Pure Mathematics 3

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Chapter 1: Modulus Functions

→ notation :

→ the modulus of $f(x)$ is denoted by $|f(x)|$.

definition:

$$|f(x)| = \begin{cases} f(x) & \text{if } f(x) \geq 0 \\ -f(x) & \text{if } f(x) \leq 0 \end{cases}$$

$|3| = 3 \quad (\because > 0)$

$|-3| = -(-3) = 3. \quad (\because -3 < 0)$

eg¹ $|x| = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{otherwise} \end{cases}$

eg² $|2x-1| = \begin{cases} 2x-1 & \text{if } x \geq \frac{1}{2} \ (2x-1 \geq 0) \\ 1-2x & \text{otherwise} \end{cases}$

eg³ $|3x+2| = \begin{cases} 3x+2 & \text{if } x \geq \frac{-2}{3} \\ -3x-2 & \text{otherwise} \end{cases}$

To solve inequalities involving modulus functions.

→ important results

① $|f(x)| \leq a$, where $a \in \mathbb{R}^+$

$\Rightarrow -a \leq f(x) \leq a.$

Proof

$|f(x)| = \begin{cases} f(x), & f(x) \geq 0 \\ -f(x), & f(x) < 0 \end{cases}$

$\therefore |f(x)| \leq a \Rightarrow f(x) \leq a$ or $-f(x) \leq a$

$\therefore f(x) \geq -a$

hence $-a \leq f(x) \leq a.$

eg¹ solve $|2x-1| < 5$ ← find the set of values of x which satisfy this.

$\Rightarrow -5 < 2x-1 < 5$

$-4 < 2x < 6$

$-2 < x < 3.$

eg² $|2^x - 5| < 3$

$\Rightarrow -3 < 2^x - 5 < 3$

$\therefore 2 < 2^x < 8$

$(\log_2 \cdot)$ $1 < x < 3.$

② $|f(x)| \geq a$

$\Rightarrow f(x) \geq a$ or $f(x) \leq -a.$

Proof

$|f(x)| = \begin{cases} f(x), & f(x) \geq 0 \\ -f(x), & f(x) < 0 \end{cases}$

$|f(x)| \geq a$

$\Rightarrow f(x) \geq a$ or $-f(x) \geq a$

$\Rightarrow f(x) \leq -a.$ QED

eg¹ Solve $|3x-2| \geq 7$

$\therefore 3x-2 \geq 7$ or $3x-2 \leq -7$

$x \geq 3, \quad x \leq \frac{-5}{3}$

④ $|f(x)|^2 = [f(x)]^2$

eg¹ Solve $|ax+b| \leq c|px+q|$, } $c > 0$

eg² Solve $|ax+b| \geq c|px+q|$

Method 1: square both sides.

⇒ we can only square both sides if both sides are positive.

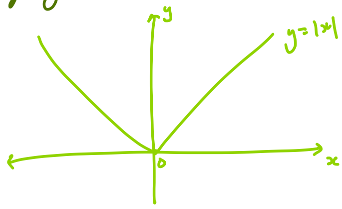
Sketching modulus graphs

→ linear functions.

eg¹ sketch the graph of $y = |x|$

→ By defⁿ,

$y = |x| \Rightarrow y = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0 \end{cases}$



⇒ the graph consists of 2 line segments:

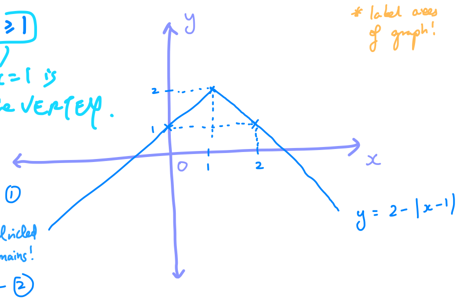
$y = x, \quad x \geq 0$

& $y = -x, \quad x < 0$

eg² sketch the graph of $y = 2 - |x-1|$

* label axes of graph!

$|x-1| = \begin{cases} x-1 & \text{if } x-1 \geq 0 \text{ ie } \boxed{x \geq 1} \\ -(x-1) & \text{if } \boxed{x < 1} \end{cases}$ — $x = 1$ is the VERTEX.



$\therefore y = 2 - |x-1|$

$\Leftrightarrow y = \begin{cases} 2-(x-1) & \text{if } x \geq 1 \quad -① \\ \quad \searrow 3-x \\ 2-(-(x-1)) & \text{if } x < 1 \quad -② \\ \quad \searrow x+1 \end{cases}$ restricted domains!

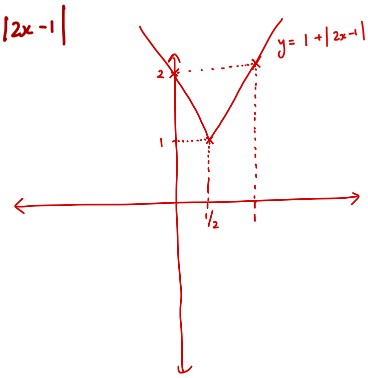
① $x = 1, \quad y = 3$

$x = 2, \quad y = 1$

eg³ sketch the graph of $y = 1 + |2x-1|$

$|2x-1| = \begin{cases} 2x-1 & \text{if } x \geq \frac{1}{2} \\ 1-2x & \text{otherwise} \end{cases}$

$\therefore 1 + |2x-1| = \begin{cases} 2x & \text{if } x \geq \frac{1}{2} \\ 2-2x & \text{if } x < \frac{1}{2} \end{cases}$



③ $|f(x) - g(x)| = |g(x) - f(x)|$

Proof

$|f(x) - g(x)| = \begin{cases} f(x) - g(x) \\ -(f(x) - g(x)) = g(x) - f(x). \end{cases}$

$|g(x) - f(x)| = \begin{cases} g(x) - f(x) \\ -[g(x) - f(x)] = f(x) - g(x) \end{cases}$

these are the same.

eg¹ solve $|1-3x| \leq 5$

$-5 \leq 1-3x \leq 5$

$-6 \leq -3x \leq 4$

$\therefore 2 \geq x \geq \frac{-4}{3}$ * must swap ineq!

alternative.

$|1-3x| = |3x-1| \ (\leq 5)$

$\therefore -5 \leq 3x-1 \leq 5$

$-4 \leq 3x \leq 6$

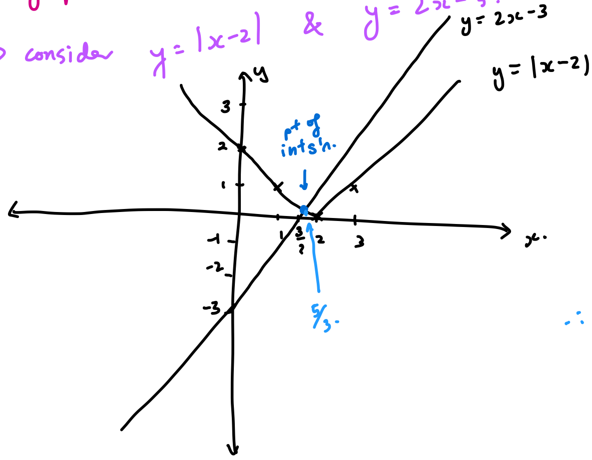
$\frac{-4}{3} \leq x \leq 2$

⑤ Solving $|ax+b| > px+q$ or $|ax+b| < px+q$.

→ CANNOT square both sides!

→ graphical method. eg: solve $|x-2|$ $2x-3$.

⇒ consider $y = |x-2|$ & $y = 2x-3$.



solve $|x-2| > 2x-3$.

pt of intersection:

$y = -(x-2)$, $y = 2x-3$.

$\downarrow$

$2 - x = 2x - 3$

$5 = 3x$ $\therefore x = \frac{5}{3}$.

$\therefore |x-2| > 2x-3$

$\Rightarrow x < \frac{5}{3}$.

Chapter 2: Remainder Theorems

Factor theorem

two methods of dividing polynomials:

① Synthetic division.

eg $\dfrac{x^3 - x^2 + x + 14}{x+2}$

$$\begin{array}{r|rrr|r} -2 & 1 & -1 & 1 & 14 \\ (+) & 0 & -2 & 6 & -14 \\ \hline & 1 & -3 & 7 & 0. \end{array}$$

$x^2 - 3x + 7$. remainder.

② By inspection.

$x^3 - x^2 + x + 14 = (x+2)(x^2 - 3x + 7)$

$+2x^2$ $-3x^2$

To factorise quartic polynomials, & to solve quartic eqns.

$\Rightarrow p(x) = ax^4 + bx^3 + cx^2 + dx + e$.

case ① : $p(x)$ is a product of 4 linear factors. $p(x) = (\quad)(\quad)(\quad)(\quad)$

case ② : $p(x)$ is a product of 2 linear factors & 1 quadratic factor. $p(x) = (rx+t)(gx+h)(Ax^2+Bx+C)$

case ③ : $p(x)$ is a product of 2 quadratic factors. $p(x) = (Ax^2+Bx+C)(Dx^2+Ex+F)$

consider a polynomial w/ a repeated linear factor.

$p(x) = (ax+b)^2 q(x)$.

$\therefore p'(x) = (ax+b)^2 q'(x) + 2(ax+b)(a)q(x)$

if $x = -\frac{b}{a}$, $p(-\frac{b}{a}) = 0$.

$\therefore p'(-\frac{b}{a}) = 0$.

Hence,
If $p(-\frac{b}{a}) = p'(-\frac{b}{a}) = 0$,
then $(ax+b)$ is a repeated factor of $p(x)$.

Chapter 3: Binomial Expansion

For $|x| < 1$,

$$(1+x)^n = 1 + nx + \frac{n(n-1)}{2!}x^2 \dots$$

Partial fractions

uses:
- expansion
- integration
- differentiation
- further maths

"rational function": quotient of two polynomials.

eg $f(x) =$ ① $\frac{x+1}{x(x-2)}$, ② $\frac{x^2+x-1}{(x-1)(x+2)(x-3)}$, ③ $\frac{2x^2-1}{(x+3)(2x-1)}$

"partial fractions": how? → express f in partial fractions.

$$\frac{2}{x+3} + \frac{3}{2x-1} = \frac{7x+7}{(x+3)(2x-1)}.$$

partial fractions easy.

Rules

#1: numerator must be at least one degree less than the denominator.

#2: corresponding to any linear factor $ax+b$ in the den. of a rational func, there exists $\frac{A}{ax+b}$.

#3: likewise, if $(ax+b)$ is repeated twice, there is pf $\frac{A}{ax+b}$ & $\frac{B}{(ax+b)^2}$.

#4: for a quadratic factor x^2+c^2, the pf is $\frac{Ax+B}{x^2+c^2}$.

eg1 $\frac{7x+7}{(x+3)(2x-1)} = \frac{A}{x+3} + \frac{B}{2x-1}$

eg2 $\frac{x^2+1}{x(x+2)(2x+1)} = \frac{A}{x} + \frac{B}{x+2} + \frac{C}{2x+1}$

eg3 $\frac{x+5}{(3x-1)(x+1)^2} \equiv \frac{A}{3x-1} \boxed{+ \frac{B}{x+1} + \frac{C}{(x+1)^2}}$

eg4 $\frac{x-4}{(2x+1)(x-1)^2} = \frac{A}{2x+1} + \frac{B}{x-1} + \frac{C}{(x-1)^2}$

$$= \frac{-2}{2x+1} + \frac{B}{x-1} \; \frac{-1}{(x-1)^2}$$

$$x-4 = -2(x-1)^2 + B(2x+1)(x-1) - (2x+1).$$

let $x=0$:

$$-4 = -2(-1)^2 + B(1)(-1) - (1)$$

$$-B = -1 \quad \therefore B = 1.$$

$$\therefore \frac{x-4}{(2x+1)(x-1)^2} = \frac{-2}{2x+1} + \frac{1}{x-1} - \frac{1}{(x-1)^2}$$

eg1 $\frac{4}{(x-3)(x+1)} = \frac{A}{x-3} + \frac{B}{x+1}$

method ①: comparison of x coeff & x^0 coeff — DON'T DO THIS

method ②: $\frac{4}{(x-3)(x+1)} \equiv \frac{A}{x-3} + \frac{B}{x+1}$

$$4 \equiv A(x+1) + B(x-3)$$

if $x-3=0$, $x=3$

⇒ when $x=3$, $4 = A(4)$ $\therefore A = 1$.

⇒ when $x=-1$

⇒ $4 = B(-4)$ $\therefore B = -1$

(*) method ③: SHORTCUT. "cover-up method"

→ strictly for linear factors only.

$$\frac{4}{(x-3)(x+1)} \equiv \frac{A}{x-3} + \frac{B}{x+1}$$

① Cover up factor under the partial fraction.

② Subst value of x that makes that factor 0.

$$\frac{4}{\cancel{(x-3)}(x+1)} \equiv \frac{1}{x-3} + \frac{-1}{x+1}$$

eg2 $\frac{2-x+8x^2}{(1-x)(1+2x)(2+x)} = \frac{A}{1-x} + \frac{B}{2x+1} + \frac{C}{x+2}$

$$= \frac{1}{1-x} + \frac{2}{2x+1} - \frac{4}{x+2}.$$

eg3 $\frac{2x^2+1}{x(x-1)^2} = \frac{A}{x} + \frac{B}{x-1} + \frac{C}{(x-1)^2}$

$$= \frac{1}{x} + \frac{B}{x-1} + \frac{3}{(x-1)^2}$$

$$2x^2+1 = (x-1)^2 + Bx(x-1) + 3x$$

let $x=2$

$$9 = 1 + B(2)(1) + 3(2)$$

$$\therefore B = 1$$

$$\therefore \frac{2x^2+1}{x(x-1)^2} = \frac{1}{x} + \frac{1}{x-1} + \frac{3}{(x-1)^2}$$

eg 5 $\dfrac{5x}{(x+2)(x^2+1)} = \dfrac{A}{x+2} + \dfrac{Bx+C}{x^2+1}$

$= \dfrac{-2}{x+2} + \dfrac{Bx+C}{x^2+1}$

$5x = -2(x^2+1) + (Bx+C)(x+2)$

if $x=0$, $0 = -2(1) + C(2)$ $\therefore C = 1.$

if $x=1$, $5 = -2(2) + (B+1)(3)$

$\therefore B = 2.$

$\therefore \dfrac{5x}{(x+2)(x^2+1)} = \dfrac{-2}{x+2} + \dfrac{2x+1}{x^2+1}.$

* $(1-f(x))^n$ is a GP

$a=1$, $r = f(x)$.

$\Rightarrow 1 + f(x) + [f(x)]^2 + [f(x)]^3 \dots$

eg 6 $\dfrac{3x+8}{(2x+1)(x^2+3)} = \dfrac{A}{2x+1} + \dfrac{Bx+C}{x^2+3}.$

$= \dfrac{2}{2x+1} + \dfrac{Bx+C}{x^2+3}.$

$\therefore 3x+8 = 2(x^2+3) + (Bx+C)(2x+1).$

$x=0 \Rightarrow 8 = 2(3) + C(1)$

$\therefore C = 2.$

$x=1 \Rightarrow 11 = 2(4) + (B+2)(3)$

$B = -1.$

$\therefore \dfrac{3x+8}{(2x+1)(x^2+3)} = \dfrac{2}{2x+1} + \dfrac{-x+2}{x^2+3}.$

eg 7 $\dfrac{x^2+3x+1}{(x+1)(x+2)}$

*since num of $\equiv$ deg of den, we must do long division first.

$$\begin{array}{r} 1 \\ x^2+3x+2 \,\big)\, x^2+3x+1 \\ (-)\ x^2+3x+2 \\ \hline -1 \end{array}$$

$\Rightarrow \dfrac{x^2+3x+1}{(x+1)(x+2)}$

$\therefore \dfrac{x^2+3x+1}{(x+1)(x+2)} = 1 + \dfrac{-1}{(x+1)(x+2)}$

$= 1 + \left[\dfrac{A}{x+1} + \dfrac{B}{x+2}\right]$

$= 1 + \left[\dfrac{-1}{x+1} + \dfrac{1}{x+2}\right]$

Chapter 4: Indices and Logs

a^x ← exponent; ↑ index

e (Euler's constant) $\simeq 2.718$

Properties

1) $a^n \equiv \underbrace{a \times a \cdots \times a}_{n}$
2) $a^0 \equiv 1$
3) $a^{-n} \equiv \frac{1}{a^n}$
4) $a^{\frac{1}{n}} \equiv \sqrt[n]{a}$.
5) $a^m a^n \equiv a^{mn}$
6) $\frac{a^m}{a^n} \equiv a^{m-n} = \frac{1}{a^{n-m}}$
7) $(a^m)^n \equiv a^{mn}$
8) $a^n b^n \equiv (ab)^n$
9) $\frac{a^n}{b^n} \equiv \left(\frac{a}{b}\right)^n$

$\boxed{\begin{matrix} ^*(a+b)^n \neq a^n+b^n \\ (a-b)^n \neq a^n-b^n. \end{matrix}}$

Exponential Eq^n s → Quadratics.

$Aa^{2x} + Ba^x + C = 0$

$u = a^x \Rightarrow Au^2 + Bu + C = 0.$

$u = \underline{\alpha}, \; u = \underline{\beta}$

$\therefore a^x = \underline{\alpha}, \; a^x = \underline{\beta}$

*we use substitution when three or more terms are present in the equation.

Log Functions

* if $a^x = y$, $\Rightarrow x = \log_a y$

Properties

1) $a^0 = 1$. $\therefore 0 = \log_a(1)$. for $a > 0, a \neq 1$
2) $a^1 = a$ $\therefore \log_a a = 1$.
3) $a^x \neq 0$ $\therefore$ $\log_a 0$ is undefined
4) $a^x \not< 0$ $\therefore$ $\log_a(-ve)$ is undefined
5) $1^x = 1 \; \forall x$ $\therefore$ $\log_1 a$ is undefined.

Laws

1) $\log_a x + \log_a y \equiv \log_a(xy)$

Proof: Let $n = \log_a x$, $m = \log_a y$

$\therefore x = a^n, \; y = a^m.$

$\therefore xy = a^{n+m}$

$\therefore \log_a(xy) = m + n$

$\log_a(xy) = \log_a x + \log_a y.$

2) $\log_a x - \log_a y \equiv \log_a\left(\frac{x}{y}\right)$
3) $\log_a x^n \equiv n \log_a x.$

Solving Eq^n s

① $a^x = b$

1) b can be expressed as a^n

$\therefore a^x = a^n \quad \therefore x = n.$

$T = \ln a \ln b$

$e^T = e^{(\ln a)(\ln b)}$

$= (e^{\ln a})^{\ln b} = (e^{\ln b})^{\ln a}$

$= a^{\ln b} = b^{\ln a}$

Deducing a non-linear relation to a linear relation.

1) $y = ab^x$ is non-linear (curve).

$\Rightarrow \ln y = \ln(ab^x)$

$\ln y = \ln a + \ln b^x$

$\therefore \ln y = (\ln b)x + \ln a$

2) $y = ax^b$, $b \neq 1$

$\ln y = \ln(ax^b)$

$\Rightarrow \ln y = \ln a + b \ln x$

$\therefore \ln y = b \ln x + \ln a.$

Exponential Eq^n s

⇒ unknown in exponent or base.

* if $a^x = a^y \Rightarrow x = y$

$2^x = 8$

$2^x = 2^3 \quad \therefore x = 3$

*other properties.

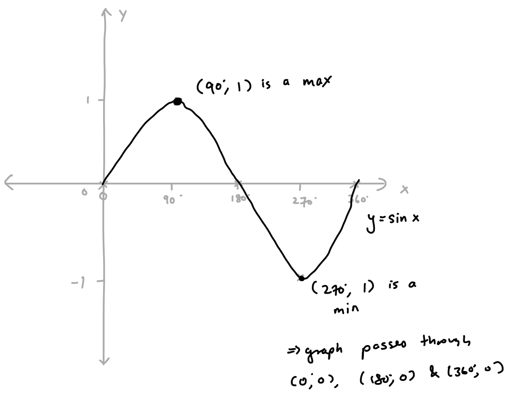
1) $y = e^x \Rightarrow \ln y = x$.
2) taking log of both sides.

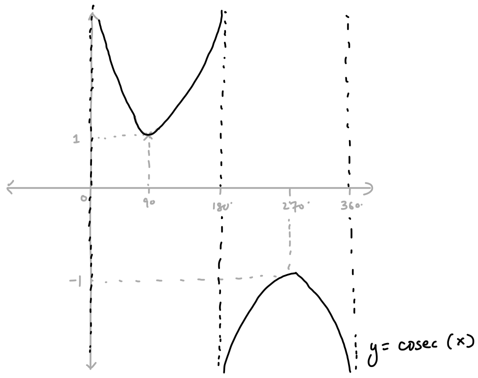
Chapter 5: Trigonometry

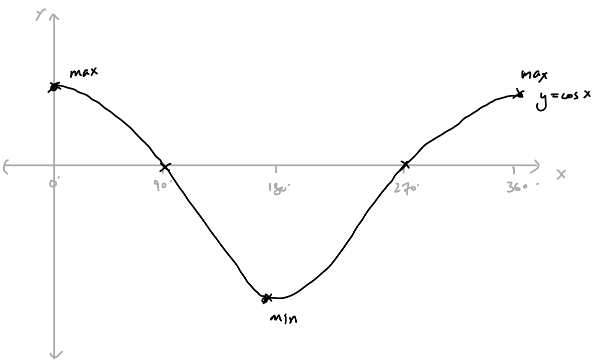
- $\sec x = \frac{1}{\cos x}$
- $\text{cosec } x = \frac{1}{\sin x}$
- $\cot x = \frac{1}{\tan x}$

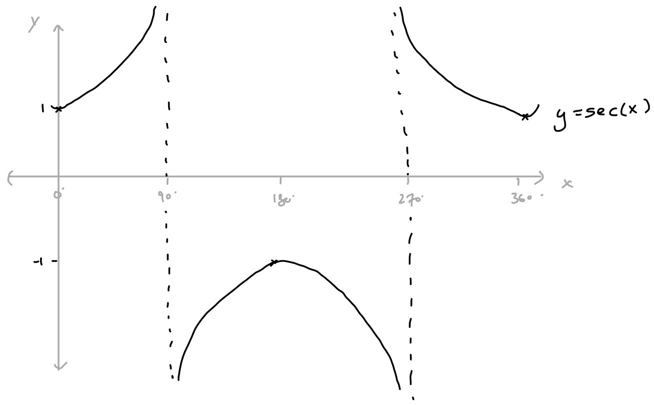
Sketch graphs of $y = \sec x$, $y = \text{cosec } x$, $y = \tan x$.

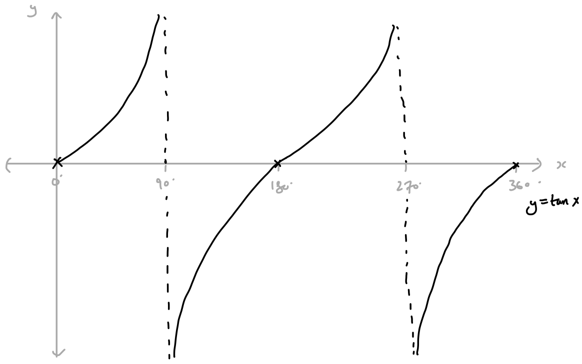
$y = f(x)$	$y = \frac{1}{f(x)}$
i) if (a, b) is a min pt; →	$(a, \frac{1}{b})$ is a max pt (given $b \neq 0$)
ii) if (a, b) is a max pt; →	$(a, \frac{1}{b})$ is a min pt (given $b \neq 0$)
iii) if $(a, 0)$ is a pt on the x-axis →	"$x = a$" is an asymptote. (FM) — a line that always touches the curve at ∞. → curve cannot cross over this line.
iv) $x = a$ is an asymptote →	the curve intersects the x-axis at $(a, 0)$.

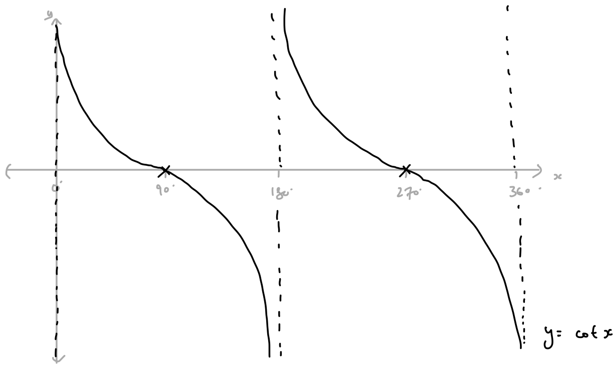


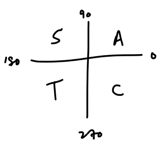












Identities

(P1) $\tan x \equiv \dfrac{\sin x}{\cos x}$

$\sin^2 x + \cos^2 x \equiv 1$

→ given in formula booklet.

① $\sin^2 x + \cos^2 x \equiv 1.$ $\searrow \div \sin^2 x$

$\div \cos^2 x$ $\quad \tan^2 x + 1 \equiv \dfrac{1}{\cos^2 x}$ $\qquad 1 + \cot^2 x \equiv \dfrac{1}{\sin^2 x}$

$\therefore \boxed{\tan^2 x + 1 \equiv \sec^2 x.}$ $\qquad \boxed{\therefore 1 + \cot^2 x = \operatorname{cosec}^2 x.}$

② Compound angle identities

i) $\sin(A+B) = \sin A \cos B + \cos A \sin B$

ii) $\cos(A+B) = \cos A \cos B - \sin A \sin B.$

→ hence, if we replace B by $-B$:

$\sin(A-B) \equiv \sin(A)\cos(-B) + \cos A \sin(-B).$

we know $\cos(-B) \equiv \cos B$ & $\sin(-B) \equiv -\sin(B).$

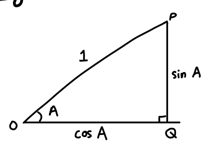
$\therefore$ iii) $\sin(A-B) \equiv \sin A \cos B - \cos A \sin B.$

→ likewise:

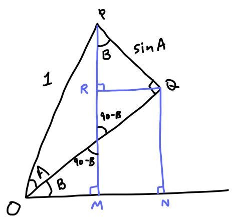
$\cos(A-B) \equiv \cos(A)\cos(-B) - \sin(A)\sin(-B)$

$\therefore$ iv) $\cos(A-B) \equiv \cos(A)\cos(B) + \sin(A)\sin(B)$

Proof



→ rotate Δ through an ∡ of B about O anticlockwise. →



$\cos QPR = \dfrac{PR}{PQ}$

$\therefore \cos B = \dfrac{PR}{\sin A}$

$\therefore PR = \sin A \cos B.$

$\sin QON = \dfrac{QN}{OQ}$

$\sin B = \dfrac{QN}{\cos A}$

$\therefore QN = \cos A \sin B.$

Let $\angle POQ = A$, and $OP = 1$.

then $OQ = \cos A$, $QP = \sin A$

$\sin(A+B) = \dfrac{PM}{OP}$

$= PM.$

$= PR + RM.$

$= PR + QN.$

$\sin(A+B) = \sin A \cos B + \cos A \sin B$

$\cos(A+B) = \dfrac{OM}{OP}$

$= OM$

$= ON - MN$

$= ON - RQ$ ⟶ $= \cos A \cos B - \sin A \sin B.$

$\sin B = \dfrac{RQ}{PQ}$

$= \dfrac{RQ}{\sin A}$

$\therefore RQ = \sin A \sin B.$

$\cos B = \dfrac{ON}{OQ}$

$= \dfrac{ON}{\cos A}$

$\therefore ON = \cos A \cos B$

Recap

$\sin(A+B) = \cos A \sin B + \sin A \cos B$

$\sin(A-B) =$

$\Rightarrow \tan(A+B) \equiv \dfrac{\sin(A+B)}{\cos(A+B)}$

$\tan(A+B). \equiv \dfrac{\sin A \cos B + \cos A \sin B}{\cos A \cos B - \sin A \sin B}$

Dividing every term on the RHS by $\cos A \cos B$:

v) $\tan(A+B) \equiv \dfrac{\tan A + \tan B}{1 - \tan A \tan B}$

$\therefore \Rightarrow \tan(A-B) \equiv \dfrac{\tan A + \tan(-B)}{1 - \tan A \tan(-B)}$

as $\tan(\theta) = -\tan\theta$:

vi) $\tan(A-B) \equiv \dfrac{\tan A - \tan B}{1 + \tan A \tan B}$

③ Double angle identities.

$\sin(A+A) \equiv \sin(2A) \equiv \cos(A)\sin(A) + \sin(A)\cos(A)$

$\therefore$ i) $\sin(2A) \equiv 2\sin(A)\cos(A)$.

$\cos(A+A) \equiv \cos(A)\cos(A) - \sin(A)\sin(A)$

$\equiv \cos^2(A) - \sin^2(A)$

$\equiv \cos^2(A) - (1-\cos^2(A)) \equiv (1-\sin^2 A) - \sin^2 A$

$\therefore$ ii) $\cos(2A) \equiv \cos^2(A) - \sin^2(A) \equiv 2\cos^2(A) - 1 \equiv 1 - 2\sin^2(A)$

$$\tan(A+A) \equiv \frac{\tan(A) + \tan(A)}{1 - \tan(A)\tan(A)}$$

$\therefore$ iii) $\tan(2A) \equiv \dfrac{2\tan(A)}{1-\tan^2(A)}$

Harmonic form

$a\sin x + b\cos x \equiv R\sin(x+\alpha)$

$a\sin x - b\cos x \equiv R\sin(x-\alpha)$

$a\cos x + b\sin x \equiv R\cos(x-\alpha)$

$a\cos x - b\sin x \equiv R\cos(x+\alpha)$

$\therefore R = \sqrt{a^2+b^2}$

$\alpha = \tan^{-1}\left(\frac{b}{a}\right)$

* 1st form sin → sin

" " cos → cos

sin: + → +, − → + cos + → −, − → +

* $a, b > 0$

* $R > 0$ $0 < \alpha < \frac{\pi}{2}$

or $0 < \alpha < 90°$

Consider:

$a\sin x + b\cos x \equiv R\sin(x+\alpha)$

$\equiv R[\sin x\cos\alpha + \cos x\sin\alpha]$

$a\sin x + b\cos x \equiv R\cos\alpha\sin x + R\cos x\sin\alpha$

compare coefficient of $\sin x$:

$R\cos\alpha = a$ — ①

compare coeff. of $\cos x$:

$R\sin\alpha = b$. — ②

$①^2 + ②^2 \Rightarrow R^2\cos^2\alpha + R^2\sin^2\alpha = a^2+b^2$

$\therefore R = \sqrt{a^2+b^2}$

$\frac{②}{①} \Rightarrow \tan\alpha = \frac{b}{a}$.

$\therefore \alpha = \tan^{-1}\left(\frac{b}{a}\right)$.

Chapter 6: Differentiation

From P1: $\frac{d}{dx}(x^n) = nx^{n-1}$

$\frac{d}{dx}(f(x))^n = n(f(x))^{n-1}\frac{d}{dx}f(x)$

→ $f(x)$ is a polynomial

P3.

$\frac{dy}{dx} = \lim_{\delta x \to 0} \frac{\delta y}{\delta x}$.

#1 $y = \log_{10} x$

$y + \delta y = \log_{10}(x + \delta x)$

$\therefore \delta y = \log_{10}(x+\delta x) - y$

$= \log_{10}(x+\delta x) - \log_{10}(x)$

$= \log_{10}\left(\frac{x+\delta x}{x}\right)$

$\delta y = \log_{10}\left(1 + \frac{\delta x}{x}\right)$

$\therefore \frac{\delta y}{\delta x} = \frac{1}{\delta x}\log_{10}\left(1+\frac{\delta x}{x}\right)$

$\lim_{\delta x \to 0}\frac{\delta y}{\delta x} = \lim_{\delta x\to 0}\frac{1}{\delta x}\log_{10}\left(1+\frac{\delta x}{x}\right)$

Let $\frac{\delta x}{x} = t$, ie $\delta x = tx$.

as $\delta x \to 0$, $t \to 0$

$\therefore \frac{dy}{dx} = \lim_{t\to 0}\frac{1}{tx}\log_{10}(1+t)$

$= \lim_{t\to 0}\left(\frac{1}{x}\log_{10}(1+t)^{\frac{1}{t}}\right)$

$= \frac{1}{x}\log_{10}\left[\underbrace{\lim_{t\to 0}(1+t)^{\frac{1}{t}}}_{}\right]$

e, Euler's constant $\approx 2.718281828 46$

$\boxed{\therefore \frac{d}{dx}\log_{10} x = \frac{1}{x}\log_{10} e}$

$\frac{d}{dx}\log_e x = \frac{1}{x}\log_e e$

$= \frac{1}{x}$.

$\therefore \boxed{\frac{d}{dx}\ln x = \frac{1}{x}.}$

$\boxed{\#2 \quad \frac{d}{dx}(\sin x) = \cos x}$

$\boxed{\#3 \quad \frac{d}{dx}(\cos x) = -\sin x}$

$\boxed{\#4 \quad \frac{d}{dx}(\tan x) = \sec^2 x}$

#5 $y = e^x$

$\ln y = x = \ln y$

$\frac{dx}{dy} = \frac{1}{y}$

$\therefore \frac{dy}{dx} = y = e^x$

$\boxed{\therefore \frac{d}{dx}(e^x) = e^x.}$

#6. $\frac{d}{dx}(\sec x)$

$= \frac{d}{dx}((\cos x)^{-1})$

$= -(\cos x)^{-2}(-\sin x)$

$= \frac{\sin x}{\cos^2 x} = \sec x \tan x$.

$\boxed{\therefore \frac{d}{dx}(\sec x) = \sec x \tan x.}$

7. $\frac{d}{dx}(\operatorname{cosec} x)$

$= \frac{d}{dx}\left(\frac{1}{\sin x}\right) = \frac{d}{dx}((\sin x)^{-1})$

$= -(\sin x)^{-2}(\cos x)$

$= \frac{-(\cos x)}{\sin^2 x}$

$= -\operatorname{cosec} x \cot x$

$\boxed{\therefore \frac{d}{dx}(\operatorname{cosec} x) = -\operatorname{cosec} x \cot x}$

#8. $\frac{d}{dx}(\cot x)$

$= \frac{d}{dx}((\tan x)^{-1})$

$= -(\tan x)^{-2}\sec^2 x$

$= \frac{-\sec^2 x}{\tan^2 x} = \frac{-1}{\cos^2 x}\cdot\frac{\cos^2 x}{\sin^2 x}$

$= -\operatorname{cosec}^2 x$

$\boxed{\therefore \frac{d}{dx}(\cot x) = -\operatorname{cosec}^2 x.}$

RULES

1) Chain rule: (composite f)

$\frac{d}{dx}f(g(x)) = \frac{df}{dg}\times\frac{dg}{dx}$

eg 1 $\frac{d}{dx}(\sin^3 x)$

$= \frac{d}{dx}([\sin x]^3)$

$= 3(\sin x)^2(\cos x)$

$= 3\sin^2 x\cos x$

eg 4 $\frac{d}{dx}(\cos 3x)$

$= -\sin 3x \ (3)$

$= -3\sin 3x$

eg 5 $\frac{d}{dx}\sin\left(2x+\frac{\pi}{4}\right)$

$= \cos\left(2x+\frac{\pi}{4}\right)(2)$

$= 2\cos\left(2x+\frac{\pi}{4}\right)$

eg 2 $\frac{d}{dx}\ln(x^2+1)$

$= \frac{1}{x^2+1}(2x)$

$= \frac{2x}{x^2+1}$

eg 6 $\frac{d}{dx}\left(\frac{1}{e^{3x}}\right)$

$= \frac{d}{dx}(e^{-3x})$

$= -3e^{-3x}$

eg 3 $\frac{d}{dx}\ln\left(\frac{1+x}{1-x}\right)$

$= \frac{d}{dx}\left[\ln(1+x) - \ln(1-x)\right]$

$= \frac{1}{1+x}(1) - \frac{1}{1-x}(-1)$

$= \frac{1}{1+x} + \frac{1}{1-x}$

Tip: if you see $\frac{d}{dx}\ln(f(x))$, try to simplify before you simplify further.

Product rule

$\frac{d}{dx}(uv) = u'v + uv'$

Quotient rule

$\frac{d}{dx}\left(\frac{u}{v}\right) = \frac{u'v - uv'}{v^2}$

Application

① Stationary pts → $\frac{dy}{dx} = 0$

$\frac{d^2y}{dx^2} > 0$ → min

$\frac{d^2y}{dx^2} < 0$ → max

② Curve gradient = $\frac{dy}{dx}$

③ Eq^n of tangent at (h,k)

• $y - k = m(x-h)$

m = value of $\frac{dy}{dx}$ at (h,k)

④ Eq^n of normal at (h,k)

→ $y - k = -\frac{1}{m}(x-h)$

Implicit Differentiation

"Implicit" → relations b/w x & y.

* imp. / difficult to make y the subject.

new syllabus !

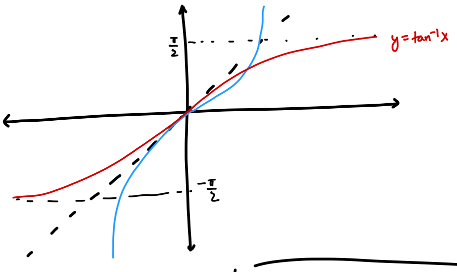
$$\boxed{\text{To find } \frac{d}{dx}(\tan^{-1}x)}$$

* $\tan^{-1}(x) \neq (\tan x)^{-1}$

** must use a restricted domain!

$f(x) = \tan^{-1}(x)$, $-\frac{\pi}{2} < x < \frac{\pi}{2}$.

for the restricted domain, $f^{-1}(x)$ exists



let $y = \tan^{-1}(x)$

∴ $\tan y = x$

Method #1:

$\frac{d}{dx}(\tan y) = \frac{d}{dx}(x)$

$\sec^2 y \frac{dy}{dx} = 1$

∴ $\frac{dy}{dx} = \frac{1}{\sec^2 y}$

$= \frac{1}{1 + \tan^2 y}$

$\boxed{\frac{dy}{dx} = \frac{1}{1+x^2}}$.

Method #2:

let $y = \tan^{-1}x$

∴ $\tan y = x$

∴ $x = \tan y$

∴ $\frac{dx}{dy} = \sec^2 y$

$= 1 + \tan^2 y$

$= 1 + x^2$

∴ $\frac{dy}{dx} = \frac{1}{1+x^2}$.

$$\boxed{\star \ \frac{d}{dx}\tan^{-1}(x) = \frac{1}{1+x^2}}$$

∴ $\frac{d}{dx}\tan^{-1}[f(x)] = \frac{1}{1+[f(x)]^2} f'(x)$

eg¹ $\frac{d}{dx}(\tan^{-1}(x^2))$

$= \frac{1}{1+x^4}(2x)$

$= \frac{2x}{1+x^4}$

eg² $\frac{d}{dx}(\tan^{-1}(\sqrt{x}))$ [$x^{\frac{1}{2}}$]

$= \frac{1}{1+x}\left(\frac{1}{2}x^{-\frac{1}{2}}\right)$

$= \frac{1}{2(x+1)\sqrt{x}}$

eg³ $\frac{d}{dx}(\tan^{-1}(e^{2x}))$

$= \frac{1}{1+e^{4x}}\left[2e^{2x}\right]$

$= \frac{2e^{2x}}{1+e^{4x}}$

eg⁴ $\frac{d}{dx}\tan^{-1}\left(\frac{1}{x}\right)$

$= \frac{1}{1+\left(\frac{1}{x}\right)^2}(-x^{-2})$

$= \frac{1}{\left(\frac{x^2+1}{x^2}\right)}\left(-\frac{1}{x^2}\right)$

$= \frac{-1}{x^2+1}$

Parametric eq^{n}s of a curve

$\begin{cases} x = f(t) \\ y = g(t) \end{cases}$

t is a parameter

↳ t is a variable & for each value of t, it corresponds to one & only one pt on the curve.

* To find the pt on the curve

↳ to find the value of t.

eg¹ $\begin{cases} x = t^2 \\ y = 2t. \end{cases}$

when $t = 1 \leftrightarrow (1, 2)$

when $t = -1 \leftrightarrow (1, -2)$

☆ To find $\frac{dy}{dx}$

$\frac{dy}{dx} = \frac{\left(\frac{dy}{dt}\right)}{\left(\frac{dx}{dt}\right)}$

→ in terms of t

Chapter 6: Integration

*trapezium rule not in syllabus

P1 : $\int x^n\,dx = \frac{x^{n+1}}{n+1} + C$

$\int (\underbrace{ax+b}_{\text{linear}})^n\,dx = \frac{(ax+b)^{n+1}}{a(n+1)} + C$

① TO INTEGRATE RATIONAL FUNCTIONS

$\int \frac{g(x)}{f(x)}\,dx$, where $f(x)$ & $g(x)$ are polynomials.

$\Rightarrow$ recall $\frac{d}{dx}(\ln[f(x)]) = \frac{1}{f(x)} f'(x)$

$= \frac{f'(x)}{f(x)} + C.$

conversely,

$\int \frac{f'(x)}{f(x)}\,dx = \ln|f(x)| + C.$

↳ the numerator is the differential of the denominator.

$\int \frac{1}{x}\,dx = \ln(x) + C, \; x>0$

" $\int \frac{-1}{-x}\,dx = \ln(-x) + C, \; x<0.$

* hence $\int \frac{1}{x}\,dx = \ln|x| + C.$

★ $\int \frac{A}{ax+b}\,dx$

$= \int \frac{ax\frac{A}{a}}{ax+b}\,dx$

$= \frac{A}{a}\int \frac{a}{ax+b}\,dx$

$= \frac{A}{a}\ln(ax+b) + C.$

$\therefore \int \frac{A}{ax+b}\,dx = \frac{A}{a}\ln\underbrace{(ax+b)}_{\text{must be linear.}} + C.$

cannot be used. why? $(1+x^2)$ is not linear.

eg¹ $\int \frac{2}{3x-1}\,dx$

$= \frac{2}{3}\ln(3x-1) + C$

eg² $\int \frac{1}{4+2x}\,dx$

$= \frac{1}{2}\ln(2x+4) + C$

eg³ $\int \frac{3}{1-4x}\,dx$

$= \frac{-3}{4}\ln(1-4x) + C$

eg⁴ $\int \frac{x}{1+x^2}\,dx$

$= \frac{1}{2}\int \frac{2x}{1+x^2}\,dx$

$= \frac{1}{2}\ln(1+x^2) + C.$

eg⁵ $\int \frac{2}{1+x^2}\,dx$

↳ cannot use the result $\int \frac{f'(x)}{f(x)}\,dx = \ln[f(x)] + C$

$= 2\int \frac{1}{1+x^2}\,dx$

$= 2\tan^{-1}x + C.$

★ $\frac{d}{dx}\left[\tan^{-1}\left(\frac{x}{a}\right)\right]$

$= \frac{1}{1+\left(\frac{x}{a}\right)^2}\frac{d}{dx}\left(\frac{1}{a}x\right)$

$= \frac{1}{1+\frac{x^2}{a^2}}\left(\frac{1}{a}\right)$

$= \frac{1}{\frac{a^2+x^2}{a^2}}\left(\frac{1}{a}\right)$

$= \frac{a}{a^2+x^2}$

$\boxed{\therefore \frac{d}{dx}\left(\tan^{-1}\left(\frac{x}{a}\right)\right) = \frac{a}{a^2+x^2}.}$

$\boxed{\therefore \int \frac{1}{a^2+x^2} = \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right) + C.}$

$\boxed{a>0}$

eg¹ $\int \frac{1}{4+x^2}\,dx$

$= \frac{1}{2}\tan^{-1}\left(\frac{x}{2}\right) + C.$

eg² $\int \frac{1}{3+x^2}\,dx$

$= \frac{1}{\sqrt{3}}\tan^{-1}\left(\frac{x}{\sqrt{3}}\right) + C.$

eg³ $\int_0^1 \frac{1}{1+3x^2}\,dx$

$= \frac{1}{3}\int_0^1 \frac{1}{\frac{1}{3}+x^2}\,dx$

$= \frac{1}{3}\left[\frac{1}{\left(\frac{1}{\sqrt{3}}\right)}\tan^{-1}\left(\frac{x}{\left(\frac{1}{\sqrt{3}}\right)}\right)\right]_0^1$

$= \frac{1}{3}\sqrt{3}\left[\tan^{-1}(\sqrt{3}x)\right]_0^1$

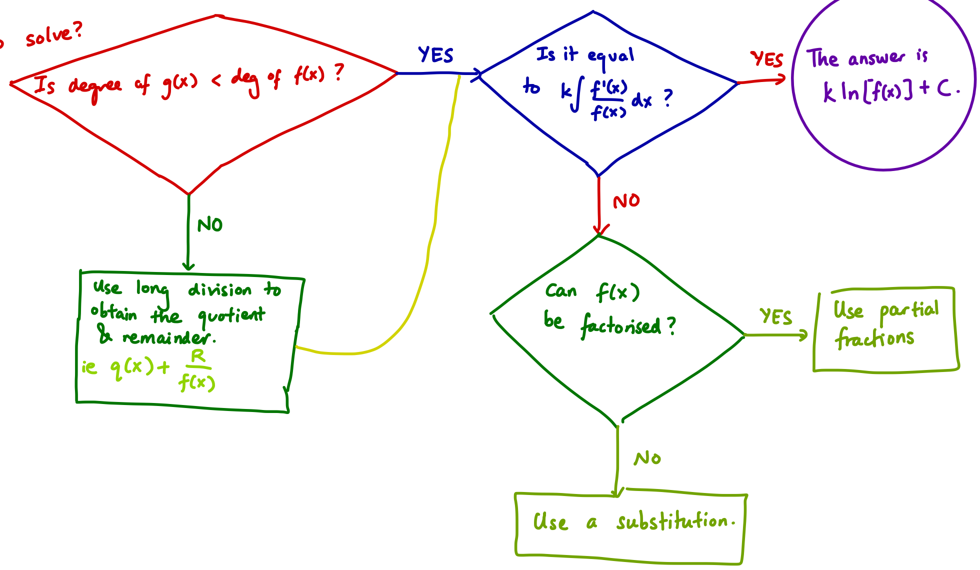
$= \frac{1}{3}\sqrt{3}\left[\underbrace{\tan^{-1}(\sqrt{3})}_{\frac{\pi}{3}} - \underbrace{\tan^{-1}(0)}_{0}\right]$

$= \frac{\sqrt{3}\pi}{9}$

$\therefore \int \frac{g(x)}{f(x)}\,dx + C$

$g(x), f(x)$ 2/3-degree polynomials.

How to solve?



For $f(x) = \dfrac{A}{ax+b} + \dfrac{Bx+C}{x^2+d^2}$ $(C \neq 0)$

$$\Rightarrow \quad \frac{A}{ax+b} + \boxed{\frac{Bx}{x^2+d^2} + \frac{C}{x^2+d^2}.}$$ split!

$$\therefore \int f(x)\,dx = \int \frac{A}{ax+b}\,dx + \frac{B}{2}\int \frac{2x}{x^2+d^2}\,dx + \boxed{C\int \frac{1}{x^2+d^2}\,dx}$$

$$= \frac{A}{a}\ln(ax+b) + \frac{B}{2}\ln(x^2+d^2) + C\frac{1}{d}\tan^{-1}\left(\frac{x}{d}\right) + c$$

② To integrate exponential functions

$$\frac{d}{dx}(e^{mx}) = me^{mx}$$

$$\frac{d}{dx}\left[\frac{e^{mx}}{m}\right] = e^{mx}$$

$$\boxed{\therefore \int \frac{e^{mx}}{m}\,dx = e^{mx} + c.}$$

③ To integrate trig functions

Recall

$$\frac{d}{dx}(\sin(mx)) = m\cos(mx)$$

$$\therefore \int \cos mx\,dx = \frac{\sin(mx)}{m} + c.$$

eg 1 $\int \cos 2x\,dx = \frac{\sin 2x}{2} + c.$

eg 2 $\int \cos \frac{1}{2}x\,dx = 2\sin\frac{1}{2}x + c$

eg 3 $\int \cos\left(2x+\frac{\pi}{3}\right) dx = \frac{\sin\left(2x+\frac{\pi}{3}\right)}{2} + c$

$$\frac{d}{dx}(\cos(mx+\alpha)) = -m\sin(mx+\alpha)$$

$$\therefore \int \sin(mx+\alpha)\,dx = \frac{-\cos(mx+\alpha)}{m} + c.$$

eg 1 $\int \sin 3x\,dx = \frac{-\cos 3x}{3} + c.$

eg 2 $\int \sin\left(x-\frac{\pi}{4}\right) dx = \frac{-\cos\left(x-\frac{\pi}{4}\right)}{1} + c.$

$$\frac{d}{dx}(\tan(mx)) = m\sec^2(mx)$$

$$\therefore \int \sec^2(mx)\,dx = \frac{\tan(mx)}{m} + c.$$

eg 1 $\int \sec^2 2x\,dx = \frac{\tan 2x}{2} + c.$

$$\boxed{\text{case 1} \quad \int \sin^n mx\,dx, \quad \int \cos^n mx\,dx, \quad n \geq 2}$$

A) n is even (esp 2)

Method : use the identity

$$\cos 2A = 2\cos^2 A - 1 = 1 - 2\sin^2 A$$

$$\therefore \cos^2 A = \frac{\cos 2A + 1}{2}, \quad \sin^2 A = \frac{1-\cos 2A}{2}.$$

B) n is odd (esp 3)

Method 1 : use an identity.
(Q will ask to prove).

Method 2 : use a given substitution
(method s).

④ Integration by parts

$\frac{d}{dx}(uv) = uv' + vu'$ (u, v are functions in x)

Integrate both sides wrt x

$\Rightarrow uv = \int uv' + vu' \, dx$

$uv = \int u\,dv + \int v\,du$

$$\boxed{\therefore \int u\,dv = uv - \int v\,du.}$$

given in formula booklet

☆ This method is used to integrate:

1) Log functions

2) Product of two functions

3) Inverse tangent

eg $\int \ln x \, dx$

$u = \ln x \quad \frac{dv}{dx} = 1$

$\frac{du}{dx} = \frac{1}{x} \quad v = x$

$= x\ln x - \int x \cdot \frac{1}{x} \, dx$

$= x\ln x - x + C$

GUIDELINES

① If there is a log function / $\tan^{-1}$ function, then let "u" be the log function / $\tan^{-1}$ function.

② If there is no log function, then let "u" be the polynomial function.

⑤ Integration by substitution, that is given.

1) Functions involving "$\sqrt[n]{\ }$"

2) Odd powers of sin, cos

3) etc.

Chapter 7:
Differential Equations

↳ an eq$^{\underline{n}}$ involving
- → an indep. var, x;
- → a dep. var, y;
- → & 1+ derivatives of y wrt x (ie $\frac{dy}{dx}$ or $\frac{d^2y}{dx^2}$). ← FM.

⇒ order of a diff eq$^{\underline{n}}$ = order of highest derivative.

★ For P3, we only consider 1st order differential eq$^{\underline{n}}$s, in which the variables are separable

↳ $\frac{dy}{dx}$ can be expressed as $\boxed{f(x)\,g(y)}$.

⇒ consider $\frac{dy}{dx} = f(x)g(y)$.

$(\div g(y))$ $\quad \frac{1}{g(y)}\frac{dy}{dx} = f(x)$

$\therefore \int \frac{1}{g(y)}\frac{dy}{dx}\,dx = \int f(x)\,dx$

$$\boxed{= \int \frac{1}{g(y)}\,dy = \int f(x)\,dx.}$$

or $\int \frac{dy}{g(y)} = f(x)\,dx$.

eg solve the diff eq$^{\underline{n}}$

$(x^2+4)\frac{dy}{dx} = 6xy$.

$\Rightarrow \int \frac{1}{y}\,dy = \int \frac{6x}{x^2+4}\,dx$

$\ln(y) + c_1 = 3\ln(x^2+4) + c_2$

$\Rightarrow \ln(y) = 3\ln(x^2+4) + \underbrace{(c_2 - c_1)}_{\text{arbitrary constant}}$

$\Rightarrow \ln(y) = 3\ln(x^2+4) + c$

★ general solution
- ↳ satisfies the diff. eq$^{\underline{n}}$
- ↳ has 1 constant of integ. on RHS.

★ particular solution
- ↳ additional info has been given, such that the constant of integration can be found.

Formulating a simple statement involving rate of Δ as a differential eq$^{\underline{n}}$

eg if volume is changing, then the rate of Δ of the volume can be represented as $\frac{dV}{dt}$.

$\frac{dV}{dt} > 0$ if rate > 0, $\frac{dV}{dt} < 0$ if rate < 0.

★ usually rate ∝ to a certain quantity.

Chapter 8: Numerical Methods

Consider a eq$^{\underline{n}}$ that we cannot solve directly.

eg $x^5 + 3x = 2$.

We can approximate the values of such values using graphical methods or iteration.

① Graphical methods

Consider $y = f(x)$. Assume at $x = a$, the graph intersects the x-axis; ie $f(a) = 0$.

⇒ Hence, the # of roots $\in \mathbb{R}$ of the eq$^{\underline{n}}$ $f(x) = 0$ is the # of pts of ints$^{\underline{n}}$ b/w the graph $y = f(x)$ & the x axis.

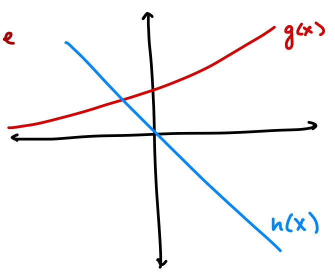
↳ the x-coord of the pts of ints$^{\underline{n}}$ is the value of the roots of the eq$^{\underline{n}}$.

However, if the graph $y = f(x)$ is cumbersome to sketch, then we need to rearrange $f(x) = 0$ to the form $g(x) = h(x)$.

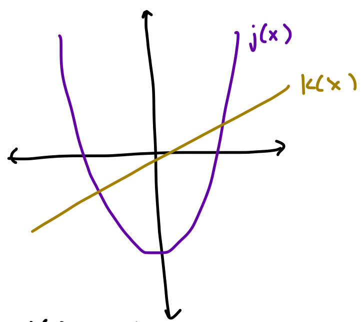
↳ ideally, $g(x)$ & $h(x)$ can be sketched easily.

↳ the # of ints$^{\underline{n}}$s b/w $g(x)$ & $h(x)$ are the # of roots of $f(x) = 0$.

ie



$g(x) = h(x)$ has 1 root $\in \mathbb{R}$



$j(x) = k(x)$ has 2 roots $\in \mathbb{R}$

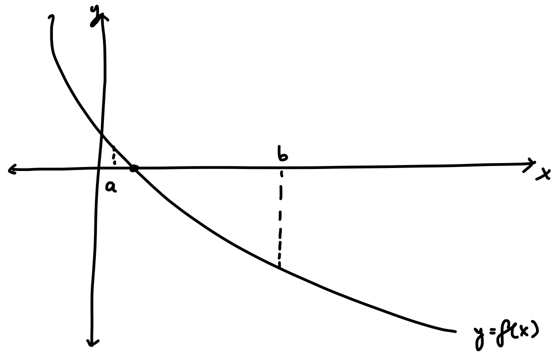
② To approximate the location of a root by searching for a sign Δ.

extra knowledge.

☆ this method fails IF:

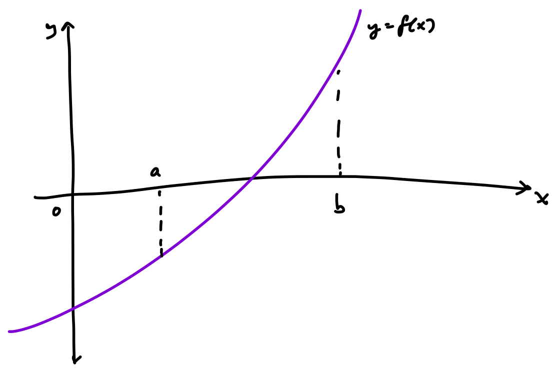
1) repeated root eg $y = x^2$

2) f is not continuous b/w a & b. eg asymptote.



$f(a) > 0$, $f(b) < 0$.

⇒ $b <$ root $< a$



$f(a) < 0$, $f(b) > 0$

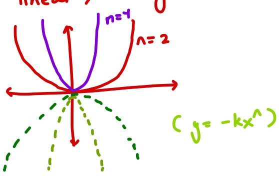
⇒ $a <$ root $< b$.

Recap: Std Curves

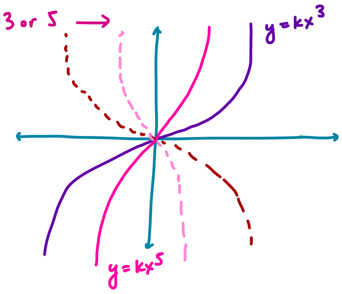
① $y = kx^n$, $k > 0$, $k \in \mathbb{Z}$

$n = 1 \longrightarrow$ linear, through $(0,0)$

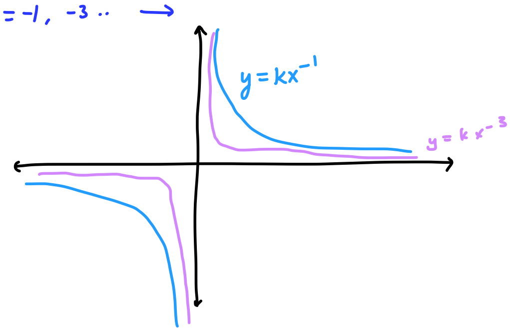
$n = 2$ or $4 \longrightarrow$



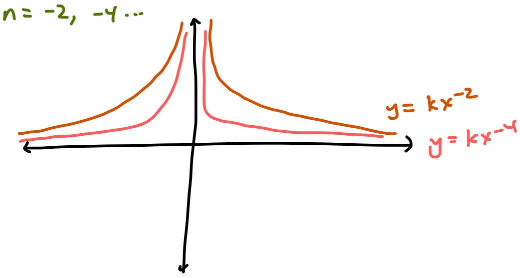
$n = 3$ or $5 \longrightarrow$



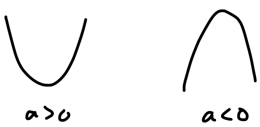
$n = -1, -3 \ldots \longrightarrow$



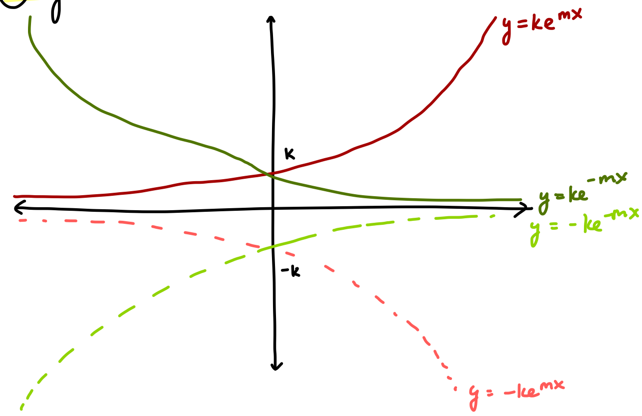
$n = -2, -4 \ldots$



② $y = ax^2 + bx + c$



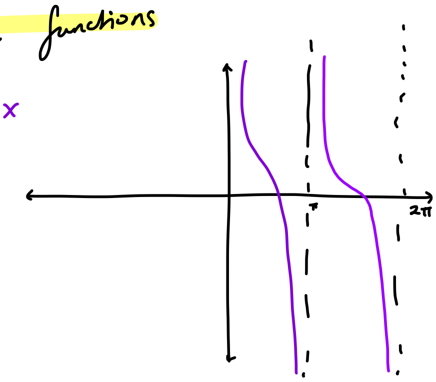
③ $y = ke^{mx}$, $k > 0$, $m > 0$



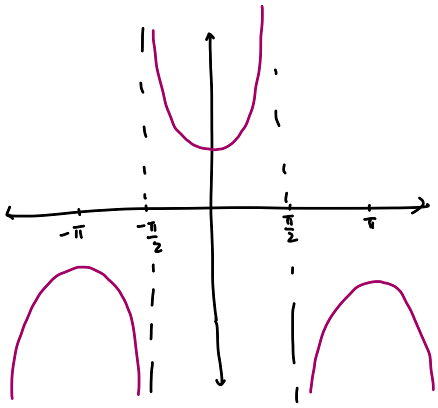
④ $y = \ln x$, $x > 0$

⑤ Trig functions

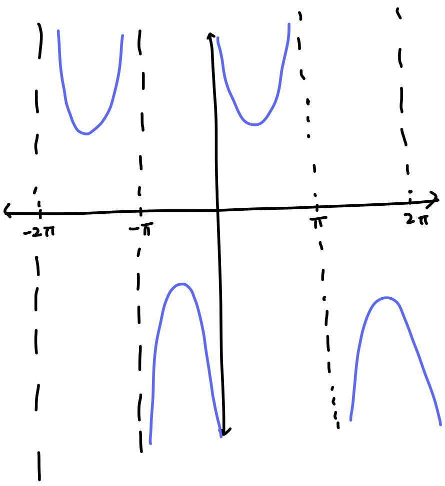
i) $\cot x$



ii) $\sec x$



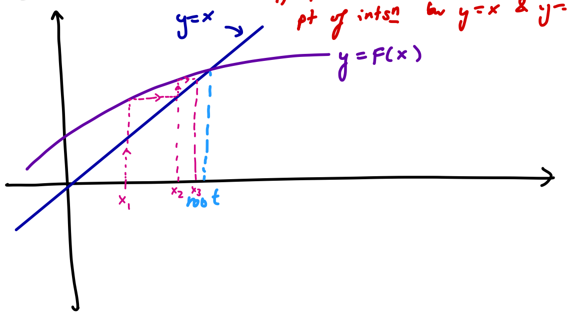
iii) $\operatorname{cosec} x$



③ Iteration

Suppose a graph $x = F(x)$.

the root is the x coord of the pt of intsⁿ bw $y=x$ & $y=F(x)$.



→ We can determine the eqⁿ that the iterative formula was meant to solve, provided the sequence $x_1, x_2 \dots$ converges to the root.

$x_{n+1} = F(x_n) \Rightarrow \alpha = F(\alpha)$.

eg $x_{n+1} = \dfrac{3}{1+\ln x_n}$

(3sf) $\alpha = 1.85$

$x = \dfrac{3}{1+\ln x}$

$\therefore 1 + \ln x - \dfrac{3}{x} = 0$.

Pick some x_1.

Then $x_2 = F(x_1)$.

$x_3 = F(x_2)$ etc.

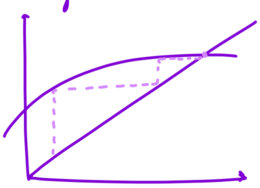
☆ We cannot be sure what will happen to the seq $x_1, x_2 \dots$

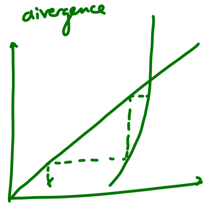
→ the terms may:

1) converge; → orig. solution.

2) diverge;

3) or oscillates.





Ex 1 using the iterative F $x_{n+1} = 2^{\frac{1}{x_n}}$, calc root of $x^x = 2$.

$x_1 = 1.5$

$x_2 = 2^{\frac{1}{1.5}}$

$= 1.5874$

$x_3 = 1.5475$

$x_4 = 1.5650$

$x_5 = 1.5572$ } agree to 2dp

$x_6 = 1.5607$

$\therefore$ root ≈ 1.56 (2dp).

using $x_{n+1} = 2x_n^{(1-x_n)}$, $x_1 = 1.5$. Calc seq of $x_1, x_2 \dots$ Comment briefly on the seq.

$x_1 = 1.5$

$x_2 = 2(1.5)^{(1-1.5)}$

$= 1.6330$

$x_3 = 1.4663$

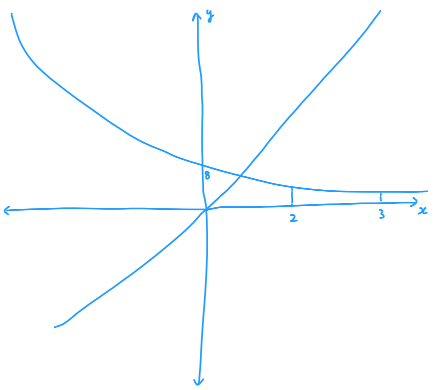
$x_4 = 1.6731$

$x_5 = 1.4144$

⋮

⇒ the sequence diverges away from the root.

Ex 2 $x = 8e^{-\frac{1}{2}x}$



∴ there is only 1 pt of intersection bw the 2 graphs

∴ the eqⁿ only has one real root.

$8e^{-\frac{1}{2}x} - x = 0$

$f(x) = 8e^{-\frac{1}{2}x} - x$.

$f(2) = 0.9430$ (>0)

$f(3) = -1.2149$ (<0)

$\therefore 2 < \text{root} < 3$.

Let $x_1 = 2.5$

$\therefore x_2 = 8e^{-\frac{1}{2}(2.5)}$

$= 2.2920$

$x_3 = 2.5431$

$x_4 = 2.2430$

$x_5 = 2.6062$

⋮

∴ the sequence does not converge to the root.

Chapter 9: Complex Numbers

Any number of the form $x+iy$ is called a complex number, where $x, y \in \mathbb{R}$, & $i=\sqrt{-1}$ (or $i^2=-1$)

eg¹ $x = 3 \pm \sqrt{-4}$
$= 3 \pm 2i$ ← complex numbers.

eg² $z = \dfrac{-2 \pm \sqrt{-3}}{2}$
$= -1 \pm \frac{\sqrt{3}}{2} i$

★ if $x=0$ → totally imaginary.
" $y=0$ → totally real.

⇒ A complex number consists of 2 parts: the real part (x) & the imaginary part (y).
$\underline{x} + i\underline{y}$

★ Let z ↦ a complex number.

Notation — • the real part of z is denoted by $Re(z)$.
ie $Re(z) = x$.
• the imaginary part of z is denoted by $Im(z)$.
ie $Im(z) = y$.

⇒ consider i^n $(n \in \mathbb{Z}^+)$

$$i^n \begin{cases} i, & n = 4k+1 \\ -1, & n = 4k+2 \\ -i, & n = 4k+3 \\ 1, & n = 4k+4 \end{cases} \quad (k \in \mathbb{Z}^+ \cup 0)$$

Fundamental Operations of Complex Numbers

+ let $u = a+bi$, $v = c+di$ (= denotes) $\}$ $a,b,c,d \in \mathbb{R}$

$u+v = (a+c) + (b+d)i$

eg $u = 3+4i$, $v = -5+i$ $u+v = 3+4i-5+i$
$= -2+5i$.

− $u-v = (a+ib) - (c+id)$
$= (a-c) + (b-d)i$.

eg $z = 7-i$, $w = 4-3i$
$z-w = (7-i) - (4-3i)$
$= 3+2i$.

× $uv = (a+bi)(c+di)$
$= (ac-bd) + (ad+bc)i$

eg¹ $z_1 = 3-2i$, $z_2 = -5+4i$
$\therefore z_1 z_2 = (3-2i)(-5+4i)$
$= (-15-(-8)) + (12+10)i$
$= -7+22i$

eg² $z = \sqrt{3}-i$ $w = 3+2\sqrt{3}i$
$zw = (\sqrt{3}-i)(3+2\sqrt{3}i)$
$= (3\sqrt{3} - (-2\sqrt{3})) + (6-3)i$
$= 5\sqrt{3} + 3i$.

÷ $\dfrac{u}{v} = \dfrac{a+bi}{c+di}$ $\dfrac{a+bi}{c+di} \cdot \dfrac{c-di}{c-di}$

$= \dfrac{ac+bd + (bc-ad)i}{c^2+d^2}$ $= \dfrac{(a+bi)(c-di)}{c^2+d^2}$

$= \dfrac{(ac+bd) + (bc-ad)i}{c^2+d^2}$.

eg $\dfrac{-4+i}{-3-2i} \cdot \dfrac{-3+2i}{-3+2i}$
$= \dfrac{(12-2) + (-8-3)i}{9+4}$
$= \dfrac{10-11i}{13} = \frac{10}{13} - \frac{11}{13}i$

★ The conjugate of a complex number, $z = x+yi$, denoted by z^*, is defined by $z^* = x-iy$.
(ie flip sign of y, don't Δ x)

eg¹ $z = 3+2i$
$z^* = 3-2i$

eg² $z = -\sqrt{3} - 4i$
$z^* = -\sqrt{3} + 4i$.

Important results

① $z + z^* = (x+iy) + (x-iy)$
$= 2x$.
↳ The sum of a complex # & its conjugate is a totally real number.

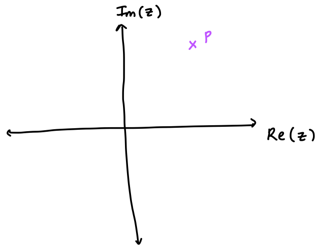
② $zz^* = (x+iy)(x-iy)$
$= x^2+y^2$.
↳ The product of a complex number & its conjugate is a totally real number.

To represent a complex number geometrically.

A complex # consists of two parts — a real part, x, & the imaginary part, y.

Let $z = x + yi$.

$\therefore Re(z) = x, \quad Im(z) = y$.



Argand diagram.

If P represents the complex num $x + iy$,

Then P is the point (x, y) in the diagram.

Definitions.

Let $z = x + iy$. and z is represented by the pt $P(x,y)$ in an Argand diagram.

① The modulus of a complex number z, denoted by $|z|$, is defined as

$$|z| = \sqrt{x^2 + y^2}.$$

eg¹ $u = 3 + 4i$

$\therefore |u| = \sqrt{3^2 + 4^2} = 5$.

eg² $v = -24 + 7i$

$\therefore |v| = \sqrt{24^2 + 7^2} = 25$.

eg³ if $w = 3$ $\quad w = 2i$

$\therefore |w| = 3. \quad |w| = 2.$

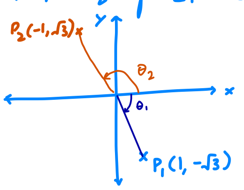
② The argument of a complex number, denoted by $\arg(z)$, is defined as

$$\arg(z) = \tan^{-1}\left(\frac{y}{x}\right).$$

$z_1 = 1 - \sqrt{3}i \qquad z_2 = -1 + \sqrt{3}i$

$\arg(z_1) = \tan^{-1}(-\sqrt{3}) = -60°$. $\qquad \arg(z_2) = \tan^{-1}(-\sqrt{3}) = -60°$.

↳ Let P_1 & P_2 resp. z_1 & z_2 resp in an Argand diagram.



$\therefore \arg(z_1) = -60°$,

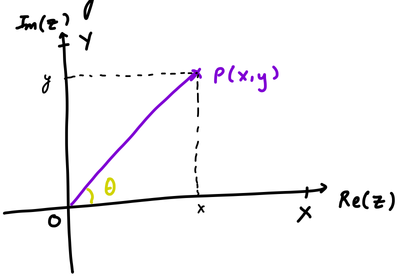
$\arg(z_2) = 120°$.

* $\tan^{-1}\left(\frac{y}{x}\right)$ is a many-value function.

The principal value of the argument of z is defined as

$$-\pi < \arg(z) \leq \pi.$$

- Interpretation of modulus & argument, geometrically



$|OP| = \sqrt{x^2 + y^2} = |z|$.

$\tan\theta = \frac{y}{x}$

$\therefore \theta = \tan^{-1}\left(\frac{y}{x}\right) = \arg(z)$.

$\theta = \angle POX$

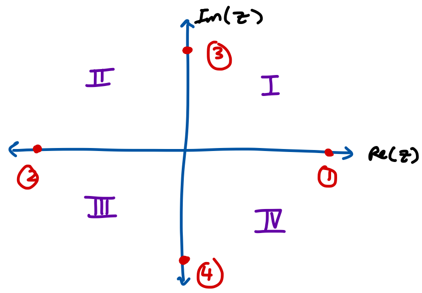
Special Cases

① $Re(z) > 0, \; Im(z) = 0 \mapsto \arg(z) = 0$

② $Re(z) < 0, \; Im(z) = 0 \mapsto \arg(z) = 180°$

③ $Re(z) = 0, \; Im(z) > 0 \mapsto \arg(z) = 90°$

④ $Re(z) = 0, \; Im(z) < 0 \mapsto \arg(z) = -90°$



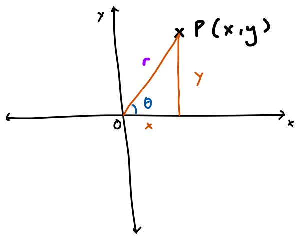
→ if P lies in I $(x > 0, \; y > 0)$ $\mapsto$ answer from calculator

→ if P lies in IV $(x > 0, \; y < 0)$ $\mapsto$ answer from calculator

→ if P lies in II $(x < 0, \; y > 0)$ $\mapsto$ $180° +$ ans from calc

→ if P lies in III $(x < 0, \; y < 0)$ $\mapsto$ $-180° +$ ans from calc

The modulus-argument form of a complex number.



let $r = |\overrightarrow{OP}|$
$= |z|$.

$\tan\theta = \frac{y}{x}$

$\therefore \theta = \tan^{-1}\left(\frac{y}{x}\right)$.

$\cos\theta = \frac{x}{r}$ $\quad$ $\sin\theta = \frac{y}{r}$

$\therefore x = r\cos\theta$ $\quad$ $\therefore y = r\sin\theta$

Hence $z = x + iy$ ⟵ algebraic form of a $\# \in \mathbb{C}$

$\Rightarrow z = r\cos\theta + ir\sin\theta$

$= r(\cos\theta + i\sin\theta)$ ⟵ modulus-argument form of a $\# \in \mathbb{C}$ (polar form - FM1)

modulus argument

$= re^{i\theta}$. $\quad$ ($e^{i\theta} = \cos\theta + i\sin\theta$, θ in rad)

exponential / Euler's form. of a $\# \in \mathbb{C}$

☆ $a + ib = 0$ if $a, b = 0$

Proof by contradiction:

Suppose $a + ib = 0$, $\quad a \neq 0,\ b \neq 0$

$a + ib = 0$

$a = -ib$

$a^2 = -b^2$

$a^2 + b^2 = 0$.

But $a^2 > 0$, $b^2 > 0$ $\quad$ $\therefore a^2 + b^2 > 0$ for $a, b \in \mathbb{R}$!

Hence $a, b = 0$ if $a + ib = 0$.

☆ $a + ib = c + id$ $\quad$ When two $\# \in \mathbb{C}$ are equal, we can equate the real parts & the imag parts.

$\Leftrightarrow a = c$ & $b = d$.

Proof $(a - c) + (b - d)i = 0$

$\therefore a - c = 0,\ b - d = 0$

$\therefore a = c,\ b = d$.

eg' The complex # u has mag 2 & arg $\frac{\pi}{6}$. Express u in the form $x + iy$, where $x, y \in \mathbb{R}$

$u = 2\left(\cos\frac{\pi}{6} + i\sin\frac{\pi}{6}\right)$

$= 2\left(\frac{\sqrt{3}}{2} + i\frac{1}{2}\right)$

$= \sqrt{3} + i$

What is the use of the modulus-argument form?

$\Rightarrow$ Consider the following example:

The complex # $\sqrt{3} + i$ is denoted by u.

Find the modulus & argument of u^4.

$u = \sqrt{3} + i = r(\cos\theta + i\sin\theta)$

$r = \sqrt{(\sqrt{3})^2 + (1)^2} = 2$

$\theta = \tan^{-1}\left(\frac{1}{\sqrt{3}}\right) = \frac{\pi}{6}$.

$\therefore u = 2\left(\cos\frac{\pi}{6} + i\sin\frac{\pi}{6}\right)$

$= 2e^{i\left(\frac{\pi}{6}\right)}$.

$u^4 = \left(2e^{i\frac{\pi}{6}}\right)^4$ $\quad$ $\therefore |u^4| = 16$

$= 16e^{i\frac{2\pi}{3}}$. $\quad$ $\therefore \arg(u^4) = \frac{2\pi}{3}$.

If $z = r(\cos\theta + i\sin\theta)$

$\Rightarrow$ $z^n = r^n(\cos n\theta + i\sin n\theta)$

$\Rightarrow |z^n| = |z|^n \ (= r^n)$

$\arg(z^n) = n\arg(z) \ (= n\theta)$

We can use these results to find the sqrt of a complex number.

To solve cubics & quartics

$az^3+bz^2+cz+d=0$

$az^4+bz^3+cz^2+dz+e$

$a,b,c,d,e \in \mathbb{R}$

Result (w/o proof)

The complex roots w/ a polynomial eqn with real coefficients occur in conjugate pairs.

In other words,

if $a+bi$ is a root,

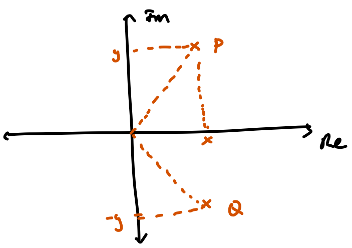
then $a-bi$ is also a root.

Geometrical effects of:

1) Conjugating a $\# \in \mathbb{C}$

Let P & Q represent z & z^*, resp, in an Argand diagram.

$\therefore \quad z = x+iy, \quad z^* = x-iy, \quad x,y \in \mathbb{R}$



1) Q is the pt of reflection of P in the real axis

2) OPQ is an isosceles $\triangle$

ie $OP = OQ$

ie $|z| = |z^*|$

3) $\arg(z^*) = -\arg(z)$

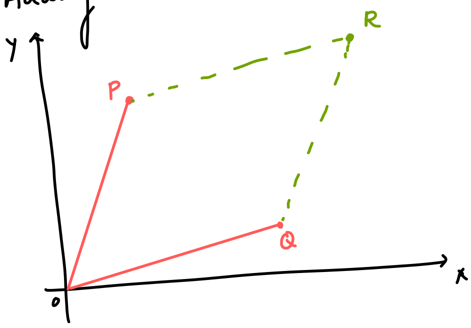
Special case

If $\arg(z) = \frac{\pi}{4}$,

then $\angle POQ = \frac{\pi}{2}$

$\therefore \triangle OPQ$ is a $\llcorner$ $\triangle$

2) Adding 2 $\# \in \mathbb{C}$



effect:

- $OPRQ$ is a parallelogram
- $|z_1+z_2| = OR$

Let P, Q rep z_1 & z_2 resp

" R rep z_1+z_2.

We can use $\vec{OP}$ & $\vec{OQ}$ to represent z_1 & z_2 resp.

Hence $\vec{OP} + \vec{OQ} \rightarrow$ addition of 2 vectors.

$\Rightarrow$ //gram law of addition:

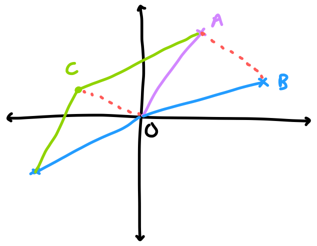
$\vec{OP} + \vec{OQ} = \vec{OR}$.

3) Geometrical effects of subtracting $\# \in \mathbb{C}$.

Let A, B, C rep z_1, z_2 & $z_1 - z_2$ resp.

$\Rightarrow z_1 - z_2 = z_1 + (-z_2)$.

$\Rightarrow \vec{OA} - \vec{OB} = \vec{OA} + (-\vec{OB})$.



$\vec{OC} = -\vec{OB} + \vec{OA}$.

effects:

1) $OC = AB$

$\therefore |z_1 - z_2| =$ dist b/w A & B.

($OBAC$ is a //gram)

2) OC // BA

$\therefore \arg(z_1 - z_2) = \measuredangle$ b/w AB & the horizontal through B.

3) Geometrical effects of multiplying 2 $\# \in \mathbb{C}$.

$z_1 = r_1(\cos\theta_1 + i\sin\theta_1)$. $|z_1| = r_1$, $\arg(z_1) = \theta_1$

$= r_1 e^{i\theta_1}$

$z_2 = r_2(\cos\theta_2 + i\sin\theta_2)$, $|z_2| = r_2$, $\arg(z_2) = \theta_2$.

$= r_2 e^{i\theta_2}$

$\Rightarrow z_1 z_2 = r_1 r_2 e^{i(\theta_1+\theta_2)}$.

Effects:

1) $|z_1 z_2| = |z_1||z_2|$

2) $\arg(z_1 z_2) = \arg(z_1) + \arg(z_2)$

Ex $z_1 = 1+i$, $z_2 = 4-3i$

$\therefore |z_1 z_2| = \sqrt{1^2+1^2}\sqrt{4^2+3^2} = 5\sqrt{2}$.

$\arg(z_1 z_2) = \tan^{-1}\left(\frac{1}{1}\right) + \tan^{-1}\left(\frac{-3}{4}\right)$

$= 45 - 36.87$

$= \underline{\underline{8.13°}}$.

4) Geometrical effects of dividing # $\in \mathbb{C}$.

$$\frac{z_1}{z_2} = \frac{r_1 e^{i\theta_1}}{r_2 e^{i\theta_2}}$$
$$= \frac{r_1}{r_2} e^{i(\theta_1 - \theta_2)}.$$

effects:

1) $\left|\frac{z_1}{z_2}\right| = \frac{|z_1|}{|z_2|}$

2) $\arg\left(\frac{z_1}{z_2}\right) = \arg(z_1) - \arg(z_2)$.

Locus problems

→ Let us consider variable # $\in \mathbb{C}$.

if $z = x+iy$,
then the pt $P(x,y)$ rep z in an Argand diagram is a variable pt.

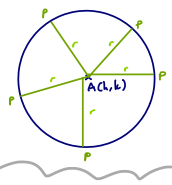
If z varies such that it satisfies certain conditions,
then P will trace out a path in the Argand diagram.
↳ this is denoted as the locus of P.

case 1: $|z - z_1| = r_1$, $z_1 \in \mathbb{C}$, $r_1 \in \mathbb{R}$
z_1, r_1 constant.

A represents z_1.
Hence $|z - z_1| = AP$. Hence $AP = r$.

↳ P varies, such that its dist from A is always equal to r.

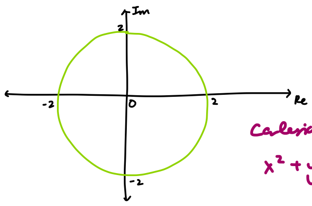
Visualisation:



The locus of P is a circle, with centre (h,k) & radius r.

eg1: $|z| = 2$

$\therefore |z - (0 + 0i)| = 2$.



Cartesian eqn:
$x^2 + y^2 = 4$

Special cases

$|z - (h+ik)| = r$.

1. If $h^2 + k^2 = r^2$ ⟹ locus passes through the origin.

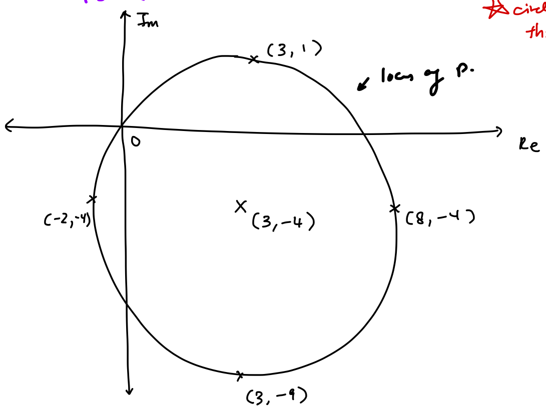
2. If $|h| = r$ ⟹ locus touches the Im axis.

3. If $|k| = r$ ⟹ locus touches the Re axis.

eg2 Sketch the locus $|z - 3 + 4i| = 5$.

$|z - 3 + 4i| = 5$

⟹ $|z - (3 - 4i)| = 5$.



☆ circle must pass through the origin.

case 2 : $|z-z_1| = |z-z_2|$

let $z_1 = a+ib$, $z_2 = c+id$.

$\Rightarrow A(a,b)$ $B(c,d)$

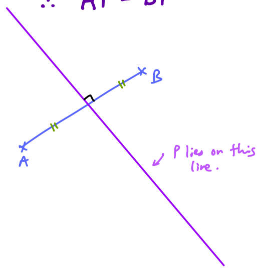
A & B are fixed

$|z-z_1| = AP$

$|z-z_2| = BP$

$\Rightarrow$ P moves, such that its dist from A = dist from B.

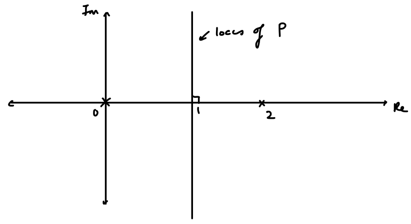
$\therefore \overline{AP} = \overline{BP}$.



✱ P lies on the perpendicular bisector of $\overline{AB}$.

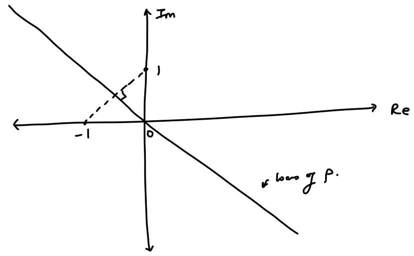
Ex 1 : $|z| = |z-2|$

$\Rightarrow |z-(0+0i)| = |z-(2+0i)|$



Ex 2 : $|z+1| = |z-i|$

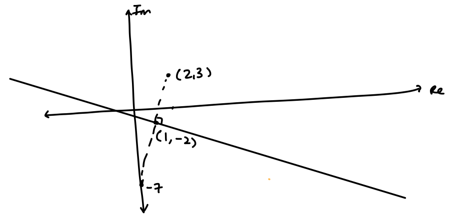
$\Rightarrow |z-(-1+0i)| = |z-(0+i)|$



Cartesian eq^n : $y = -x$

Ex 3 : $|z-2-3i| = |z+7i|$

$\Rightarrow |z-(2+3i)| = |z-(0-7i)|$



$m_{AB} = \frac{3-(-7)}{2-0} = 5.$

$\therefore m_{\perp bis} = -\frac{1}{5}.$

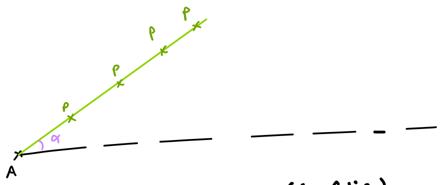
eq^n is $\therefore$ $y-(-2) = -\frac{1}{5}(x-1)$

ie $y-2 = -\frac{1}{5}(x-1)$

case 3 : $\arg(z-z_1) = \alpha$

let A rep z_1 in an Argand diag.

$\therefore \arg(z-z_1) = \angle$ b/w PA & horizontal line through A



The locus is a line segment (half line) drawn from A & having a directed angle α w/ the real axis

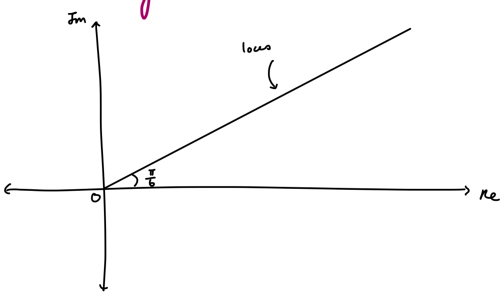
$m = \tan\alpha$

$\Rightarrow y-k = \tan\alpha\,(x-h)$, $x > h$

if we let $z_1 = h+ki$

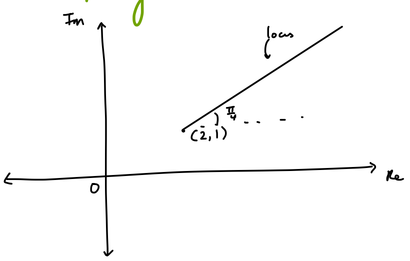
Ex 1 : $\arg z = \frac{\pi}{6}$

$\therefore \arg(z-(0+0i)) = \frac{\pi}{6}.$



Ex 2 : $\arg(z-2-i) = \frac{\pi}{4}$

$\Rightarrow \arg(z-(2+i)) = \frac{\pi}{4}$



$m = \tan(\frac{\pi}{4})$
$= 1.$

$\therefore$ eq^n of locus

$\Rightarrow y-1 = x-2$

ie $y = x-1$, $\underline{\underline{x > 2.}}$

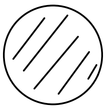
case 4 : $\arg(\frac{1}{z}) = \alpha.$

$\arg \underline{1} - \arg z = \alpha$

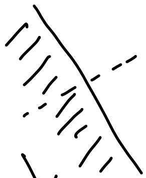
0 .

$\therefore \arg z = -\alpha.$

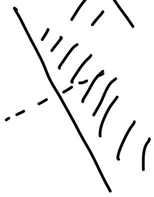
Inequalities of a # $\in \mathbb{C}$.

$|z-z_1| \leq r \Rightarrow$ 

$|z-z_1| < r \Rightarrow$ 

$|z-z_1| \leq |z-z_2| \Rightarrow$ 

$AP \leq BP$

$|z-z_1| \geq |z-z_2| \Rightarrow$ 

$AP \geq BP$

$\alpha \leq \arg(z-z_1) \leq \beta$

Euler's / exponential form

$z = r(\cos\theta + i\sin\theta)$

$\quad = re^{i\theta}$.

$(r = |z|,\ \theta = \arg(z)$ in radians$)$.

- Sqrt of a complex # in exponential form.

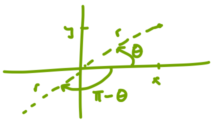
(& give ans also in exp form)

$z = \pm(x+iy)$ $\quad \therefore z = x+iy,\ z = -x-iy$

r is the same

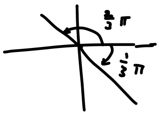
θ is not

1



$\boxed{\therefore \arg(-x-iy) = \theta - \pi.}$

Ex $\quad z^2 = 16e^{-\frac{2}{3}\pi i}$.



$\therefore z = \pm 4e^{-\frac{1}{3}\pi i}$

$\therefore z = 4e^{-\frac{1}{3}\pi i}, \quad z = 4e^{\frac{2}{3}\pi i}$.

Chapter 10: Vectors

→ consists of P1 vectors + other stuff

Recall :

* Position vector of A ⇒ $\overrightarrow{OA}$
  If A has coords (a_1, a_2, a_3)
  Then $\overrightarrow{OA} = a_1\underset{\sim}{i} + a_2\underset{\sim}{j} + a_3\underset{\sim}{k} = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix}$

* Displacement vector $\overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA}$

* Pos vector of midp of $\overrightarrow{AB} = \frac{1}{2}(\overrightarrow{OA} + \overrightarrow{OB})$

* Scalar product : $\underset{\sim}{a} \cdot \underset{\sim}{b} = |a||b| \cos\theta$
  $\begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} \cdot \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix} = a_1b_1 + a_2b_2 + a_3b_3$

* To find $\angle ABC$ → consider $\overrightarrow{BA} \cdot \overrightarrow{BC}$, find θ.

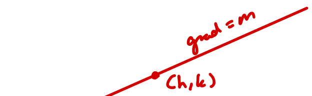
* if $\underset{\sim}{a} \cdot \underset{\sim}{b} = 0$ → $a \perp b$

Additional Theory

→ To find a vector eq^{n} of a straight line

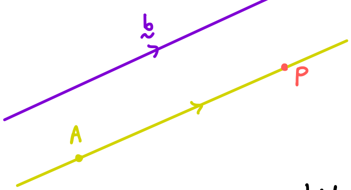
A straight line is defined by a vector parallel to the line.
1) a pt on the line, &
or 2) 2 pts on the line.

Recall : Cartesian coords



→ $y - k = m(x - h)$

Consider the system



which consists of a line, which passes thru the pt A, w/ pos vector $\underset{\sim}{a}$ & is // to $\underset{\sim}{b}$.

Let P be a general pt on the line.
∴ $\overrightarrow{AP}$ is // to $\underset{\sim}{b}$.
ie $\overrightarrow{AP} = \lambda\underset{\sim}{b}$, λ = scalar

Then $\overrightarrow{OP} - \overrightarrow{OA} = \lambda\underset{\sim}{b}$

$\boxed{\Rightarrow \overrightarrow{OP} = \overrightarrow{OA} + \lambda\underset{\sim}{b}.}$

↳ notation :

• $\underset{\sim}{r}$ is used to denote the pos vector of any pt on the line.

↳ $\overrightarrow{OP} = \underset{\sim}{r}$.

$\boxed{↳ \underset{\sim}{r} = \underset{\sim}{a} + \lambda\underset{\sim}{b}.}$ ⇒ [This is referred to as the $\boxed{\text{vector eq}^{n}}$ of the line.]

pv of any pt on ℓ

pv on a particular pt on ℓ

a vector // to ℓ / direction vector of ℓ

a scalar / parameter.

To find a pt on ℓ ⟺ find value of λ.

eg^{1} $\underset{\sim}{r} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} + \lambda\begin{pmatrix} -2 \\ 0 \\ 5 \end{pmatrix}$

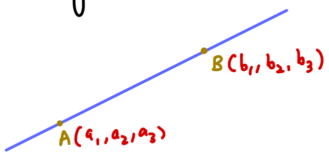
This is the vector eq^{n} of a straight line through the pt (1,2,3), and // to the vector $\begin{pmatrix} -2 \\ 0 \\ 5 \end{pmatrix}$.

$\lambda = 1$ → $\underset{\sim}{r} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} + \begin{pmatrix} -2 \\ 0 \\ 5 \end{pmatrix} = \begin{pmatrix} -1 \\ 2 \\ 8 \end{pmatrix}$ ⇒ (−1,2,8) is a pt on ℓ

eg^{2} $\underset{\sim}{r} = t\begin{pmatrix} 1 \\ -4 \\ 2 \end{pmatrix}$

→ Line ℓ passes through origin & // to $\begin{pmatrix} 1 \\ -4 \\ 2 \end{pmatrix}$

→ To find a vector eqⁿ of the line through A & B



$$\underset{\sim}{r} = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} + \lambda \begin{pmatrix} b_1 - a_1 \\ b_2 - a_2 \\ b_3 - a_3 \end{pmatrix}$$

$\vec{OA}$ or $\vec{OB}$ $\qquad$ $\vec{AB}$ or $\vec{BA}$

⇒ To det whether a given pt lies on the line:

Coord geometry

eg $y + 2x = 5$ $\qquad$ $(1,3)$ is on the line bc $3 + 2(1) = 5$

Vectors

eg $\underset{\sim}{r} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} + \lambda \begin{pmatrix} -3 \\ 0 \\ 4 \end{pmatrix}$.

Is $(4,2,-1)$ on the line?

$$\Rightarrow \left.\begin{matrix} 1 - 3\lambda = 4 \ \therefore \lambda = -1 \\ 2 + 0 = 2 \quad \lambda \in \mathbb{R} \\ 3 + 4\lambda = -1 \ \therefore \lambda = -1 \end{matrix}\right\} \therefore \lambda = -1, \text{ only.}$$

Hence $(4,2,-1)$ is on the line.

⇒ To show that a pt does not lie on a line:

eg $\underset{\sim}{r} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} + \lambda \begin{pmatrix} -3 \\ 0 \\ 4 \end{pmatrix}$

Ex: show that $(5,8,-11)$ does not lie on the line.

Proof by contradiction:

If $(5,8,-11)$ lies on the ℓ_1, λ has a definite value.

ie $\begin{pmatrix} 5 \\ 8 \\ -11 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} + \lambda \begin{pmatrix} -3 \\ 0 \\ 4 \end{pmatrix}$

Then $5 = 1 - 3\lambda \rightarrow \lambda = -\frac{4}{3}$

$8 = 2 \rightarrow$ contradiction!

Hence $(5,8,-11)$ does not lie on the line.

How to determine whether 2 vector eqⁿs are identical?

Consider:

$$\ell_1 = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} + \lambda \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}$$

$$\ell_2 = \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix} + \mu \begin{pmatrix} d_1 \\ d_2 \\ d_3 \end{pmatrix}$$

If:

1) $\begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix} = k \begin{pmatrix} d_1 \\ d_2 \\ d_3 \end{pmatrix}$ (k is scalar)

$\Rightarrow \ell_1 // \ell_2$.

2) If $\vec{AC} // \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}$ or C lies on ℓ_1.

$\Rightarrow \ell_1 \equiv \ell_2$.

→ To determine whether 2 lines in 3-D space intersect / non-intersecting (skew lines)

Let $\ell_1 \mapsto \underset{\sim}{r} = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} + \lambda \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}$

$\ell_2 \mapsto \underset{\sim}{r} = \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix} + \mu \begin{pmatrix} d_1 \\ d_2 \\ d_3 \end{pmatrix}$

1) If $\begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix} = k \begin{pmatrix} d_1 \\ d_2 \\ d_3 \end{pmatrix}$, then $\ell_1 // \ell_2$

2) If $\begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix} \neq k \begin{pmatrix} d_1 \\ d_2 \\ d_3 \end{pmatrix}$, then either:

i) ℓ_1 intersects ℓ_2, or

ii) ℓ_1 & ℓ_2 are skew lines / non-intersecting lines.

Consider the 2nd case.

If the 2 lines intersect:

then at the pt of intersectⁿ,

$$\begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} + \lambda \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix} = \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix} + \mu \begin{pmatrix} d_1 \\ d_2 \\ d_3 \end{pmatrix}.$$

Equating components:

$$a_1 + b_1\lambda = c_1 + d_1\mu$$
$$a_2 + b_2\lambda = c_2 + d_2\mu$$
$$a_3 + b_3\lambda = c_3 + d_3\mu.$$

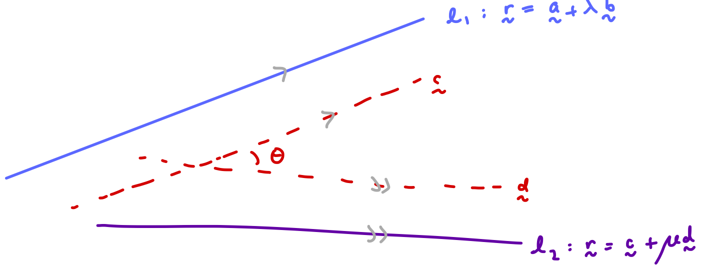
↳ solve 2 of the 3 eqⁿs to find λ, μ, and check whether these values satisfy the remaining eqⁿ.

→ if they satisfy the eqⁿ,
↳ ℓ_1 & ℓ_2 intersect.

if they don't,
↳ they are skew lines.

→ To find the acute ∡ b/w 2 non-parallel lines. (intersecting or not)

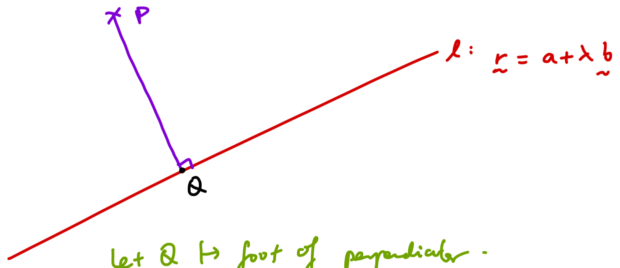
↳ scalar product.



Acute ∡ b/w ℓ_1 & ℓ_2 ≡ acute ∡ b/w their direction vectors.

ie $\underset{\sim}{b} \cdot \underset{\sim}{d} = |\underset{\sim}{b}||\underset{\sim}{d}| \cos\theta$

To find the coords of the foot of the perpendicular from a pt to a line & hence find the perpendicular dist.



Let Q $\mapsto$ foot of perpendicular.

Since Q lies on l, $\overrightarrow{OQ} = \underset{\sim}{a} + \lambda \underset{\sim}{b}$ for a particular value of λ to be determined.

$PQ \perp l \Rightarrow \overrightarrow{PQ} \perp \underset{\sim}{b}$.

$\therefore \overrightarrow{PQ} \cdot \underset{\sim}{b} = 0 \rightarrow$ we can find λ.
"nice number".

The $\perp$ dist $= |\overrightarrow{PQ}|$.