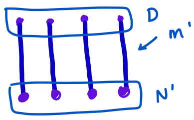
Proof. ($\Rightarrow$) Suppose M is a matching of size $|A|$.

Then every vertex of A is saturated.

Let $D \subseteq A$, and let

$M' = \{e \in M \mid e$ incident with a vertex in $D\}$

$N' = \{v \in B \mid v$ incident with an edge in $M'\}$.



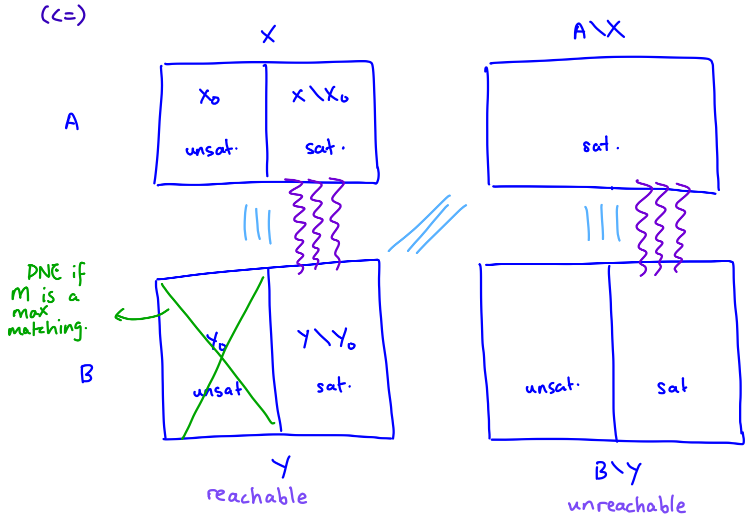
Then $|D| = |M'| = |N'|$.

But since $N' \subseteq N(D)$, it follows that

$|D| \leq |N(D)|$,

as needed. #

($\Leftarrow$)



Suppose that G does not have a matching of size $|A|$.

Let M be a max matching. Then $|M| < |A|$ by assumption.

Consider X, Y, etc. from Konig's Theorem.

Since M is a max matching, $Y_0 = \emptyset$.

Then $|M| < |A| \Rightarrow X_0 \neq \emptyset$. (ie $\exists$ an unsaturated vertex).

$\therefore |X| = |X_0| + |X \setminus X_0|$

$> |X \setminus X_0|$

$= |Y \setminus Y_0|$ (matching gives a bijection)

$= |Y|$.

But also notice $N(X) \subseteq Y$. Otherwise, we would have edges that go from $X \to B \setminus Y$, which by the diagram cannot occur.

So

$|X| > |N(X)|$.

Thus

$\exists D \subseteq A$ s.t. $|N(D)| < |D|$, as required. ☒

EXAMPLE 1: RANKS OF PILES IN A DECK OF CARDS

Problem:

> You have a deck of 52 cards.
> Deal the cards as evenly as possible into k piles, where $k \leq 13$.
> Then, it is always possible to choose 1 card from each pile, with no two same rank.
> (rank = A, 2, 3, …, K.)

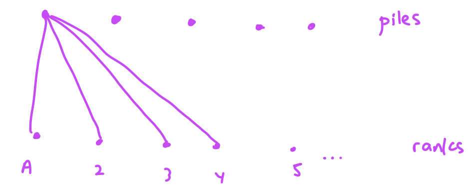
Proof. Use Hall's Thm.

Let G be a graph with bipartition (A, B):

A = set of piles

$B = \{A, 2, \ldots, K\}$,

and we have an edge pr, where $p \in A$, $r \in B$, iff there exists a card of rank r in pile p.



Let $D \subseteq A$.

Consider cards in piles from D.

Each pile has ≥ 4 cards (since $k \leq 13$).

Thus, there are at least $4|D|$ cards in these piles, combined.

$\therefore$ There are only 4 cards of each rank in the deck;

$\therefore$ There are at least $|D|$ ranks represented in these piles.

In particular, we've proved $|D| \leq |N(D)|$.

By Hall's Theorem, G has a matching of size $|A|$, which is exactly the solution to our problem. #

$k \geq 1 \Rightarrow$ EVERY k-REGULAR BIPARTITE GRAPH HAS A PERFECT MATCHING

If $k \geq 1$, then every k-regular bipartite graph has a perfect matching.

Proof. Let G be a k-regular graph with bipartition (A, B).

Let $D \subseteq A$. Then

$$\underset{\substack{\| \\ k|D|}}{\text{\# edges incident with a vertex in } D} \leq \underset{\substack{\| \\ k|N(D)|}}{\text{\# edges incident with a vertex in } N(D)}.$$

since the graph is k-regular.

Thus $|D| \leq |N(D)|$.

So, by Hall's Theorem, there exists a matching of size $|A|$.

We can apply the same reasoning, switching roles of A & B.

$\therefore$ There exists a matching of size $|B|$.

Thus $\underbrace{|A| = |B|}$ and these are perfect matchings. ☒

can also be shown by edge-counting arguments.

CONVEX HULL : conv(S)

Let $S \subseteq \mathbb{R}^m$, say $S = \{s_1, \dots, s_k\}$.

Then the "convex hull" of S is

$$\boxed{\text{conv}(S) = \{a_1 s_1 + \dots + a_k s_k \mid a_i \geq 0 \ \forall i, \ \sum_{i=1}^{k} a_i = 1\}.}$$

eg

DOUBLY STOCHASTIC MATRIX

A "doubly stochastic matrix" A is a $n \times n$ matrix such that

① $A_{ij} \geq 0 \quad \forall i, j$;

② $A\underline{1} = \underline{1}$, $\underline{1}$ is the "all-ones" vector;

(ie sum of each row = 1)

③ $A^T \underline{1} = \underline{1}$.

(ie sum of each column = 1).

DSM_n

We write

$$\boxed{DSM_n = \text{set of all } n \times n \text{ doubly stochastic matrices.}}$$

PERMUTATION MATRIX

A "permutation matrix" P is an $n \times n$ matrix such that

① Every entry is 0 or 1;

② There is exactly one 1 in each row; &

③ There is exactly one 1 in each column.

PM_n

We write

$$\boxed{PM_n = \text{set of all permutation matrices.}}$$

$DSM_n = \text{conv}(PM_n)$

<< BIRKHOFF - VON NEUMANN THEOREM >>

We claim $DSM_n \subseteq \text{conv}(PM_n)$.

Proof. Easy: $\text{conv}(PM_n) \subseteq DSM_n$.

Hard: Show that if $A \in DSM_n \Rightarrow A \in \text{conv}(PM_n)$.

Form a bipartite graph G with bipartition (A, B).

Let

$A = \{R_1, \dots, R_n\}$

$B = \{C_1, \dots, C_n\}$

where $R_i = i^{th}$ row of A, & $C_j = j^{th}$ column of A.

Then let

edge joining $R_i C_j \iff A_{ij} \neq 0$.

Claim: G has a perfect matching.

Let $D \subseteq A$. Then

$$\sum_{\substack{i,j \text{ s.t.} \\ R_iC_j \in E(G), \\ R_i \in D}} A_{ij} \leq \sum_{\substack{i,j \text{ s.t.} \\ R_iC_j \in E(G) \\ C_j \in N(D)}} A_{ij}$$

this set $\subseteq$ is a subset of this set!

Then

$$\sum_{\substack{i,j \text{ s.t.} \\ R_iC_j \in E(G), \\ R_i \in D}} A_{ij} = \sum_{\substack{i,j, \\ R_i \in D}} A_{ij} = |D|$$

&

$$\sum_{\substack{i,j \text{ s.t.} \\ R_iC_j \in E(G), \\ C_j \in N(D)}} A_{ij} = |N(D)|.$$

So, G has a perfect matching.

To finish the argument, induct on the # of non-zero entries in A.

Base case: $A \in PM_n$. (✓)

Inductive step: Find a perfect matching $M = \{R_{i_1}C_{j_1}, R_{i_2}C_{j_2}, \dots, R_{i_n}C_{j_n}\}$.

Let P = permutation matrix with 1s at $(i_1, j_1), (i_2, j_2), \dots$.

Let $q = \min\{A_{ij} \mid R_iC_j \in M\}$.

Then

$$\frac{A - qP}{1 - q}$$

is a DSM with fewer non-zero entries. ∎