STAT 240
Personal Notes

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Chapter 1: What is Probability?

RANDOM EXPERIMENTS (1.1)

A "random experiment" is the process of obtaining a random observed result.

Random experiments can be split into two types:

1. Controlled experiments; and
   eg flipping a coin, rolling a die
2. Observational studies.
   eg # of students taking STAT 240 in F2021

FEATURES OF RANDOM EXPERIMENTS

Note that random experiments have the following common features:

1. The outcomes/results cannot be predicted with certainty; and
2. All the possible outcomes are known beforehand with certainty.

SAMPLE SPACE (1.2)

OUTCOME

An "outcome" is an observed result of interest from a random experiment.

eg the number rolled after rolling a die.

SAMPLE SPACE

The "sample space" of a random experiment is the set of all possible distinct outcomes of said experiment.

eg when rolling a 6-sided die:

$S = \{1, 2, 3, 4, 5, 6\}$

EVENTS

An "event" of a random experiment is a group or set of outcomes of said experiment; ie subsets of the sample space.

There are two types of events:

1. Simple events – consist of one outcome
   eg rolling a 1 on a 6-sided die:
   $S = \{1, \dots, 6\}$ $E = \{1\}$
2. Compound events – consist of multiple outcomes
   eg rolling an odd number on a 6-sided die:
   $S = \{1, \dots, 6\}$ $E = \{1, 3, 5\}$

Note that

1. Two simple events will never occur simultaneously;
   eg can never roll a 1 & 3 at the same time with one die
2. A compound event occurs if and only if one of "its" simple events occurs; and
   eg odd # rolled $\Leftrightarrow$ 1 rolled or 3 rolled or 5 rolled (on a 6-sided die)
3. Two compound events can occur simultaneously.
   eg 3 rolled $\Rightarrow$ $\begin{cases} \text{odd number rolled } (E = \{1,3,5\}); \text{ and} \\ \text{multiple of 3 rolled } (E = \{3,6\}). \end{cases}$

DEFINITIONS OF PROBABILITY (1.3)

"Probability" is a quantitative measure of how likely an event is to occur.

CLASSICAL DEFINITION

The "classical definition" of probability states that each distinct outcome in the sample space is equally likely to occur.

In this case, the probability of an event E is equal to

$$P(E) = \frac{\#\text{ of ways event can occur}}{\#\text{ of outcomes in the sample space}}.$$

eg roll a 6-sided die once.
E = number is odd.
$\Rightarrow E = \{1,3,5\}$, $S = \{1,2,3,4,5,6\}$.
So $P(E) = \frac{3}{6} = \frac{1}{2}$.

RELATIVE FREQUENCY DEFINITION

The "relative frequency" definition of probability states that the probability of an event occurring is the proportion it occurs in a very long series of repetitions of the experiment.

eg rolling a 6-sided die 300 times
$\Rightarrow$ 3 shows up 49 of those 300 times
$\Rightarrow$ so $P(\text{die} = 3) \approx \frac{49}{300} \approx 1/6$.

SUBJECTIVE PROBABILITY DEFINITION

In the "subjective probability" definition of probability, the probability of an event is determined by an opinion (ie what a person thinks the probability is).

eg the probability of COVID-19 being eradicated by 2022.

Note that this plays a role in fields like "Bayesian Statistics".

DISCRETE PROBABILITY MODELS (1.4)

In discrete probability models:

1. The sample space S satisfies $|S| \leq |\mathbb{N}|$;
   ie there are either a finite or countably infinite number of basic events; and
2. Each probability p_i satisfies $0 \leq p_i \leq 1$; and
3. The probabilities of each basic event sum to 1;
   ie $\sum p_i = 1$.

CLASSIC DISCRETE MODELS (1.5)

In classic discrete models:

1. The sample space S satisfies $|S| < |\mathbb{N}|$ (ie it is finite); and
2. All basic events are equally likely to occur;
   ie $P(a_1) = \dots = P(a_{|S|}) = \frac{1}{|S|}$.

Chapter 2: Counting Techniques

FULL FACTORIAL: $n!$ (2.1)

The factorial of n, denoted as "$n!$" and defined to be

$$n! = n(n-1)\cdots 1$$

is the number of ways to put n distinguishable objects in a row.

COMBINATIONS: C^r_n OR ${}^nC_r$ (2.2)

"n choose r", denoted as "C^r_n" or "${}^nC_r$", defined to be

$$C^r_n = \frac{n!}{r!(n-r)!} = \frac{n(n-1)\cdots(n-(r-1))}{r!}$$

is the number of ways to select r objects from n distinguishable objects.

PERMUTATIONS: P^r_n OR ${}^nP_r$ (2.3)

"P^r_n" or "${}^nP_r$", defined to be

$$P^r_n = \frac{n!}{(n-r)!} = n(n-1)\cdots(n-(r-1)) = C^n_r \cdot r!$$

is the number of ways to select r objects from n distinguishable objects and put them in a row.

GENERALIZATION OF COMBINATIONS (2.4)

We can show the number of ways to arrange n objects in a row, where n_1 objects are of type 1, n_2 objects are of type 2, .. , n_k objects are of type k, where $n_1 + n_2 + \cdots + n_k = n$, is

$$\#\text{ of outcomes} = \frac{n!}{n_1! \cdots n_k!} = C^{n_1}_n C^{n_2}_{n-n_1} C^{n_3}_{n-n_1-n_2} \cdots C^{n_{k-1}}_{n_k+n_{k-1}} C^{n_k}_{n_k}.$$

eg Roll a die 4 times. Find P(the sum = 10).

Sol$^{\underline{n}}$. This is equivalent to distributing 10 balls into 4 sections, where each section has at least 1 ball.

9 different spaces for the "dividers", 4 "dividers"

$\Rightarrow C^9_4$ ways of "positioning" the dividers.

But, we exclude the options where one of the sections has 7 balls, ie 7111, 1711, 1171, 1117.

Hence $P(\text{event}) = \dfrac{C^9_4 - 4}{6^4}$, since there are 6^4 outcomes of rolling a 6 sided die twice. ⊠

STARS & BARS WITHOUT "EMPTY" SECTIONS

Given n stars, the # of ways to divide them up into k sections with $k-1$ rods without one of the sections containing zero elements is

$$\# = C^{n-1}_{k-1}.$$

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eg $n=5$, $k=4$

STARS & BARS WITH "EMPTY" SECTIONS

Given n stars, the # of ways to divide them up into k sections with $k-1$ rods with one (or more) sections containing zero elements is

$$\# = C^{k-1}_{n+k-1}.$$

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eg $n=5$, $k=4$

2nd section has no elements.

Chapter 3: Probability Rules

RELATIONS AMONGST EVENTS (3.1)

EVERY EVENT $\subseteq S$ (THE "CERTAIN" EVENT)

Let A be an event.
Then necessarily
$A \subseteq S = \{$the event that $\underline{\text{always}}$ occurs$\}$.

$\emptyset$ (THE "IMPOSSIBLE" EVENT)

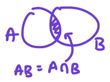
We use "$\emptyset$" to denote the event that never occurs.

UNION OF EVENTS: $A \cup B$

Let A, B be events.
Then "$A \cup B$" is the event that $\underline{\text{at least}}$ $\underline{\text{one}}$ of the two occurs.

INTERSECTION OF EVENTS: $A \cap B$

Let A, B be events.
Then, "$A \cap B$" is the event that $\underline{\text{both}}$ A & B occur.
We also denote $A \cap B = AB$.



MUTUALLY EXCLUSIVE / DISJOINT

Let A, B be events.
Then, we say A & B are "mutually exclusive" (or "disjoint") if $A \cap B = \emptyset$.

INCLUSION: $A \subseteq B$

Let A, B be events.
Then, we say "$A \subseteq B$" if B occurs whenever A occurs; ie
A occurs $\Rightarrow$ B occurs.

COMPLEMENT: $A^c = \bar{A}$

Let A be an event.
Then, $\bar{A}$ is the event such that
$\bar{A}$ occurs $\Leftrightarrow$ A $\underline{\text{does not}}$ occur.

PARTITION OF S

Let $B_1, \dots, B_n$ be events.
Then, we say $B_1, \dots, B_n$ form a "partition" of S if
$B_1 \cup \dots \cup B_n = S$ & $B_i \cap B_j = \emptyset \quad \forall i \neq j$.

PROBABILITY RULES (3.2)

A probability function $P: \mathcal{P}(S) \to [0,1]$ is any function that satisfies the following for any $A, B \subseteq S$:

① $P(\emptyset) = 0$;
② $P(S) = 1$;
③ $P(A) \geq 0 \quad \forall A \subseteq S$; (non-negativity)
④ $A \subseteq B \Rightarrow P(A) \leq P(B)$;
⑤ $P(A \cup B) = P(A) + P(B) - P(A \cap B)$; & (addition law of probability)
- this generalizes to more variables as well.

⑥ $P(A^c) = 1 - P(A)$.

Chapter 4: Conditional Probability and Event Independence

CONDITIONAL PROBABILITY (4.1)

Let A, B be events.
Then, the probability that A happens given B already happens, denoted as "$P(A|B)$", is equal to

$$P(A|B) = \frac{P(A \cap B)}{P(B)}.$$

*note $P(B) \neq 0$ necessarily.

INDEPENDENCE [OF TWO EVENTS] (4.2)

1. Let A, B be events.
Then, we say A & B are "independent" if and only if

$$P(A \cap B) = P(A)\,P(B).$$

2. Note that if $P(A), P(B) \neq 0$, then A & B cannot be mutually exclusive (ie $P(A \cap B) = 0$) if they are independent.

3. If A & B are independent, then
① A & B^c are independent;
② A^c & B are independent; and
③ A^c & B^c are independent.

4. Note that independence arises from independent random events.

INDEPENDENCE [OF > TWO EVENTS] (4.3)

1. Let $A_1, \dots, A_n$ be n events.
Then, we say $A_1, \dots, A_n$ are (mutually) independent if

$$P(A_{n_1} A_{n_2} \cdots A_{n_k}) = P(A_{n_1}) \cdots P(A_{n_k}) \quad \forall \{n_1, \dots, n_k\} \in \mathcal{P}(\{1, \dots, n\}).$$

2. For the $n=3$ case, A_1, A_2 & A_3 are independent if
① $P(A_1 A_2) = P(A_1)\,P(A_2)$;
② $P(A_1 A_3) = P(A_1)\,P(A_3)$;
③ $P(A_2 A_3) = P(A_2)\,P(A_3)$; and
④ $P(A_1 A_2 A_3) = P(A_1)\,P(A_2)\,P(A_3)$.

$A_1, \dots, A_n$ ARE INDEPENDENT $\Rightarrow P\left(\prod_{i=1}^{n} A_i\right) = \prod_{i=1}^{n} P(A_i | A_1 \cdots A_{i-1})$ (THE MULTIPLICATION FORMULA) (4.4.1)

Let $A_1, \dots, A_n$ be independent events.
Then necessarily

$$P(A_1 \cdots A_n) = P(A_1)\,P(A_2|A_1)\,P(A_3|A_1 A_2) \cdots P(A_n | A_1 \cdots A_{n-1}).$$

Proof. Note that for any $k = 1, \dots, n$, we have

$$P(A_k | A_1 \cdots A_{k-1}) = \frac{P(A_1 \cdots A_{k-1} A_k)}{P(A_1 \cdots A_{k-1})} = P(A_k).$$

The proof follows trivially. ⊠

$A_1, \dots, A_n$ PARTITION $S \Rightarrow P(B) = \sum_{i=1}^{\infty} P(A_i)\,P(B|A_i)$ (TOTAL PROBABILITY FORMULA) (4.4.2)

Let $A_1, A_2, \dots$ form a partition of S, ie we have that $A_i A_j = \emptyset \ \forall i \neq j$ & $\bigcup_{i=1}^{\infty} A_i = S$.
Let B be an event. Then necessarily

$$P(B) = P(BS) = \sum_{i=1}^{\infty} P(A_i B) = \sum_{i=1}^{\infty} P(A_i)\,P(B|A_i).$$

* this also works for finite collections of events as well.

$A_1, \dots, A_n$ PARTITION $S \Rightarrow P(A_i|B) = \frac{P(A_i)P(B|A_i)}{\sum_{j=1}^{\infty} P(A_j)P(B|A_j)}$ (THE BAYES FORMULA) (4.4.3)

Let $A_1, A_2, \dots$ form a partition of S, and let B be such that $P(B) \neq 0$.
Then necessarily, for any $i \in \mathbb{N}$, we have that

$$P(A_i|B) = \frac{P(A_i B)}{P(B)} = \frac{P(A_i)\,P(B|A_i)}{P(B)} = \frac{P(A_i)\,P(B|A_i)}{\sum_{j=1}^{\infty} P(A_j)\,P(B|A_j)}.$$

*again, this also generalises to the finite case.

Chapter 5: Discrete Random Variables and Probability Models

RANDOM VARIABLES (5.1)

RANDOM VARIABLE (RV) (DI)

1. Let S be a sample space. Then, a "random variable" is defined to be some $X : S \to \mathbb{R}$.
2. Note that we usually denote random variables by capital letters. (eg X, Y, Z, etc)

DISCRETE [r.v.]

1. Let $X \in \mathbb{R}^S$ be a r.v. Then, we say X is "discrete" if $|\text{range}(X)| \leq |\mathbb{N}|$.

PROBABILITY MASS FUNCTION (PMF)

1. Let $X \in \mathbb{R}^S$ be a r.v. Then, the "probability mass function" (or p.m.f) of X is defined to be the function $f : \text{range}(X) \to [0,1]$ by
$$f(x) = P[X = x] \quad \forall x \in \text{range}(X).$$
2. By construction of f, note that
$$\sum_{x \in X} f(x) = 1.$$

CUMULATIVE DISTRIBUTION FUNCTION (CDF)

1. Let $X \in \mathbb{R}^S$ be a r.v. Then, the "cumulative distribution function" (or "cdf") of X is defined to be the function $F : \mathbb{R} \to [0,1]$ by
$$F(x) = P[X \leq x] \quad \forall x \in \mathbb{R}.$$
2. Properties of cdf:
   (1) $F(x_1) \leq F(x_2) \iff x_1 \leq x_2 \quad \forall x_1, x_2 \in \mathbb{R}$; and
   (2) $\lim_{x \to -\infty} F(x) = 0$ & $\lim_{x \to +\infty} F(x) = 1$.

PMF CAN BE OBTAINED BY CDF, AND VICE VERSA

1. Let $X \in \mathbb{R}^S$ be discrete. Then, given the pmf f of X, we can obtain X's cdf F, and vice versa.

Proof. Let $x \in \text{range}(X)$. See that
$f(x) = P[X=x] = P[X \leq x] - P[X \leq x - \varepsilon] = F(x) - F(x-\varepsilon)$,
where $\varepsilon > 0$ is such that $\text{range}(X) \cap [x-\varepsilon, x] = \{x\}$.
(Since X is discrete, such an ε will exist.)

FINDING PMF (EI)

Let X be the number of heads after flipping a fair coin n times. Find the pmf of X.

Sol^{n}. See that $\text{range}(X) = \{0, 1, \dots, n\}$.
Then
$$P[X=k] = C_n^k \left(\tfrac{1}{2}\right)^k \left(\tfrac{1}{2}\right)^{n-k} = C_n^k \left(\tfrac{1}{2}\right)^n,$$
and so the pmf of X is $f : \{0, \dots, n\} \to [0,1]$ given by
$$f(k) = P[X=k] = C_n^k \left(\tfrac{1}{2}\right)^n \quad \forall k = 0, \dots, n. \quad \#$$

BERNOULLI TRIALS & RELATED RV (5.2)

BERNOULLI TRIALS (5.2.1)

1. A "Bernoulli trial" focuses on a particular random experiment with only two possible outcomes: success or failure.
2. We call the random variables and the experiment obtained from Bernoulli trials as "Bernoulli random variables" and "Bernoulli experiment" respectively.

BERNOULLI RV (5.2.2)

1. In particular, if B is a Bernoulli rv:
   (1) then $P[B = \text{Success}]$, or $P(B)$, is equal to $P(B) = p$ (where p = probability of success); and
   (2) $P[B = \text{Failure}]$, or $P(B^c)$, is equal to $P(B^c) = 1-p$.
2. Thus, the pmf of B is $f : \{0,1\} \to [0,1]$ by $f(0) = 1-p$ & $f(1) = p$, or equivalently by
$$f(x) = p^x (1-p)^{1-x} \quad \forall x \in \{0,1\}.$$

BERNOULLI SEQUENCE (5.2.3)

1. A "Bernoulli sequence" occurs when
   (1) we repeat a Bernoulli trial many times;
   (2) the results are all independent; and
   (3) the success probability p stays the same.

BINOMIAL DISTRIBUTION: $X \sim \text{Binomial}(n,p)$ / $X \sim \text{Bin}(n,p)$ (5.2.4)

1. Let X be the rv equal to the number of successes after repeating a Bernoulli trial n times independently, with probability of success p.
   Then, we say X follows a binomial distribution, and write $X \sim \text{Binomial}(n,p)$.
2. In this case, the pmf of X is equal to
$$f : \{0, \dots, n\} \to [0,1] \text{ by } f(k) = C_n^k p^k (1-p)^{n-k} \quad \forall 0 \leq k \leq n.$$

GEOMETRIC DISTRIBUTION: $X \sim \text{Geometric}(p)$ / $X \sim \text{Geo}(p)$ (5.2.5)

1. Repeat independent Bernoulli trials, with success probability p, until a trial is successful. Let the rv X be equal to the number of failures before the success was reached.
   Then, we say X follows a geometric distribution, and write $X \sim \text{Geometric}(p)$.
2. In this case, the pmf of X is equal to
$$f : \mathbb{N} \to [0,1] \text{ by } f(k) = (1-p)^k p \quad \forall k \in \mathbb{N}.$$
3. Note that
   (1) $P(X \geq n) = (1-p)^n \quad \forall n \in \mathbb{N}$; and

   Proof. $P(X \geq n) = \sum_{k=n}^{\infty} (1-p)^k p = (1-p)^n p \sum_{k=0}^{\infty} (1-p)^k = (1-p)^n p \left(\frac{1}{p}\right) = (1-p)^n$. $\boxtimes$

   (2) $P(X \geq m+n \mid X \geq n) = P(X \geq m) \quad \forall m, n \in \mathbb{N}$ (the memory-less property).

   Proof. $P(X \geq m+n \mid X \geq n) = \frac{P(X \geq m+n \,\cap\, X \geq n)}{P(X \geq n)} = \frac{P(X \geq m+n)}{P(X \geq n)} = \frac{(1-p)^{m+n}}{(1-p)^n} = (1-p)^m = P(X \geq m)$. $\boxtimes$

NEGATIVE BINOMIAL DISTRIBUTION:

$X \sim$ Negative Binomial(k,p) / $X \sim NB(k,p)$ (5.2.6)

1. Repeat independent Bernoulli trials, with success probability p, until the k^{th} success is reached.
   Let the rv X be the number of failures before the k^{th} success.
   Then, we say X follows a negative binomial distribution, and write $X \sim$ Negative Binomial(k,p).

2. In this case, the pmf of X is equal to $f: \mathbb{N} \to [0,1]$ by $f(n) = C^n_{n+k-1}\, p^k (1-p)^n \quad \forall n \in \mathbb{N}$.

Proof. See that

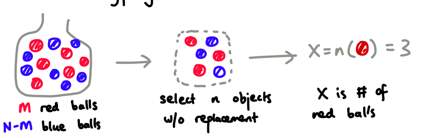
$$\begin{aligned} P[X=n] &= P[\text{having } n \text{ failures before } k^{th} \text{ success}] \\ &= P[n \text{ failures } \& \text{ } k-1 \text{ successes, followed by } k^{th} \text{ success}] \\ &= \frac{(n+k-1)!}{n!(k-1)!}(1-p)^n (p)^{k-1} \cdot p \end{aligned}$$

$$\therefore P[X=n] = C^n_{n+k-1}(1-p)^n p^k. \quad \blacksquare$$

HYPERGEOMETRIC DISTRIBUTION:

$X \sim$ Hypergeometric(N, M, n) / $X \sim Hyp(N,M,n)$ (5.3)

1. Suppose we have a collection of N objects; M of one type, and $N-M$ of another (distinct) type.
   Randomly select n objects without replacement, where $n \leq \min\{M, N-M\}$.
   Let the rv X be the number of objects of the first type in these n objects.
   Then, we say X follows a "hypergeometric distribution", and write
   $X \sim$ Hypergeometric(N, M, n).



2. In this case, the pmf of X is equal to $f: \{0, \dots, n\} \to [0,1]$ by $f(k) = \dfrac{\binom{M}{k}\binom{N-M}{n-k}}{\binom{N}{n}} \quad \forall k = 0, \dots, n$.

VANDERMONDE'S IDENTITY: $\sum\limits_{k=0}^{n} \binom{M}{k}\binom{N-M}{n-k} = \binom{N}{n}$

Let $n \leq M, N-M$.
Then necessarily
$\binom{N}{n} = \sum\limits_{k=0}^{n} \binom{M}{k}\binom{N-M}{n-k}$.

POISSON DISTRIBUTION:

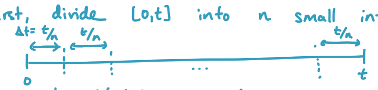
$X \sim$ Poisson(λ) / $X \sim$ Poi(λ) (5.4)

1. In some observational studies, events happen over time or space.
   We say such an event follows a Poisson process if the following conditions are satisfied:
   ① Events in non-overlap time intervals are independent; } independence
   ② $P[\geq 2 \text{ events in } [t, t+\Delta t)] = o(\Delta t)$, where $\lim\limits_{\Delta t \to 0} \frac{o(\Delta t)}{\Delta t} = 0$ and $\Delta t \ll t$; and } individuality
   ③ $P[\text{one event in } [t, t+\Delta t)] = \lambda \Delta t + o(\Delta t)$, $\lambda \in \mathbb{R}$. } homogeneity
   Note that we call "λ" in ③ the "intensity parameter".

2. Let the rv X be the number of events in $[0,t]$.
   Then we say X follows a Poisson distribution, and write
   $X \sim$ Poisson(λ).

3. In this case, the pmf of X is given by $f: \mathbb{N} \to [0,1]$ by $f(x) = \dfrac{e^{-\lambda t}(\lambda t)^x}{x!}$.

Proof. First, divide $[0,t]$ into n small intervals:



Note that $\Delta t \to 0$ as $n \to \infty$.
Let the events
$B_1^{(n,x)}$ = there are x small intervals each with one event; &
$B_2^{(n)}$ = ≥ 1 small interval exists with two or more events.

Then, see that

$$\begin{aligned} P(B_1^{(n,x)}) &= \binom{n}{x}\left(P(\text{one event in interval of length } \tfrac{t}{n} = \Delta t)\right)^x (1-p)^{n-x} \quad \text{(by binomial dist}^{\underline{n}}) \\ &= \binom{n}{x}\left(\lambda \tfrac{t}{n} + o(\tfrac{t}{n})\right)^x \left(1 - \lambda\tfrac{t}{n} - o(\tfrac{t}{n})\right)^{n-x}. \quad \text{(by point ③ of defn)} \end{aligned}$$

Notice that since we want to consider infinitely small periods of time for our Poisson variable, we can deduce that

$$\begin{aligned} P(X=x) &= \lim_{n\to\infty} P(B_1^{(n,x)}) \\ &= \lim_{n\to\infty}\left[\binom{n}{x}\left(\lambda\tfrac{t}{n} + o(\tfrac{t}{n})\right)^x\left(1-\lambda\tfrac{t}{n} - o(\tfrac{t}{n})\right)^{n-x}\right] \\ &= \lim_{n\to\infty}\left[\frac{n!}{x!(n-x)!}\left(\lambda\tfrac{t}{n} + o(\tfrac{t}{n})\right)^x\left(1-\lambda\tfrac{t}{n} - o(\tfrac{t}{n})\right)^{n-x}\right] \\ &= \lim_{n\to\infty}\left[\frac{1}{x!}\frac{n(n-1)\cdots(n-x+1)}{n^x}\left(\lambda t + n\,o(\tfrac{t}{n})\right)^x\left(1 - \tfrac{\lambda t}{n} - o(\tfrac{t}{n})\right)^n \cdot \left(1 - \tfrac{\lambda t}{n} - o(\tfrac{t}{n})\right)^{-x}\right] \\ &= \frac{1}{x!}(1)(\lambda t)^x \lim_{n\to\infty}\left(1 - \frac{\lambda t}{n}\right)^n (1) \\ &= \frac{1}{x!}(\lambda t)^x e^{-\lambda t} \qquad \text{(using the identity } e^x = \lim_{n\to\infty}(1+\tfrac{x}{n})^n) \end{aligned}$$

$$\therefore P(X=x) = \frac{e^{-\lambda t}(\lambda t)^x}{x!},$$

as needed $\blacksquare$

Chapter 6: Expectation and Variance

EXPECTED VALUE / EXPECTATION [OF A DISC RV] (6.1)

Let X be a drv, with $\text{ran}(X) = A$ & pmf $f(x)$. Then, the "expectation" or "expected value" of X, denoted as "$E(X)$", is defined to be equal to

$$E(X) = \sum_{x \in A} x f(x). \quad \text{(D1)}$$

Note to calculate expectations, we need to:

① Identify the rv X involved;
② Find the pmf of X; and
③ Compute $E(X)$.

$X \sim \text{Bernoulli}(p) \Rightarrow E(X) = p$ (6.2.1)

Let $X \sim \text{Bernoulli}(p)$. Then necessarily $E(X) = p$.

Proof.
$$\begin{aligned} E(X) &= \sum_{x \in \{0,1\}} x f(x) \\ &= 0P(X^c) + 1P(X) \\ &= p. \quad \blacksquare \end{aligned}$$

$X \sim \text{Binomial}(n,p) \Rightarrow E(X) = np$ (6.2.2)

Let $X \sim \text{Binomial}(n,p)$. Then necessarily $E(X) = np$.

Proof.
$$\begin{aligned} E(X) &= \sum_{k=0}^{n} k P[X=k] \\ &= \sum_{k=0}^{n} k\left(\binom{n}{k} p^k (1-p)^{n-k}\right) \\ &= \sum_{k=1}^{n} k \frac{n!}{k!(n-k)!} p^k (1-p)^{n-k} \\ &= np \sum_{k=1}^{n} \frac{(n-1)!}{(k-1)!(n-k)!} p^{k-1} (1-p)^{n-k} \\ &= np \sum_{k=0}^{n-1} \frac{(n-1)!}{k!(n-k-1)!} p^k (1-p)^{n-k-1} \\ &= np(1) \quad \text{(by Bin Formula)} \end{aligned}$$

$\therefore E(X) = np.$ $\blacksquare$

$X \sim \text{Geometric}(p) \Rightarrow E(X) = \frac{1-p}{p}$ (6.2.3)

Let $X \sim \text{Geometric}(p)$. Then necessarily $E(X) = \frac{1-p}{p}$.

Proof.
$$\begin{aligned} E(X) &= \sum_{k=0}^{\infty} k \cdot P(X=k) \\ &= \sum_{k=0}^{\infty} k(1-p)^k p. \\ &= p(1-p)\sum_{k=1}^{\infty} k(1-p)^{k-1}. \end{aligned}$$

Recall the identity

$$\frac{1}{(1-x)^2} = 1 + 2x + 3x^2 + \dots + kx^{k-1} + \dots, \quad |x|<1$$

Since $|1-p|<1$, thus

$$\frac{1}{p^2} = \frac{1}{(1-(1-p))^2} = 1 + 2(1-p) + 3(1-p)^2 + \dots = \sum_{k=1}^{\infty} k(1-p)^{k-1},$$

and so

$$E(X) = p(1-p)\left(\frac{1}{p^2}\right) = \frac{1-p}{p}. \quad \blacksquare$$

$X \sim NB(k,p) \Rightarrow E(X) = \frac{k(1-p)}{p}$ (6.2.4)

Let $X \sim NB(k,p)$. Then necessarily $E(X) = \frac{k(1-p)}{p}$.

Proof.
$$\begin{aligned} E(X) &= \sum_{n=0}^{\infty} n \cdot P(X=n) \\ &= \sum_{n=1}^{\infty} n\binom{n+k-1}{n} p^k (1-p)^n \\ &= \sum_{n=1}^{\infty} n \frac{(n+k-1)!}{n!(k-1)!} (1-p)^n p^k \\ &= \sum_{n=1}^{\infty} k\frac{(1-p)}{p} \cdot \frac{((n-1)+(k+1)-1)!}{(n-1)!\,k!} (1-p)^{n-1} p^{k+1} \\ &= \frac{k(1-p)}{p} \sum_{n=1}^{\infty} \binom{(n-1)+(k+1)-1}{k+1} (1-p)^{n-1} p^{k+1} \\ &= \frac{k(1-p)}{p} \sum_{n=0}^{\infty} \binom{n+(k+1)-1}{k+1} (1-p)^n p^{k+1} \\ &= \frac{k(1-p)}{p}(1) \end{aligned}$$

$\therefore E(X) = \frac{k(1-p)}{p}.$ $\blacksquare$

$X \sim \text{Hyp}(N,M,n) \Rightarrow E(X) = \frac{nM}{N}$ (6.2.5)

Let $X \sim \text{Hyp}(N,M,n)$. Then necessarily $E(X) = \frac{nM}{N}$.

Proof.
$$\begin{aligned} E(X) &= \sum_{k=0}^{n} k \cdot P(X=k) \\ &= \sum_{k=1}^{n} k \cdot \frac{\binom{M}{k}\binom{N-M}{n-k}}{\binom{N}{n}} \\ &= \frac{n!(N-n)!}{N!} \sum_{k=1}^{n} k \cdot \frac{M!}{k!(M-k)!} \frac{(N-M)!}{(n-k)!(N-M-n+k)!} \\ &= \frac{n!(N-n)!}{N!} \sum_{k=1}^{n} M \cdot \frac{(M-1)!}{(k-1)!(M-k)!} \frac{(N-M)!}{(n-k)!(N-M-n+k)!} \\ &= \frac{M(n!)(N-n)!}{N!} \sum_{k=1}^{n} \binom{M-1}{k-1}\binom{N-M}{n-k} \\ &= M\frac{n!(N-n)!}{N!} \sum_{k=0}^{n-1} \binom{M-1}{k}\binom{N-M}{n-k-1} \\ &= M\frac{n!(N-n)!}{N!}\binom{N-1}{n-1} \quad \text{(by Vandermonde's Identity)} \\ &= M\frac{n!(N-n)!}{N!} \cdot \frac{(N-1)!}{(n-1)!(N-n)!} \end{aligned}$$

$\therefore E(X) = \frac{Mn}{N}.$ $\blacksquare$

$X \sim \text{Po}(\mu) \Rightarrow E(X) = \mu$ (6.2.6)

Let $X \sim \text{Po}(\mu)$. Then necessarily $E(X) = \mu$.

Proof.
$$\begin{aligned} E(X) &= \sum_{n=0}^{\infty} n \cdot \frac{e^{-\mu}\mu^n}{n!} \\ &= \sum_{n=1}^{\infty} \frac{e^{-\mu}\mu^n}{(n-1)!} \\ &= \mu \sum_{n=0}^{\infty} \frac{e^{-\mu}\mu^n}{n!} \\ &= \mu(1) = \mu. \quad \blacksquare \end{aligned}$$

$E[g(X)] = \sum_{x \in A} g(x) f(x)$ ⟪THE LAW OF THE UNCONSCIOUS STATISTICIAN⟫ (6.3)

Let g be a function on the drv X, which has range A and pmf X.

Then necessarily $E[g(X)] = \sum_{x \in A} g(x) f(x)$.

Proof. Let $Y = g(X)$, and let $D_y = \{x \in X : g(x) = y\}$, and let $B = \text{ran}(Y)$.

Then
$$P[Y=y] = P[g(X)=y] = \sum_{x \in D_y} P[X=x].$$

Hence
$$\begin{aligned} E(Y) &= \sum_{y \in B} y \cdot P[g(X)=y] \\ &= \sum_{y \in B} y \cdot \sum_{x \in D_y} P[X=x] \\ &= \sum_{y \in B} \sum_{x \in D_y} g(x) P[X=x] \end{aligned}$$

$\therefore E(Y) = E(g(X)) = \sum_{x \in A} g(x) f(x).$ $\blacksquare$

$E[ag(X)+b] = aE[g(X)] + b$; $E[ag_1(X) + bg_2(X)] = aE[g_1(X)] + bE[g_2(X)]$ ⟪LINEAR PROPERTIES OF EXPECTATION⟫

Let g, g_1, g_2 be functions on the drv X, and let $a, b \in \mathbb{R}^+$.

Then necessarily

① $E[ag(X) + b] = aE[g(X)] + b$; and
② $E[ag_1(X) + bg_2(X)] = aE[g_1(X)] + bE[g_2(X)]$.

Proof. Follows almost directly from LOUS. $\blacksquare$

VARIANCE (6.4)

Let X be a drv.
Then, the "variance" of X, denoted as "$Var(X)$" or "σ^2", is defined to be equal to

$$Var(X) = E[(X-E[X])^2].$$

* we use "μ" to denote "$E(X)$", so

$$Var(X) = E[(X-\mu)^2].$$

$Var(X)$ is used to measure the "variability" of a sample, ie how "concentrated" the data is.

The "standard deviation" of X, denoted as "σ", is defined to be

$$\sigma = \sqrt{Var(X)}.$$

$Var(X) = \sum_{x\in A} (x-E(X))^2 f(x) = E(X^2) - [E(X)]^2$

Let X be a drv, with $ran(X) = A$ & pmf f.
Then necessarily
① $Var(X) = \sum_{x\in A} (x-E(X))^2 f(x)$; and
② $Var(X) = E(X^2) - [E(X)]^2$.

Proof. By defn,

$$Var(X) = E[(x-E(X))^2]$$
$$= \sum_{x\in A} (x-E(X))^2 f(x) \quad \text{(by LOUS)},$$

showing ①.
Then

$$\sum_{x\in A} (x-E(X))^2 f(x) = \sum_{x\in A} (x^2 - 2xE(X) + E(X)^2) f(x)$$
$$= \sum_{x\in A} x^2 f(x) - 2E(X) \sum_{x\in A} x + E(X)^2$$
$$= \sum_{x\in A} x^2 f(x) - 2E(X)E(X) + E(X)E(X)$$
$$= E(X^2) - [E(X)]^2,$$

showing ②. ∎

PROPERTIES OF VARIANCE

Let X be a drv. Note the following:
① $Var(X) \geq 0$;
② $E(X^2) \geq (E(X))^2$;
③ $Var(X) = 0 \iff P(X=c) = 1$ for some constant c; and
④ $Var(aX+b) = a^2 Var(X)$.

Proof. By defn of variance,

$$Var(X) = \sum_{x\in A} (x-E(X))^2 f(x) = \sum_{x\in A} (x-E(X))^2 P[X=x],$$

and as $P[X=x]$, $(x-E(X))^2 \geq 0$ $\forall x \in A$, ① follows.
Thus

$$E(X^2) - (E(X))^2 \geq 0 \implies E(X)^2 \geq (E(X))^2, \text{ showing ②.}$$

Next,

$$Var(X) = 0 \iff \sum_{x\in A} \underbrace{(x-\mu)^2}_{\geq 0} \underbrace{f(x)}_{>0} = 0$$
$$\iff (x-\mu)^2 = 0 \quad \forall x \in A$$
$$\iff x = \mu \quad \forall x \in A \quad (\text{ie } A = \{\mu\})$$
$$\iff P[X=\mu] = 1, \text{ showing ③;}$$

and

$$Var(aX+b) = E[(aX+b)^2] - (E(aX+b))^2$$
$$= a^2E(X^2) + 2abE(X) + b^2 - (aE(X)+b)^2$$
$$= a^2E(X^2) + 2abE(X) + b^2 - a^2E(X)^2 - 2abE(X) - b^2$$
$$= a^2E(X^2) - a^2E(X)^2$$
$$= a^2 Var(X),$$

showing ④. ∎

$X \sim Bernoulli(p) \Rightarrow Var(X) = p(1-p)$ (6.5.1)

Let $X \sim Bernoulli(p)$. Then necessarily $Var(X) = p(1-p)$.

Proof.

$$Var(X) = E(X^2) - (E(X))^2$$
$$= \sum_{x=0}^{1} x^2 P[X=x] - p^2$$
$$= p - p^2$$
$$\therefore Var(X) = p(1-p). \ ∎$$

$X \sim Binomial(n,p) \Rightarrow Var(X) = np(1-p)$ (6.5.2)

Let $X \sim Binomial(n,p)$. Then necessarily $Var(X) = np(1-p)$.

Proof. First, see that

$$Var(X) = E(X^2) - (E(X))^2$$
$$= E(X(X-1)) - (E(X))^2 + E(X).$$

Then,

$$E(X(X-1)) = \sum_{k=0}^{n} k(k-1) P[X=k]$$
$$= \sum_{k=2}^{n} k(k-1) \frac{n!}{k!(n-k)!} p^k (1-p)^{n-k}$$
$$= \sum_{k=2}^{n} n(n-1) \cdot \frac{(n-2)!}{(k-2)!(n-k)!} p^k (1-p)^{n-k}$$
$$= n(n-1)p^2 \sum_{k=0}^{n-2} \frac{(n-2)!}{k!(n-k-2)!} p^k (1-p)^{n-2-k}$$
$$= n(n-1)p^2(1)$$
$$\therefore E(X(X-1)) = n(n-1)p^2.$$

Thus

$$Var(X) = E(X(X-1)) - (E(X))^2 + E(X)$$
$$= n(n-1)p^2 - (np)^2 + np$$
$$= np[(n-1)p - np + 1]$$
$$\therefore Var(X) = np(1-p). \ ∎$$

$X \sim Geometric(p) \Rightarrow Var(X) = \frac{(1-p)^2}{p^2} + \frac{1-p}{p}$ (6.5.3)

Let $X \sim Geometric(p)$. Then necessarily $Var(X) = \frac{(1-p)^2}{p^2} + \frac{1-p}{p}$.

Proof. See that

$$Var(X) = E(X^2) - (E(X))^2$$
$$= E(X(X-1)) - (E(X))^2 + E(X).$$

Then

$$E(X(X-1)) = \sum_{n=0}^{\infty} n(n-1) \cdot P[X=n]$$
$$= \sum_{n=2}^{\infty} n(n-1) \cdot (1-p)^n p$$
$$= p(1-p)^2 [(1\times2) + (2\times3)(1-p) + (3\times4)(1-p)^2 + \cdots]$$
$$= 2p(1-p)^2 [1 + 3(1-p) + 6(1-p)^2 + \cdots]$$
$$= 2p(1-p)^2 \left(\frac{1}{(1-(1-p))^3}\right)$$
$$\therefore E(X(X-1)) = \frac{2(1-p)^2}{p^2}.$$

Thus

$$Var(X) = E(X(X-1)) - (E(X))^2 + E(X)$$
$$= \frac{2(1-p)^2}{p^2} - \left(\frac{1-p}{p}\right)^2 + \left(\frac{1-p}{p}\right)$$
$$= \left(\frac{1-p}{p}\right)^2 + \left(\frac{1-p}{p}\right). \ ∎$$

$X \sim Poisson(\mu) \Rightarrow Var(X) = \mu$ (6.5.4)

Let $X \sim Poisson(\mu)$. Then necessarily $Var(X) = \mu$.

Proof. $Var(X) = E(X(X-1)) - (E(X))^2 + E(X)$.

Then

$$E(X(X-1)) = \sum_{n=0}^{\infty} n(n-1) P[X=n]$$
$$= \sum_{n=2}^{\infty} n(n-1) \frac{e^{-\mu}\mu^n}{n!}$$
$$= \mu^2 \sum_{n=2}^{\infty} \frac{e^{-\mu}\mu^{n-2}}{(n-2)!}$$
$$= \mu^2 \sum_{n=0}^{\infty} \frac{e^{-\mu}\mu^n}{n!}$$
$$= \mu^2(1) = \mu^2.$$

Hence

$$Var(X) = E(X(X-1)) - (E(X))^2 + E(X)$$
$$= \mu^2 - (\mu)^2 + \mu$$
$$= \mu. \ ∎$$

Chapter 7: Discrete Multivariate Distributions

BIVARIATE DISTRIBUTIONS (7.1)

"Bivariate distributions" are probability distributions that deal with <u>two</u> random variables.

JOINT PMF : $(X,Y) \sim f(x,y)$

1 Let X, Y be drv.

Then, the "joint probability mass function", ie the "joint pmf", of X & Y is the function f defined by

$$f(x,y): \mathrm{ran}(X) \times \mathrm{ran}(Y) \to [0,1] \quad \text{by} \quad f(x,y) = P[X=x, Y=y].$$

2 In this case, we write $(X,Y) \sim f(x,y)$.

3 Properties of joint pmf :

① $f(x,y) \geq 0$ (by def$^{\underline{n}}$ of f); and

② $\sum_y \sum_x f(x,y) = 1.$

} similar to properties for single variable pmf.

4 In general, for drv $X_1, \dots, X_n$, the joint pmf of $X_1, \dots, X_n$ is defined by

$$f: \prod_{i=1}^{n} \mathrm{ran}(X_i) \to [0,1] \quad \text{by} \quad f(x_1,\dots,x_n) = P[X_1=x_1, \dots, X_n=x_n].$$

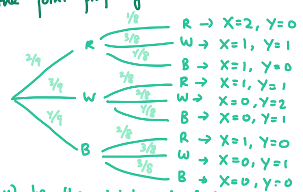
eg' A box contains 2 red, 3 white & 4 black ones.
Randomly select 2 balls w/o replacement.
Let

X = # of red balls selected; &

Y = # of white balls selected.

Find the joint pmf of X & Y.

Sol$^{\underline{n}}$.



Let $f(x,y)$ be the joint pmf of f.
By adding the probabilities for each case, you eventually get that

$f(x,y)$	$x=0$	$x=1$	$x=2$
$y=0$	$6/36$	$8/36$	$1/36$
$y=1$	$12/36$	$6/36$	0
$y=2$	$3/36$	0	0

MARGINAL PMF: $f_X(x)$, $f_i(x_i)$ (7.1.1)

1 Let f be the joint pmf of some drv X & Y.

Then, the "marginal pmfs" of X & Y, denoted as "f_X" & "f_Y" respectively, is defined to be equal to

$$f_X(x) = P[X=x] = \sum_y P[X=x, Y=y] = \sum_y f(x,y),$$

and

$$f_Y(y) = P[Y=y] = \sum_x P[X=x, Y=y] = \sum_x f(x,y).$$

* we also denote

$f_X := f_1$ & $f_Y := f_2$.

2 In general, for drv $X_1, \dots, X_n$, we have that

$$f_{X_i}(x_i) = P[X_i = x_i] = \sum_{x_1} \cdots \sum_{x_{i-1}} \sum_{x_{i+1}} \cdots \sum_{x_n} P[X_1=x_1, \dots, X_n=x_n]$$
$$= \sum_{x_1} \cdots \sum_{x_{i-1}} \sum_{x_{i+1}} \cdots \sum_{x_n} f(x_1,\dots,x_n).$$

* we also denote

$f_{X_i} := f_i$.

3 Note that

① $f(x,y)$ determines $f_X(x)$ & $f_Y(y)$; but

② We <u>cannot</u> generally find $f(x,y)$ from $f_X(x)$ & $f_Y(y)$.

CONDITIONAL PMF : $f_1(x|y)$ (7.1.2)

1 Let X & Y be drv.

Then, the "conditional pmf of X given $Y=y$", denoted as "$f_1(x|y)$", is defined to be

$$f_1(x|y) = \frac{f(x,y)}{f_2(y)} = \frac{P[X=x, Y=y]}{P[Y=y]},$$

given that $f_2(y) > 0$.

2 Note that :

① $f_1(x|y) \geq 0$ (as $f(x,y) \geq 0$, $f_2(y) > 0$); &

② $\sum_x f_1(x|y) = 1 \quad \forall y \in \mathrm{ran}(Y).$

<u>Proof</u>. $\sum_x f_1(x|y) = \sum_x \frac{f(x,y)}{f_2(y)}$

$= \frac{1}{f_2(y)} \sum_x f(x,y)$

$= \frac{1}{f_2(y)} f_2(y)$ (by def$^{\underline{n}}$)

$= 1.$ ∎

INDEPENDENT [RANDOM VARIABLES] (7.1.3)

1 Let X & Y be rv, with pmf f.

Then, we say X and Y are "independent" if

$$f(x,y) = P[X=x, Y=y] = P[X=x]P[Y=y] = f_X(x) f_Y(y)$$

for each $x \in \mathrm{ran}(X)$, $y \in \mathrm{ran}(Y)$.

2 So, to show X & Y are <u>not</u> independent, it suffices to find some $x' \in \mathrm{ran}(X)$, $y' \in \mathrm{ran}(Y)$ such that

$$f(x', y') \neq f_X(x') f_Y(y').$$

eg $X \sim \mathrm{Geo}(p)$; $Y = X^2$: Show X & Y are not independent.

→ Let $x' = 0$, $y' = 1$.

Then $f(0,1) = P[X=0, Y=1] = P[X=0, X^2=1]$

$= 0$ (since this is impossible).

But $f_X(0) f_Y(1) = P[X=0] P(Y=1)$

$= P[X=0] P(X^2=1)$

$= P[X=0] P(X=1)$

$= p(1-p) \cdot p$ $(\neq 0)$. #

3 In general, the rv $X_1, \dots, X_n$ (with pmf f) are independent if

$$f(x_1, \dots, x_n) = f_1(x_1) \cdots f_n(x_n) \quad \forall x_i \in \mathrm{ran}(X_i),\ 1 \leq i \leq n.$$

DISTRIBUTION OF A FUNCTION OF RANDOM VARIABLES (7.2)

To find the pmf for $T=g(X,Y)$, we use the following method:

1. Evaluate $\text{ran}(T) = \text{ran}(g(X,Y))$;
2. Find the values of (x,y) such that $g(X,Y)\big|_{X=x, Y=y} = r$ for each $r \in \text{ran}(T)$; then
3. Use $f(x,y)$ to get the respective probabilities of each (x,y), and "merge" them accordingly under each $r \in \text{ran}(T)$ to obtain the pmf for T.

eg' A box contains 2 red, 3 white & 4 black ones.
Randomly select 2 balls w/o replacement.
Let
X = # of red balls selected; &
Y = # of white balls selected.
Let $T = 2XY-1$. Find the pmf of T. (E3)

Sol^n.

T		$x=0$	$x=1$	$x=2$
	0	-1	-1	-1
y	1	-1	1	3
	2	-1	3	7

From earlier we found the pmf of X,Y in a tabular form:

$f(x,y)$		$x=0$	$x=1$	$x=2$
	0	$6/36$	$8/36$	$1/36$
y	1	$12/36$	$6/36$	0
	2	$3/36$	0	0

By "comparing" the two tables, see that

$$P[T=-1] = P(X=0,Y=0) + P(X=0,Y=1) + P(X=0,Y=2) + P(X=0,Y=1) + P(X=0,Y=2)$$
$$= 6/36 + 8/36 + 1/36 + 12/36 + 3/36 = 30/36.$$

$P[T=1]$, $P[T=3]$ & $P[T=7]$ are calculated similarly.

$X \sim Bin(n,p)$, $Y \sim Bin(m,p)$, X,Y INDEPENDENT $\Rightarrow$ $X+Y \sim Bin(n+m, p)$ (E4)

Let $X \sim Bin(n,p)$ be independent from $Y \sim Bin(m,p)$. We can show $X+Y \sim Bin(n+m,p)$ by a pmf argument.

Proof. Let f be the joint pmf of X & Y.
Since X,Y are indpt, thus $f(x,y) = f_X(x) f_Y(y)$ $\forall x,y$.
Let $T = X+Y$. See that

$$P[T=t] = \sum_{x=0}^{t} P(X=x)P(Y=t-x)$$
$$= \sum_{x=0}^{t} \binom{n}{x} p^x (1-p)^{n-x} \binom{m}{t-x} p^{t-x} (1-p)^{m-(t-x)}$$
$$= \sum_{x=0}^{t} \binom{n}{x}\binom{m}{t-x} p^t (1-p)^{n+m-x-t+x}$$
$$= p^t (1-p)^{n+m-t} \sum_{x=0}^{t} \binom{n}{x}\binom{m}{t-x}$$
$$= p^t (1-p)^{n+m-t} \binom{n+m}{t} \quad \text{(by Vandermonde's identity),}$$

and so this suffices to show
$T \sim Bin(n+m, p)$. ∎

Let's find the conditional pmf of X given $T=t$ also.

Sol^n. See that

$$P[X=x \mid T=t] = \frac{P(X=x, T=t)}{P(T=t)}$$
$$= \frac{P(X=x, X+Y=t)}{P(T=t)}$$
$$= \frac{P(X=x, Y=t-x)}{P(T=t)}$$
$$= \frac{f_X(x) f_Y(t-x)}{P(T=t)}$$
$$= \frac{\binom{n}{x} p^x (1-p)^{n-x} \binom{m}{t-x} p^{t-x} (1-p)^{m-(t-x)}}{\binom{n+m}{t} p^t (1-p)^{n+m-t}}$$

$$\therefore P[X=x \mid T=t] = \frac{\binom{n}{x}\binom{m}{t-x}}{\binom{n+m}{t}}.$$

$\Rightarrow$ see that by construction, this is what a hypergeometric distribution is "finding"!

$X \sim Po(\mu_1)$, $Y \sim Po(\mu_2)$ $\Rightarrow$ $X+Y \sim Po(\mu_1+\mu_2)$ (E5)

Let $X \sim Po(\mu_1)$ be independent to $Y \sim Po(\mu_2)$.
Then we can similarly show that $X+Y \sim Po(\mu_1+\mu_2)$.

Proof. Let $T = X+Y$.
See that

$$P[T=t] = \sum_{x=0}^{t} P[X=x, Y=t-x]$$
$$= \sum_{x=0}^{t} P[X=x]P[Y=t-x] \quad \text{(as } X,Y \text{ are independent)}$$
$$= \sum_{x=0}^{t} \frac{e^{-\mu_1}\mu_1^x}{x!} \frac{e^{-\mu_2}\mu_2^{t-x}}{(t-x)!}$$
$$= e^{-\mu_1-\mu_2} \sum_{x=0}^{t} \frac{\mu_1^x \mu_2^{t-x}}{x!(t-x)!}$$
$$= \frac{e^{-\mu_1-\mu_2}}{t!} \sum_{x=0}^{t} \frac{t!}{x!(t-x)!} \mu_1^x \mu_2^{t-x}$$
$$= \frac{e^{-\mu_1-\mu_2}}{t!} \sum_{x=0}^{t} \binom{t}{x} \mu_1^x \mu_2^{t-x}$$
$$= \frac{e^{-\mu_1-\mu_2}}{t!} (\mu_1+\mu_2)^t \quad \text{(by the binomial formula)}$$

which suffices to show $T \sim Po(\mu_1+\mu_2)$. ∎

We can similarly find $f_X(x|t)$.

$$f_X(x|t) = P[X=x \mid T=t]$$
$$= \frac{P(X=x, T=t)}{P(T=t)}$$
$$= \frac{P(X=x, X+Y=t)}{P(T=t)}$$
$$= \frac{P(X=x, Y=t-x)}{P(T=t)}$$
$$= \frac{P(X=x)P(Y=t-x)}{P(T=t)} \quad \text{(by independence of } X,Y\text{)}$$
$$= \frac{\left(\frac{e^{-\mu_1}\mu_1^x}{x!}\right)\left(\frac{e^{-\mu_2}\mu_2^{t-x}}{(t-x)!}\right)}{\left(\frac{e^{-\mu_1-\mu_2}(\mu_1+\mu_2)^t}{t!}\right)}$$
$$= \frac{t!}{x!(t-x)!} \cdot \frac{e^{-\mu_1-\mu_2}}{e^{-\mu_1-\mu_2}} \cdot \frac{\mu_1^x \mu_2^{t-x}}{(\mu_1+\mu_2)^t}$$
$$= \binom{t}{x} \frac{\mu_1^x \mu_2^{t-x}}{(\mu_1+\mu_2)^t}.$$

TRINOMIAL DISTRIBUTION: $(X_1, X_2, X_3) \sim Tri(n, p_1, p_2, p_3)$ (7.3)

In a "trinomial distribution":
① there are three possible outcomes A, B, C for a trial;
and
② n trials occur independently.

In particular,
① If $P(A) = p_1$, $P(B) = p_2$ & $P(C) = p_3$, then $p_1 + p_2 + p_3 = 1$,
and
② If $X_1 = \#(A)$, $X_2 = \#(B)$ & $X_3 = \#(C)$, then $X_1 + X_2 + X_3 = n$.

In this case, we write
$$(X_1, X_2, X_3) \sim \text{Trinomial}(n, p_1, p_2, p_3).$$
or
$$(X_1, X_2, X_3) \sim Tri(n, p_1, p_2, p_3).$$

JOINT PMF [OF TRINOMIAL DISTRIBUTIONS]: $f(x_1, x_2, x_3) = \frac{n!}{x_1! x_2! x_3!} p_1^{x_1} p_2^{x_2} p_3^{x_3}$ (7.3.1)

Let the rv X_1, X_2, X_3 form a trinomial distribution, with $X_1 + X_2 + X_3 = n$.
Then the joint pmf is necessarily the function f where
$$f(x_1, x_2, x_3) = P[X_1 = x_1, X_2 = x_2, X_3 = x_3]$$
$$= P[X_1 = x_1, X_2 = x_2] = P[X_1 = x_1, X_3 = x_3] = P[X_2 = x_2, X_3 = x_3]$$
$$= \frac{n!}{x_1! x_2! x_3!} p_1^{x_1} p_2^{x_2} p_3^{x_3}.$$

Why? $f(x_1, x_2, x_3) = \underbrace{\frac{n!}{x_1! x_2! x_3!}}_{\substack{\#\text{ of ways to arrange } n \text{ things} \\ \text{w/ } x_1 \text{ type 1, } x_2 \text{ type 2} \\ \text{\& } x_3 \text{ type 3}}} \underbrace{p_1^{x_1} p_2^{x_2} p_3^{x_3}}_{\text{the respective probabilities multiplied.}}$

Note that
$$\sum_{x_1=0}^{n} \sum_{x_2=0}^{n-x_1} P(X_1 = x_1, X_2 = x_2, X_3 = x_3) = \sum_{x_1=0}^{n} \sum_{x_2=0}^{n-x_1} \frac{n!}{x_1! x_2! x_3!} p_1^{x_1} p_2^{x_2} p_3^{x_3}$$
$$= (p_1 + p_2 + p_3)^n$$
$$= 1 \quad (\text{as } p_1 + p_2 + p_3 = 1).$$

MARGINAL PMFS [OF TRINOMIAL DISTRIBUTIONS]: $f_{X_1}(x_1) = \sum_{x_2=0}^{n-x_1} f(x_1, x_2, x_3) = \sum_{x_3=0}^{n-x_1} f(x_1, x_2, x_3)$ (7.3.2)

Let X_1, X_2, X_3 form a trinomial distribution, with joint pmf f.
Denote the marginal pmf of f with X_1 as f_{X_1}.
Then
$$f_{X_1}(x_1) = P[X_1 = x_1] = \sum_{x_2=0}^{n-x_1} f(x_1, x_2, x_3)$$
and
$$f_{X_1}(x_1) = P[X_1 = x_1] = \sum_{x_3=0}^{n-x_1} f(x_1, x_2, x_3),$$
and f_{X_2}, f_{X_3} are defined similarly.

Proof. $f_{X_1}(x_1) = \sum_{x_2=0}^{n-x_1} f_{1,2}(x_1, x_2)$ (where $f_{1,2}(x_1, x_2) = P[X_1 = x_1, X_2 = x_2]$)
$$= \sum_{x_2=0}^{n-x_1} \frac{n!}{x_1! x_2! (n - x_1 - x_2)!} p_1^{x_1} p_2^{x_2} p_3^{n - x_1 - x_2}$$
$$= \sum_{x_2=0}^{n-x_1} \frac{n!}{x_1! x_2! x_3!} p_1^{x_1} p_2^{x_2} p_3^{x_3} \quad (\text{as } x_1 + x_2 + x_3 = n)$$
$$= \sum_{x_2=0}^{n-x_1} f(x_1, x_2, x_3),$$
and the other sum is proved similarly. ☒

$(X_1, X_2, X_3) \sim Tri(n, p_1, p_2, p_3) \Rightarrow X_i \sim Bin(n, p_i)$ (7.3.2)

Let X_1, X_2, X_3 form a trinomial distribution.
Then necessarily $X_i \sim Bin(n, p_i)$ $\forall 1 \leq i \leq 3$.

Proof. We prove it for $i = 1$; the other cases are similar.
See that
$$P[X_1 = x_1] = f_{X_1}(x_1)$$
$$= \sum_{x_2=0}^{n-x_1} \frac{n!}{x_1! x_2! x_3!} p_1^{x_1} p_2^{x_2} p_3^{x_3}$$
$$= p_1^{x_1} \frac{n!}{(n-x_1)! x_1!} \sum_{x_2=0}^{n-x_1} \frac{(n-x_1)!}{x_2! ((n-x_1) - x_2)!} p_2^{x_2} p_3^{(n-x_1) - x_2}$$
$$= p_1^{x_1} \binom{n}{x_1} (p_2 + p_3)^{n - x_1} \quad (\text{by bin formula})$$
$$= p_1^{x_1} \binom{n}{x_1} (1 - p_1)^{n - x_1},$$
which suffices to show that
$$X_1 \sim Bin(n, p_1)$$
as needed. ☒

CONDITIONAL PMFS [FOR TRINOMIAL DISTRIBUTIONS]: $(X_1 \mid X_3 = x_3) \sim Bin(n - x_3, \frac{p_1}{p_1 + p_2})$ (7.3.3)

Let X_1, X_2, X_3 form a trinomial distribution.
Then necessarily
$$(X_1 \mid X_3 = x_3) \sim Bin\left(n - x_3, \frac{p_1}{p_1 + p_2}\right),$$
and the other cases (ie $(X_i \mid X_j = x_j)$) are defined similarly.

Proof. $P[X_1 = x_1 \mid X_3 = x_3] = \frac{P(X_1 = x_1, X_3 = x_3)}{P(X_3 = x_3)}$
$$= \frac{\frac{\cancel{n!}}{x_1! x_2! \cancel{x_3!}} p_1^{x_1} p_2^{x_2} \cancel{p_3^{x_3}}}{\frac{\cancel{n!}}{(x_1 + x_2)! \cancel{x_3!}} (p_1 + p_2)^{x_1 + x_2} \cancel{p_3^{x_3}}} \quad (\text{since } X_3 \sim Bin(n, p_3))$$
$$= \frac{(x_1 + x_2)!}{x_1! x_2!} \cdot \frac{p_1^{x_1} p_2^{x_2}}{(p_1 + p_2)^{x_1 + x_2}}$$
$$= \binom{x_1 + x_2}{x_2} \left(\frac{p_1}{p_1 + p_2}\right)^{x_1} \left(\frac{(p_1 + p_2) - p_1}{p_1 + p_2}\right)^{x_2}$$
$$= \binom{n - x_3}{n - x_3 - x_1} p'^{x_1} (1 - p')^{n - x_3 - x_1} \quad \left(p' = \frac{p_1}{p_1 + p_2}\right),$$
which is sufficient to prove the claim. ☒

$(X_1, X_2, X_3) \sim Tri(n, p_1, p_2, p_3) \Rightarrow X_1 + X_2 \sim Bin(n, p_1 + p_2)$ (7.3.4)

Let $(X_1, X_2, X_3) \sim Tri(n, p_1, p_2, p_3)$.
Then necessarily $X_1 + X_2 \sim Bin(n, p_1 + p_2)$.

Proof. $P[X_1 + X_2 = t] = \sum_{x_1=0}^{t} P(X_1 = x_1, X_2 = t - x_1)$
$$= \sum_{x_1=0}^{t} \frac{n!}{x_1! (t - x_1)! (n-t)!} p_1^{x_1} p_2^{t - x_1} p_3^{n-t}$$
$$= \frac{n!}{t! (n-t)!} p_3^{n-t} \sum_{x_1=0}^{t} \frac{t!}{x_1! (t - x_1)!} p_1^{x_1} p_2^{t - x_1}$$
$$= \binom{n}{t} (n - (p_1 + p_2))^{n-t} (p_1 + p_2)^t \quad (\text{by bin formula}),$$
which suffices to prove the claim. ☒

MULTINOMIAL DISTRIBUTION: $(X_1, \ldots, X_k) \sim Mul(n, p_1, \ldots, p_k)$ (7.4)

In a "multinomial distribution":
① Each trial has k outcomes, say $A_1, \ldots, A_k$ (where $k \geq 2$);
and
② we repeat said trial independently n times.

In this case, we say
$$(X_1, \ldots, X_k) \sim \text{Multinomial}(n, p_1, \ldots, p_k)$$
or
$$(X_1, \ldots, X_k) \sim Mult(n, p_1, \ldots, p_k),$$
where $p_i = P(A_i)$ $\forall 1 \leq i \leq n$ and $X_i = \#(A_i \text{ occurred in the trials})$.

JOINT PMF [OF MULTINOMIAL DISTRIBUTIONS]: $P[X_1 = x_1, \ldots, X_k = x_k] = \frac{n!}{x_1! \cdots x_k!} p_1^{x_1} \cdots p_k^{x_k}$

Let $(X_1, \ldots, X_k) \sim Mult(n, p_1, \ldots, p_k)$.
Then the joint pmf of $X_1, \ldots, X_k$ is given by f,
where
$$f(x_1, \ldots, x_k) = P[X_1 = x_1, \ldots, X_k = x_k] = \frac{n!}{x_1! \cdots x_k!} p_1^{x_1} \cdots p_k^{x_k}.$$

Why? $f(x_1, \ldots, x_k) = \underbrace{\frac{n!}{x_1! \cdots x_k!}}_{\text{generalization of combinations}} \underbrace{p_1^{x_1} \cdots p_k^{x_k}}_{\text{multiplying the probabilities out}}$

Week 8: Expectation and Variance of Multiple Variables

$E(g(X_1,X_2)) = \sum_{x_1}\sum_{x_2} g(x_1,x_2) f(x_1,x_2)$ « LAW OF UNCONSCIOUS STATISTICIAN » (8.1)

Let X_1, X_2 be drv, and let $(X_1,X_2) \sim f(x_1,x_2)$.

Then necessarily

$$E(g(X_1,X_2)) = \sum_{x_1}\sum_{x_2} g(x_1,x_2) f(x_1,x_2).$$

$E(aX+bY+c) = aE(X) + bE(Y) + c$ (8.2)

Let X,Y be drv, and let $a,b,c \in \mathbb{R}$.

Then necessarily

$$E(aX+bY+c) = aE(X) + bE(Y) + c.$$

Proof. Let $(X,Y) \sim f(x,y)$.

$$\Rightarrow E(aX+bY+c) = \sum_x\sum_y (ax+by+c) f(x,y)$$

$$= \sum_x\sum_y ax f(x,y) + \sum_x\sum_y by f(x,y) + \sum_x\sum_y c f(x,y)$$

$$= a\sum_x x \underbrace{\sum_y f(x,y)}_{P[X=x]} + b\sum_y y \underbrace{\sum_x f(x,y)}_{P[Y=y]} + c\underbrace{\sum_x\sum_y f(x,y)}_{1}$$

$$= aE(X) + bE(y) + c. \quad \blacksquare$$

X & Y ARE INDEPENDENT ⇒ $E[g(X)h(Y)] = E[g(X)]E[h(Y)]$ (8.2)

Let X,Y be independent drv.

Then necessarily

$$E[g(X)h(Y)] = E[g(X)]\,E[h(Y)].$$

Proof. Let $(X,Y) \sim f(x,y)$.

Then

$$E[g(X)]\,E[h(Y)] = \Big(\sum_x g(x) f_X(x)\Big)\Big(\sum_y h(y) f_Y(y)\Big)$$

$$= \sum_x\sum_y g(x)h(y) f_X(x) f_Y(y)$$

$$= \sum_x\sum_y g(x)h(y) f(x,y) \quad \text{(since } X,Y \text{ are independent, see 7.1.3)}$$

$$= E[g(X)h(Y)] \quad \text{(by LOUS)},$$

as needed. $\blacksquare$

COVARIANCE: Cov(X,Y) (8.3)

Let X,Y be drv.

Then, the "covariance" between X & Y, denoted as "$\mathrm{Cov}(X,Y)$", is defined to be

$$\mathrm{Cov}(X,Y) = E[(X-E(X))(Y-E(Y))]$$

$$= E(XY) - E(X)E(Y).$$

Proof. Let $(X,Y) \sim f(x,y)$. Then

$$\mathrm{Cov}(X,Y) = \sum_x\sum_y (x-E(X))(y-E(Y)) f(x,y) \quad \text{(by LOUS)}$$

$$= \sum_x\sum_y xy f(x,y) - \sum_x\sum_y E(X) y f(x,y) - \sum_x\sum_y E(Y) x f(x,y) + \sum_x\sum_y E(X)E(Y) f(x,y)$$

$$= E(XY) - E(X)E(Y) - E(Y)E(X) + E(X)E(Y)(1)$$

$$= E(XY) - E(X)E(Y). \quad \blacksquare$$

$\mathrm{Var}(X+Y) = \mathrm{Var}(X) + \mathrm{Var}(Y) + 2\mathrm{Cov}(X,Y)$ (8.3)

Let X,Y be drv.

Then necessarily

$$\mathrm{Var}(X+Y) = \mathrm{Var}(X) + \mathrm{Var}(Y) + 2\mathrm{Cov}(X,Y).$$

Proof.

$$\mathrm{Var}(X+Y) = E((X+Y)^2) - (E(X+Y))^2$$

$$= E(X^2) + 2E(XY) + E(Y^2) - (E(X) + E(Y))^2$$

$$= E(X^2) + 2E(XY) + E(Y^2) - E(X)^2 - 2E(X)E(Y) - E(Y)^2$$

$$= (E(X^2) - E(X)^2) + (E(Y^2) - E(Y)^2) + 2(E(XY) - E(X)E(Y))$$

$$= \mathrm{Var}(X) + \mathrm{Var}(Y) + 2\mathrm{Cov}(X,Y). \quad \blacksquare$$

$\mathrm{Cov}(X,X) = \mathrm{Var}(X)$

Let X be a drv.

Then necessarily $\mathrm{Cov}(X,X) = \mathrm{Var}(X)$.

Proof.

$$\mathrm{Cov}(X,X) = \frac{2\mathrm{Var}(X) - \mathrm{Var}(X) - \mathrm{Var}(X)}{2} \quad \text{(by rearranging the above equality)}$$

$$= \mathrm{Var}(X). \quad \blacksquare$$

X, Y ARE INDEPENDENT ⇒ Cov(X,Y) = 0

1. Let X,Y be independent drv.

Then necessarily $\mathrm{Cov}(X,Y) = 0$.

Proof.

$$\mathrm{Cov}(X,Y) = E(XY) - E(X)E(Y)$$

$$= E(XY) - E(XY) \quad \text{(by 8.2)}$$

$$= 0. \quad \blacksquare$$

2. However, the converse is ~~not necessarily true!~~

$\mathrm{Var}(aX+bY+c) = a^2\mathrm{Var}(X) + b^2\mathrm{Var}(Y) + 2ab\,\mathrm{Cov}(X,Y)$

1. Let X,Y be drv, and let $a,b,c \in \mathbb{R}$.

Then necessarily

$$\mathrm{Var}(aX+bY+c) = a^2\mathrm{Var}(X) + b^2\mathrm{Var}(Y) + 2ab\,\mathrm{Cov}(X,Y).$$

Proof.

$$\mathrm{Var}(aX+bY+c) = E((aX+bY+c)^2) - (E(aX+bY+c))^2$$

$$= E(a^2X^2 + b^2Y^2 + c^2 + 2abXY + 2acX + 2bcY) - (aE(X) + bE(Y) + c)^2$$

$$= a^2E(X^2) + b^2E(Y^2) + \cancel{c^2} + 2abE(XY) + \cancel{2acE(X)} + \cancel{2bcE(Y)} - a^2E(X)^2 - b^2E(Y)^2 - \cancel{c^2} - 2abE(X)E(Y) - \cancel{2acE(X)} - \cancel{2bcE(Y)}$$

$$= a^2(E(X^2) - E(X)^2) + b^2(E(Y^2) - E(Y)^2) + 2ab(E(XY) - E(X)E(Y))$$

$$= a^2\mathrm{Var}(X) + b^2\mathrm{Var}(Y) + 2ab\,\mathrm{Cov}(X,Y). \quad \blacksquare$$

CORRELATION COEFFICIENT: ρ OR $\rho_{X,Y}$ (8·4)

Let X, Y be drv.
Then, the "correlation coefficient" of X & Y, denoted as "ρ", is defined to be equal to

$$\rho = \frac{Cov(X,Y)}{\sqrt{Var(X)Var(Y)}},$$

ie

$$\rho = \frac{E[(X-E(X))(Y-E(Y))]}{\sqrt{E[(X-E(X))^2]}\sqrt{E[(Y-E(Y))^2]}}.$$

$|\rho| \le 1$; $|\rho| = 1 \iff Y = aX+b$; $a>0 \Rightarrow \rho = 1$, $a<0 \Rightarrow \rho = -1$

Let X, Y be drv, and let ρ be the correlation coefficient of X & Y.
Then necessarily $|\rho| \le 1$.
Furthermore, $|\rho| = 1$ iff $Y = aX+b$ for some $a, b \in \mathbb{R}$.
① If $a > 0$, then $\rho = 1$;
② If $a < 0$, then $\rho = -1$.

Proof. Let $S = X + tY$.
Then

$$0 \le Var(S) = Var(X+tY)$$
$$= Var(X) + t^2 Var(Y) + 2t\, Cov(X,Y),$$

which is a quadratic function in t.
Since $0 \le Var(S)$, it follows that this quadratic has at most one real root.
Let's evaluate the discriminant:

$$\Delta = (2Cov(X,Y))^2 - 4(Var(X))(Var(Y)) \quad (\le 0)$$
$$\Rightarrow 4Cov(X,Y)^2 - 4Var(X)Var(Y) \le 0$$
$$\Rightarrow \frac{Cov(X,Y)^2}{Var(X)Var(Y)} \le 1$$
$$\Rightarrow |\rho| \le 1 \quad \text{(as needed)}.$$

In particular, $\Delta = 0$ iff $Var(S) = 0$, ie

$$4Cov(X,Y)^2 - 4Var(X)Var(Y) = 0,$$

ie $|\rho| = 1$.
In particular, $\exists t \in \mathbb{R} \ni Var(X+tY) = 0$, ie that $X + tY = c$ for some $c \in \mathbb{R}$, ie that

$Y = aX + b$ for some $a, b \in \mathbb{R}$.
Then

$$\rho = \frac{Cov(X,Y)}{\sqrt{Var(X)Var(Y)}} = \frac{E[(X-E(X))(aX+b-aE(X)-b)]}{\sqrt{E[(X-E(X))^2]E[(aX+b-aE(X)-b)^2]}}$$
$$= \frac{E[a(X-E(X))^2]}{\sqrt{E[(X-E(X))^2]E[(aX-aE(X))^2]}}$$
$$= \frac{a\,E[(X-E(X))^2]}{\sqrt{a^2(E[(X-E(X))^2])^2}}$$
$$= \frac{aE[(X-E(X))^2]}{|a|E[(X-E(X))]}$$
$$= \frac{a}{|a|},$$

so $\rho = 1 \iff a > 0$ & $\rho = -1 \iff a < 0$. ☒

POSITIVE / NEGATIVE CORRELATION

Let X, Y be drv.
Then,
① We say X & Y follow a "positive correlation" if large values of X tend to be associated with large values of Y, and small values of X tend to be associated with small values of Y.
② Conversely, we say X & Y follow a "negative correlation" if large values of X tend to be associated with small values of Y, and small values of X tend to be associated with large values of Y.

$(X_1, X_2, X_3) \sim Tri(n, p_1, p_2, p_3) \Rightarrow \rho_{X_1,X_2} = -\sqrt{\frac{p_1 p_2}{(1-p_1)(1-p_2)}}$

Let $(X_1, X_2, X_3) \sim Tri(n, p_1, p_2, p_3)$.
Then necessarily

$$\rho_{X_1,X_2} = -\sqrt{\frac{p_1 p_2}{(1-p_1)(1-p_2)}},$$

and a similar result holds for ρ_{X_1,X_3} & ρ_{X_2,X_3} too.

Proof. Recall that

$$(X_1, X_2, X_3) \sim Tri(n, p_1, p_2, p_3) \Rightarrow X_i \sim Bin(n, p_i).$$

Hence

$$E(X_i) = np_i, \quad Var(X_i) = np_i(1-p_i).$$

Then, see that

$$E(X_1X_2) = \sum_{x_1=0}^{n}\sum_{x_2=0}^{n} x_1 x_2 \cdot \frac{n!}{x_1!x_2!x_3!} p_1^{x_1} p_2^{x_2} p_3^{x_3}.$$

On the other hand,

$$X_1 + X_2 \sim Bin(n, p_1+p_2),$$

so

$$Var(X_1+X_2) = n(p_1+p_2)(1-(p_1+p_2)).$$

Thus

$$Cov(X_1, X_2) = \tfrac{1}{2}(Var(X_1+X_2) - Var(X_1) - Var(X_2))$$
$$= \tfrac{1}{2}(n(p_1+p_2)(1-(p_1+p_2)) - np_1(1-p_1) - np_2(1-p_2))$$
$$= -np_1p_2.$$

Hence

$$\rho_{X_1,X_2} = \frac{Cov(X_1,X_2)}{\sqrt{Var(X_1)}\sqrt{Var(X_2)}} = \frac{-np_1p_2}{\sqrt{np_1(1-p_1)}\sqrt{np_2(1-p_2)}}$$
$$= \frac{-np_1p_2}{n\sqrt{p_1p_2}\sqrt{(1-p_1)(1-p_2)}}$$
$$= -\sqrt{\frac{p_1p_2}{(1-p_1)(1-p_2)}}.$$ ☒

$(X_1, \dots, X_k) \sim Mult(n, p_1, \dots, p_k) \Rightarrow Cov(X_i, X_j) = -np_ip_j$

Let $(X_1, \dots, X_k) \sim Mult(n, p_1, \dots, p_k)$.
Then necessarily $Cov(X_i, X_j) = -np_ip_j \quad \forall 1 \le i, j \le k$.
Proof. Similar to trinomial case.

$(X_1, \dots, X_k) \sim Mult(n, p_1, \dots, p_k) \Rightarrow \rho_{X_i,X_j} = -\sqrt{\frac{p_ip_j}{(1-p_i)(1-p_j)}}$

Let $(X_1, \dots, X_k) \sim Mult(n, p_1, \dots, p_k)$.
Then necessarily $\rho_{X_i,X_j} = -\sqrt{\frac{p_ip_j}{(1-p_i)(1-p_j)}}$.
Proof. Similar to trinomial case. ☒

$E(\sum_{i=1}^{n} c_iX_i) = \sum_{i=1}^{n} c_iE(X_i)$

Let $X_1, \dots, X_n$ be drv, and let $c_1, \dots, c_n \in \mathbb{R}$.
Then necessarily

$$E(\sum_{i=1}^{n} c_iX_i) = \sum_{i=1}^{n} c_iE(X_i).$$

Proof. This can be proved via induction pretty easily. ☒

$Var(\sum_{i=1}^{n} c_iX_i) = \sum_{i=1}^{n} c_i^2 Var(X_i) + 2\sum_{i<j} c_ic_j Cov(X_i, X_j)$

Let $X_1, \dots, X_n$ be drv, and let $c_1, \dots, c_n \in \mathbb{R}$.
Then necessarily

$$Var(\sum_{i=1}^{n} c_iX_i) = \sum_{i=1}^{n} c_i^2 Var(X_i) + 2\sum_{i<j} c_ic_j Cov(X_i, X_j)$$
$$= \sum_{i=1}^{n} c_i^2 Var(X_i) + 2(c_1c_2Cov(X_1,X_2) + \dots + c_1c_nCov(X_1,X_n)$$
$$+ c_2c_3Cov(X_2,X_3) + \dots + c_2c_nCov(X_2,X_n)$$
$$+ \dots c_{n-1}c_nCov(X_{n-1},X_n)).$$

Proof.

$$Var(\sum_{i=1}^{n} c_iX_i) = E((\sum_{i=1}^{n} c_iX_i)^2] - E[\sum_{i=1}^{n} c_iX_i]^2$$
$$= E\left[\sum_{i=1}^{n}\sum_{j=1}^{n} c_ic_jX_iX_j\right] - \left(\sum_{i=1}^{n} c_iE(X_i)\right)^2$$
$$= \sum_{i=1}^{n}\sum_{j=1}^{n} c_ic_j E(X_iX_j) - \sum_{i=1}^{n}\sum_{j=1}^{n} c_ic_j E(X_i)E(X_j)$$
$$= \sum_{i=j} c_ic_j \underbrace{(E(X_iX_j) - E(X_i)E(X_j))}_{Var(X_i)} + \sum_{i<j} c_ic_j \underbrace{(E(X_iX_j) - E(X_i)E(X_j))}_{Cov(X_i,X_j)} + \sum_{i>j} c_ic_j \underbrace{(E(X_iX_j) - E(X_i)E(X_j))}_{Cov(X_i,X_j)\ (=Cov(X_j,X_i))}$$
$$= \sum_{i=1}^{n} c_i^2 Var(X_i) + 2\sum_{i<j} c_ic_j Cov(X_i, X_j).$$ ☒

$X_1, \dots, X_n$ ARE INDEPENDENT $\Rightarrow Var(\sum_{i=1}^{n} c_iX_i) = \sum_{i=1}^{n} c_i^2 Var(X_i)$

Let $X_1, \dots, X_n$ be drv, and let $c_1, \dots, c_n \in \mathbb{R}$.
Suppose $X_1, \dots, X_n$ are independent.
Then necessarily

$$Var(\sum_{i=1}^{n} c_iX_i) = \sum_{i=1}^{n} c_i^2 Var(X_i).$$

Proof. This follows from the fact that

$$X_i, X_j \text{ are independent} \Rightarrow Cov(X_i, X_j) = 0,$$

and applying this to the previous theorem. ☒

INDICATOR FUNCTION TECHNIQUES

USING IFS ON $Bin(n,p)$

Let $X \sim Bin(n,p)$. Find $E(X)$ & $Var(X)$ using indicator variable techniques.

Sol^n. Let $X_i = 1$ if the i^{th} trial is a success & 0 otherwise, ie

$X_i = I[\text{the } i^{th} \text{ trial is a success}]$.

Then

$X_i \sim Bernoulli(p)$,

and since $X_1, \dots, X_n$ are independent, thus

$$E(X) = E\left(\sum_{i=1}^{n} X_i\right) = \sum_{i=1}^{n} E(X_i) = np,$$

and

$$\begin{aligned} Var(X) &= Var\left(\sum_{i=1}^{n} X_i\right) \\ &= \sum_{i=1}^{n} Var(X_i) + 2\sum_{1\le i<j\le n} Cov(X_i, X_j) \\ &= \sum_{i=1}^{n} Var(X_i) + 2\sum_{1\le i<j\le n} (0) \quad \text{(by independence of } X_1,\dots,X_n) \\ &= \sum_{i=1}^{n} p(1-p) \end{aligned}$$

$\therefore Var(X) = np(1-p)$.

USING IFS ON $Hyp(N,M,n)$

Suppose there are M red balls & $N-M$ blue balls in a box.

Randomly select n balls without replacement from the box.

Let X be the # of red balls selected, so that

$X \sim Hyp(N,M,n)$.

We can find $E(X)$ & $Var(X)$ using IF techniques.

Sol^n. Let $X_i = 1$ if the i^{th} selection is a red ball, and 0 otherwise; ie $X_i = I(i^{th} \text{ trial is a success})$.

Then

$X_i \sim Bernoulli\left(\frac{M}{N}\right)$, but note the X_i's are not independent!

$\Rightarrow E(X_i) = \frac{M}{N}$, $Var(X_i) = \frac{M}{N}\left(1-\frac{M}{N}\right)$.

So

$$E(X) = E\left(\sum_{i=1}^{n} X_i\right) = \sum_{i=1}^{n} E(X_i) = \frac{nM}{N}.$$

Then,

$$Var(X) = Var\left(\sum_{i=1}^{n} X_i\right) = \sum_{i=1}^{n} Var(X_i) + 2\sum_{1\le i<j\le n} Cov(X_i, X_j).$$

See that

$$\begin{aligned} E(X_iX_j) &= \sum_{x_i=0}^{1}\sum_{x_j=0}^{1} x_ix_j P[X_i=x_i, X_j=x_j] \\ &= 1\cdot 1\cdot P[X_i=1, X_j=1] \\ &= P[X_i=1]\,P[X_j=1 \mid X_i=1] \quad \text{(by total prob formula)} \\ &= \frac{M}{N}\cdot\frac{M-1}{N-1}, \end{aligned}$$

and so

$$\begin{aligned} Var(X) &= \sum_{i=1}^{n} Var(X_i) + 2\sum_{1\le i<j\le n} Cov(X_i, X_j) \\ &= n\cdot\frac{M}{N}\cdot\left(1-\frac{M}{N}\right) + 2\binom{n}{2}\left[\frac{M}{N}\cdot\frac{M-1}{N-1} - \left(\frac{M}{N}\right)^2\right] \\ &= n\cdot\frac{M}{N}\cdot\left(\frac{N-M}{N}\right) + n(n-1)\left[\frac{M}{N}\right]\left[\frac{M-1}{N-1} - \frac{M}{N}\right] \\ &= n\cdot\frac{M}{N}\cdot\frac{N-M}{N} + n(n-1)\frac{M}{N}\left(\frac{NM-N-MN+M}{N(N-1)}\right) \\ &= n\cdot\frac{M}{N}\cdot\frac{N-M}{N} + n(n-1)\frac{M}{N}\left(\frac{M-N}{N(N-1)}\right) \\ &= n\cdot\frac{M}{N}\cdot\frac{N-M}{N}\left[1 - \frac{(n-1)}{N-1}\right] \\ &= n\cdot\frac{M}{N}\cdot\frac{N-M}{N}\left[\frac{(N-1)-(n-1)}{N-1}\right] \\ &= n\cdot\frac{M}{N}\cdot\frac{N-M}{N}\cdot\frac{N-n}{N-1}. \quad \blacksquare \end{aligned}$$

Example 10 *Suppose 10 decoration lights are arranged in a row. Each light can be ON with probability 0.7 or OFF with probability 0.3, independent of each other. Two adjacent lights are called a SWITCH if they are ON-OFF or OFF-ON.*

Denote X = number of SWITCHes. For $i = 1, \cdots, n$, let $X_i = 1$ if light i and $i+1$ form a switch; 0 otherwise. Therefore, $X = \sum_{i=1}^{9} X_i$. Find $E(X)$ and $Var(X)$.

Sol^n. eg

L1	L2	L3	L4	L5	L6	L7	L8	L9	L10
ON	OFF	ON	ON	ON	OFF	OFF	ON	ON	ON

$\underbrace{L1\,L2}_{SW1}$, $\underbrace{L2\,L3}_{SW2}$, $\underbrace{L5\,L6}_{SW3}$, $\underbrace{L7\,L8}_{SW4}$

See that

$$X_i \sim Ber(\underbrace{0.7\times 0.3}_{i^{th}\text{ light on, } (i+1)^{th}\text{ light off}} + \underbrace{0.3\times 0.7}_{i^{th}\text{ light off, } (i+1)^{th}\text{ light on}}) = Ber(0.42).$$

$\nearrow E(X_i) = 0.42$

$\searrow E(X_i^2) = 0.42(1-0.42)$

Then

$$E(X) = E\left(\sum_{i=1}^{9} X_i\right) = \sum_{i=1}^{9} E(X_i) = \sum_{i=1}^{9} 0.42 = 9(0.42).$$

Next,

$$Var(X) = Var\left(\sum_{i=1}^{9} X_i\right) = \sum_{i=1}^{9} Var(X_i) + 2\sum_{1\le i<j\le 9} Cov(X_i, X_j)$$

However, note that

→ since any lights that cannot form a switch (ie not directly adjacent to one another) do not have any "overlap",

X_i, X_j are independent if $|j-i|>1$.

Thus

$$\begin{aligned} \sum_{1\le i<j\le 9} Cov(X_i, X_j) &= \sum_{i=1}^{8} Cov(X_i, X_{i+1}) \\ &= \sum_{i=1}^{8}\left[E(X_iX_{i+1}) - E(X_i)E(X_{i+1})\right] \\ &= \sum_{i=1}^{8}\left[(\underset{\text{OFF ON OFF}}{0.7\times 0.3\times 0.7} + \underset{\text{ON OFF ON}}{0.3\times 0.7\times 0.3}) - 0.42^2\right] \\ &= 8\cdot(0.21 - 0.42^2), \end{aligned}$$

and so

$$\begin{aligned} Var(X) &= \sum_{i=1}^{9} Var(X_i) + 2\sum_{1\le i<j\le 9} Cov(X_i, X_j) \\ &= 9(0.42)(1-0.42) + 2\cdot 8\cdot(0.21 - (0.42)^2). \end{aligned}$$

Chapter 9: Continuous Probability Distributions

CONTINUOUS RV / CRV

Let X be a crv.
We say X is "continuous" if
$|\text{range}(X)| > |\mathbb{N}|$.

eg $\text{range}(X) = \mathbb{R}$, $[0,1]$, etc.

PROBABILITY DENSITY FUNCTION / PDF: $f(x)$ (9.1.1)

Let X be a crv.
Then, the probability density function of X, ie $f(x)$, describes the "distribution" of probabilities of X.
In particular:

① $f(x) \geq 0 \quad \forall x \in (-\infty, \infty)$;

② $P[a \leq X \leq b] = \int_a^b f(x)\,dx$;

③ $P[X > b] = \int_b^\infty f(x)\,dx$;

④ $P[X < a] = \int_{-\infty}^a f(x)\,dx$; &

⑤ $P[-\infty < X < \infty] = \int_{-\infty}^{\infty} f(x)\,dx = 1$.

CUMULATIVE DISTRIBUTION FUNCTION / CDF: $F(x)$ (9.1.2)

Let X be a crv, with pdf $f(x)$.
Then the "cumulative distribution function" of X, denoted by "$F(x)$", is defined to be equal to

$$F(x) = P[X \leq x] = \int_{-\infty}^{x} f(x).$$

Note that
① $0 \leq F(x) \leq 1 \quad \forall x \in \mathbb{R}$;
② $F(-\infty) = 0$ & $F(\infty) = 1$;
③ $F(x_1) \leq F(x_2) \iff x_1 \leq x_2$; &
④ If f is cts at x, then $\frac{dF(x)}{dx} = f(x)$.

$X \sim f(x)$, $Y = h(X)$ IS 1-1 $\Rightarrow$ $g(y) = f(h^{-1}(y))\left|\frac{d}{dy}h^{-1}(y)\right|$ <<CHANGE OF VARIABLES>> (9.1.3)

Let $X \sim f(x)$, and let $Y = h(X)$ such that h has a unique inverse (ie h is 1-1).
Then necessarily $Y \sim g(y)$ by

$$g(y) = f(h^{-1}(y))\left|\frac{d}{dy}h^{-1}(y)\right|.$$

eg[1] $X \sim f(x) = 3x^2$, $x \in [0,1]$; $Y = X^2$.

Sol$^{\underline{n}}$. $P[Y \leq y] = P[X^2 \leq y] = P[X \leq \sqrt{y}] = F(\sqrt{y})$.

$$\Rightarrow g(y) = \frac{d}{dy}P[Y \leq y] = \frac{d}{dy}F(\sqrt{y})$$
$$= F'(\sqrt{y}) \cdot \frac{d}{dy}(\sqrt{y})$$
$$= f(\sqrt{y}) \cdot \frac{d}{dy}(\sqrt{y})$$
$$= 3(\sqrt{y})^2 \cdot \frac{1}{2}y^{-\frac{1}{2}}$$
$$= \frac{3}{2}y^{\frac{1}{2}}.$$

eg[2] $X \sim f(x) = 3x^2$, $x \in [0,1]$; $Y = -\log X$.

Sol$^{\underline{n}}$.
$$P[Y \leq y] = P[-\log(x) \leq y]$$
$$= P[\log(x) \geq -y]$$
$$= P[x \geq e^{-y}]$$
$$= 1 - P[x < e^{-y}]$$
$$= 1 - P[x \leq e^{-y}]$$
$$= 1 - F(e^{-y}).$$

Hence
$$g(y) = \frac{d}{dy}P[Y \leq y] = -F'(e^{-y}) \cdot \frac{d}{dy}e^{-y}$$
$$= f(e^{-y}) \cdot \left|\frac{d}{dy}e^{-y}\right|$$
$$= 3(e^{-y})^2 e^{-y}$$
$$= 3e^{-3y}. \quad (y > 0)$$

EXPONENTIAL DISTRIBUTION: $X \sim Exp(\lambda)$ (9.2.1)

We say the crv $X \sim Exp(\lambda)$ iff
$f(x) = \lambda e^{-\lambda x}$, $x \geq 0$,
where $\lambda > 0$.

In particular,
$$F(x) = \int_{-\infty}^{x} \lambda e^{-\lambda t}\,dt = 1 - e^{-\lambda t} \quad \forall x \geq 0.$$

Exponential distributions are often used to model waiting times.

eg Say λ is the intensity parameter for a Poisson process.
Let $X :=$ waiting time for next event
Then $\text{range}(X) = [0, \infty)$ &
$$F(X) = 1 - P[X > x]$$
$$= 1 - P[\text{no events in } [0,x)]$$
$$= 1 - \frac{e^{-\lambda x}(\lambda x)^0}{0!} \quad \text{(using Poisson dist}^{\underline{n}}\text{)}$$
$$= 1 - e^{-\lambda x}.$$
so $X \sim Exp(\lambda)$. ☑

Also note the "memory-less property":
$$P[X > t+s \mid X > s] = P[X > t].$$

GAMMA DISTRIBUTION: $X \sim Gam(\alpha, \beta)$ (9.2.2)

We say $X \sim Gam(\alpha, \beta)$ iff
$$f(x) = \frac{1}{\beta^\alpha \Gamma(\alpha)} x^{\alpha-1} \exp\left(\frac{-x}{\beta}\right) \quad \forall x \geq 0,$$
where $\alpha, \beta > 0$.

Here, the gamma function Γ is given by
$$\Gamma(\alpha) = \int_0^\infty x^{\alpha-1}e^{-x}\,dx, \quad \alpha > 0.$$

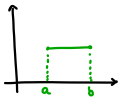
Note that
① $\Gamma(\alpha+1) = \alpha\Gamma(\alpha)$; and
② $\Gamma(\frac{1}{2}) = \sqrt{\pi}$.

Proof. ① can be shown via integration by parts fairly easily.
Note that
$$\Gamma(\tfrac{1}{2}) = \int_0^{+\infty} x^{-\frac{1}{2}}e^{-x}\,dx.$$
Let $x = u^2$. $\Rightarrow dx = 2u\,du$.
$$\Rightarrow \Gamma(\tfrac{1}{2}) = \int_0^\infty 2e^{-u^2}\,du.$$
$$\Rightarrow \Gamma^2(\tfrac{1}{2}) = 4\int_0^{+\infty}\int_0^{+\infty} e^{-u^2-v^2}\,du\,dv.$$
Let $\left.\begin{array}{l} u = r\cos\theta \\ v = r\sin\theta \end{array}\right\} \Rightarrow \Gamma^2(\tfrac{1}{2}) = 4\int_0^{+\infty}\int_0^{\frac{\pi}{2}} e^{-r^2}\,dr\,d\theta$
$$= \pi.$$
$$\Rightarrow \Gamma(\tfrac{1}{2}) = \sqrt{\pi}. \quad ☑$$

UNIFORM DISTRIBUTION: $X \sim Unif(a,b)$ (9.2.3)

1. We say $X \sim Unif(a,b)$ iff

$$f(x) = \frac{1}{b-a} \quad \forall x \in [a,b].$$



2. In this case,

$$F(x) = \frac{x-a}{b-a} \quad \forall x \in [a,b].$$

* in particular, if $a=0$ & $b=1$, then $F(x) = x$.

F_X IS INCREASING $\Rightarrow F_X(X) \sim Unif(0,1)$

Let X have cdf $F_X(x)$, and suppose F_X is strictly increasing.
Let $Y = F_X(X)$. Then $Y \sim Unif(0,1)$.

Proof.
$$\begin{aligned} P[Y \le y] &= P[F_X(x) \le y] \\ &= P[F_X^{-1}(F_X(x)) \le F_X^{-1}(y)] \\ &= P[x \le F_X^{-1}(y)] \\ &= F_X(F_X^{-1}(y)) \\ &= y \\ &= \frac{y-0}{1-0}. \end{aligned}$$

so $Y \sim Unif(0,1)$. ☒

BETA DISTRIBUTION: $X \sim Beta(\alpha_1, \alpha_2)$ (9.2.4)

1. Let $\alpha_1, \alpha_2 > 0$. Then, the beta function of α_1, α_2, denoted as "$B(\alpha_1, \alpha_2)$", is defined to be

$$B(\alpha_1, \alpha_2) = \int_0^1 x^{\alpha_1 - 1}(1-x)^{\alpha_2 - 1}\, dx,$$

or equivalently,

$$B(\alpha_1, \alpha_2) = \frac{\Gamma(\alpha_1)\Gamma(\alpha_2)}{\Gamma(\alpha_1 + \alpha_2)}.$$

2. Then, we say $X \sim Beta(\alpha_1, \alpha_2)$ iff

$$f(x) = \frac{1}{B(\alpha_1, \alpha_2)} x^{\alpha_1 - 1}(1-x)^{\alpha_2 - 1}, \quad x \in [0,1].$$

* in particular, when $\alpha_1 = 1$ & $\alpha_1 = 1$, then

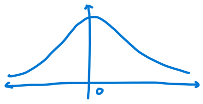
$$\begin{aligned} f(x) &= \frac{1}{B(0,1)} x^0 (1-x)^0 \\ &= \frac{\Gamma(1)}{\Gamma(1)\Gamma(0)} \\ &= \frac{1!}{1!\,0!} \\ &= 1, \end{aligned}$$

so $F(x) = \int_0^x 1\, dx = x$, so $X \sim Unif(0,1)$. ☒

STANDARD NORMAL DISTRIBUTION: $X \sim N(0,1)$ (9.2.5)

1. We say $Z \sim N(0,1)$ iff

$$f(z) = \frac{1}{\sqrt{2\pi}} e^{-\frac{z^2}{2}} \quad \forall z \in (-\infty, \infty).$$



2. In this case, the cdf of Z, denoted by Φ, is equal to

$$\Phi(z) = P[Z \le z] = \int_{-\infty}^{z} \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}\, dx.$$

* note that $\Phi(0) = \frac{1}{2}$ by symmetry of $f(x)$.

GENERAL NORMAL DISTRIBUTION: $X \sim N(\mu, \sigma^2)$

1. We say $X \sim N(\mu, \sigma^2)$ iff

$$f(x) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{(x-\mu)^2}{2\sigma^2}\right) \quad \forall x \in (-\infty, \infty),$$

where $\mu \in \mathbb{R}$ & $\sigma > 0$.

2. Note that in this case, we have $f(x+\delta) = f(x-\delta)$.

STANDARDIZATION

Let $X \sim N(\mu, \sigma^2)$. Let $Z = \frac{X-\mu}{\sigma}$.
Then see that

$$\begin{aligned} F_Z(z) = P[Z \le z] &= P\left[\frac{X-\mu}{\sigma} \le z\right] \\ &= P[X - \mu \le z\sigma] \\ &= P[X \le z\sigma + \mu] \\ &= F_X(\mu + \sigma z), \end{aligned}$$

and so

$$\begin{aligned} f_Z(z) &= \frac{d}{dz} F_Z(z) \\ &= \frac{d}{dz} F_X(\mu + \sigma z) \\ &= f_X(\mu + \sigma z)\,\sigma \\ &= \frac{\sigma}{\sqrt{2\pi\sigma^2}} \exp\left\{-\frac{(\mu + \sigma z - \mu)^2}{2\sigma^2}\right\} \\ &= \frac{1}{\sqrt{2\pi}} e^{-z^2/2}. \end{aligned}$$

In other words,

$$X \sim N(\mu, \sigma^2) \Rightarrow Z = \frac{X-\mu}{\sigma} \sim N(0,1).$$

PDF/CDF OF MULTIVARIATE CONTINOUS DISTRIBUTIONS

Let $X_1, \dots, X_n$ be crv, with joint pdf $f(x_1, \dots, x_n)$.
(this is similar to the discrete case).
Then, by definition, we have

$$P(a_1 < X_1 < b_1, \dots, a_n < X_n < b_n) = \int_{a_1}^{b_1} \cdots \int_{a_n}^{b_n} f(x_1, \dots, x_n)\, dx_n \cdots dx_1.$$

$$E(g(X_1, \dots, X_n)) = \underbrace{\int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty}}_{n} g(x_1, \dots, x_n) f(x_1, \dots, x_n)\, dx_n \cdots dx_1$$

« LAW OF THE UNCONSCIOUS STATISTICIAN »

Let $(X_1, \dots, X_n) \sim f(x_1, \dots, x_n)$ be crv.
Then necessarily

$$E(g(X_1, \dots, X_n)) = \underbrace{\int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty}}_{n} g(x_1, \dots, x_n) f(x_1, \dots, x_n)\, dx_n \cdots dx_1.$$

eg $(X, Y) \sim f(x,y)$: then

$$E(XY) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} xy\, f(x,y)\, dy\, dx.$$

Chapter 10.1: Expectation and Variance of Continuous Random Variables

EXPECTATION [OF CRV]: $E(X) = \int_{-\infty}^{\infty} x f(x)\,dx$ (10.1.1)

Let X be a crv, with pdf $f(x)$.
Then
$$E(X) = \int_{-\infty}^{\infty} x f(x).$$

$E(g(X)) = \int_{-\infty}^{\infty} g(x) f(x)\,dx$ ≪LAW OF UNCONSCIOUS STATISTICIAN≫ (10.1.1.1)

Let X be a crv, with pdf $f(x)$.
Then
$$E[g(X)] = \int_{-\infty}^{\infty} g(x) f(x)\,dx.$$

VARIANCE [OF CRV]: $Var(X) = \int_{-\infty}^{\infty} [x - E(X)]^2 f(x)\,dx$ (10.1.1.2)

Let X be a crv, with pdf $f(x)$.
Then
$$Var(X) = \int_{-\infty}^{\infty} (x - E(X))^2 f(x)\,dx.$$

Proof. Follows from LOUS as $Var(X) := E((X - E(X))^2)$. ☒

$X \sim Exp(\lambda) \Rightarrow E(X) = \frac{1}{\lambda}$, $Var(X) = \frac{1}{\lambda^2}$ (10.1.2.1)

Let $X \sim Exp(\lambda)$.
Then necessarily $E(X) = \frac{1}{\lambda}$ and $Var(X) = \frac{1}{\lambda^2}$.

Proof. $X \sim Exp(\lambda) \Rightarrow f(x) = \lambda e^{-\lambda x}$.
Thus
$$E(X) = \int_0^{\infty} x \cdot \lambda e^{-\lambda x}\,dx = \lambda \int_0^{+\infty} x e^{-\lambda x}\,dx.$$
Let $Y \sim Gamma(\alpha, \beta)$, so that
$$g(y) = \frac{1}{\beta^\alpha \Gamma(\alpha)} y^{\alpha-1} \exp\left(-\frac{y}{\beta}\right). \quad y \geq 0,\ \alpha > 0,\ \beta > 0$$
Then
$$1 = \int_0^{+\infty} g(y)\,dy = \int_0^{+\infty} \frac{1}{\beta^\alpha \Gamma(\alpha)} y^{\alpha-1} \exp\left(-\frac{y}{\beta}\right) dy.$$
$$\Rightarrow \beta^\alpha \Gamma(\alpha) = \int_0^{+\infty} y^{\alpha-1} \exp\left(-\frac{y}{\beta}\right) dy.$$
Let $\beta = \frac{1}{\lambda}$, $\alpha = 2$. The eqn above becomes
$$\int_0^{+\infty} y \exp(-\lambda y)\,dy = \left(\frac{1}{\lambda}\right)^2 \Gamma(2) = \left(\frac{1}{\lambda}\right)^2 (1!) = \frac{1}{\lambda^2},$$
and so
$$E(X) = \lambda \cdot \int_0^{+\infty} x \exp(-\lambda x)\,dx = \lambda \cdot \frac{1}{\lambda^2} = \frac{1}{\lambda}.$$
Let $\beta = \frac{1}{\lambda}$, $\alpha = 3$. The eqn above becomes
$$\int_0^{+\infty} y^2 \exp(-\lambda y)\,dy = \left(\frac{1}{\lambda}\right)^3 \Gamma(3) = \frac{2}{\lambda^3}.$$
and so
$$E(X^2) = \lambda \int_0^{+\infty} x^2 \exp(-\lambda x)\,dx = \lambda \cdot \frac{2}{\lambda^3} = \frac{2}{\lambda^2}.$$
Thus
$$Var(X) = E(X^2) - (E(X))^2 = \frac{2}{\lambda^2} - \left(\frac{1}{\lambda}\right)^2 = \frac{1}{\lambda^2},$$
as required. ☒

$X \sim Gamma(\alpha, \beta) \Rightarrow E(X) = \alpha\beta$, $Var(X) = \alpha\beta^2$ (10.1.2.2)

Let $X \sim Gamma(\alpha, \beta)$.
Then necessarily $E(X) = \alpha\beta$ & $Var(X) = \alpha\beta^2$.

Proof. Recall $X \sim Gamma(\alpha, \beta) \Rightarrow f(x) = \frac{1}{\beta^\alpha \Gamma(\alpha)} x^{\alpha-1} e^{-\frac{x}{\beta}}$, $x \geq 0$, $\alpha, \beta > 0$
Thus
$$E(X) = \int_0^{+\infty} x \cdot \frac{1}{\beta^\alpha \Gamma(\alpha)} x^{\alpha-1} e^{-\frac{x}{\beta}}\,dx = \frac{1}{\beta^\alpha \Gamma(\alpha)} \int_0^{+\infty} x^\alpha e^{-\frac{x}{\beta}}\,dx.$$
Let $u = \frac{x}{\beta} \Rightarrow du = \frac{1}{\beta} dx$
So
$$E(X) = \frac{1}{\beta^\alpha \Gamma(\alpha)} \int_{u=0}^{u=+\infty} (\beta u)^\alpha e^{-u} (\beta\,du)$$
$$= \frac{\beta}{\Gamma(\alpha)} \int_0^{+\infty} u^\alpha e^{-u}\,du$$
$$= \frac{\beta}{\Gamma(\alpha)} (\Gamma(\alpha+1)) \qquad \left(\text{recall } \Gamma(\alpha) = \int_0^{+\infty} x^{\alpha-1} e^{-x}\,dx\right)$$
$$= \alpha \cdot \beta.$$
Similarly,
$$E(X^2) = \frac{1}{\beta^\alpha \Gamma(\alpha)} \int_0^{+\infty} x^{\alpha+1} e^{-\frac{x}{\beta}}\,dx.$$
Using $u = \frac{x}{\beta}$ again,
$$\Rightarrow E(X^2) = \frac{1}{\beta^\alpha \Gamma(\alpha)} \int_{u=0}^{u=+\infty} (\beta u)^{\alpha+1} e^{-u} (\beta\,du)$$
$$= \frac{\beta^2}{\Gamma(\alpha)} \int_0^{+\infty} u^{\alpha+1} e^{-u}\,du$$
$$= \frac{\beta^2}{\Gamma(\alpha)} (\Gamma(\alpha+2)) = \beta^2 \alpha(\alpha+1).$$
Thus
$$Var(X) = E(X^2) - (E(X))^2 = \beta^2 \alpha(\alpha+1) - (\alpha\beta)^2$$
$$\therefore Var(X) = \alpha\beta^2. \quad ☒$$

$X \sim N(0,1) \Rightarrow E(X) = 0$, $Var(X) = 1$ (10.1.2.3)

Let $X \sim N(0,1)$.
Then necessarily $E(X) = 0$ & $Var(X) = 1$.

Proof. Recall $X \sim N(0,1) \Rightarrow f(x) = \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{x^2}{2}\right)$, $x \in \mathbb{R}$.
Then
$$E(X) = \int_{-\infty}^{+\infty} x \cdot \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{x^2}{2}\right) dx = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \underbrace{x \cdot \exp\left(-\frac{x^2}{2}\right) dx}_{\text{odd function}} = 0.$$
Next,
$$E(X^2) = \int_{-\infty}^{\infty} \underbrace{x^2 \cdot \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{x^2}{2}\right)}_{\text{even function}} dx = 2 \int_0^{\infty} x^2 \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{x^2}{2}\right) dx$$
Let $u = x^2$ $(x > 0)$ $\Rightarrow du = 2x\,dx \Rightarrow dx = \frac{du}{2x} = \frac{1}{2} u^{-\frac{1}{2}} du$
$$\Rightarrow E(X^2) = 2 \int_{u=0}^{u=+\infty} u \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{u}{2}\right) \frac{1}{2} u^{-\frac{1}{2}}\,du = \frac{1}{\sqrt{2\pi}} \int_0^{+\infty} u^{\frac{1}{2}} \exp\left(-\frac{u}{2}\right) du$$
note $U \sim Gamma(\frac{3}{2}, 2) \Rightarrow \int_0^{+\infty} g(u) = 1 \Rightarrow \int_0^{+\infty} \frac{1}{2^{3/2}\Gamma(\frac{3}{2})} u^{\frac{3}{2}-1} \exp\left(-\frac{u}{2}\right) du = 1$
$$\Rightarrow \int_0^{+\infty} u^{\frac{1}{2}} \exp\left(-\frac{u}{2}\right) du = 2^{\frac{3}{2}} \Gamma\left(\frac{3}{2}\right).$$
$$\therefore E(X^2) = \frac{1}{\sqrt{2\pi}} \left(2^{\frac{3}{2}} \Gamma\left(\frac{3}{2}\right)\right) = \frac{1}{\sqrt{2\pi}} \left(2^{\frac{3}{2}} \left(\frac{1}{2} \cdot \sqrt{\pi}\right)\right) \qquad \left(\Gamma\left(\frac{3}{2}\right) = \frac{1}{2} \cdot \Gamma\left(\frac{1}{2}\right) = \frac{1}{2} \cdot \sqrt{\pi}\right)$$
$$= 2^{-\frac{3}{2}} \cdot \frac{1}{\sqrt{\pi}} \cdot 2^{\frac{3}{2}} \cdot \sqrt{\pi} = 1. \quad ☒$$

$X \sim N(\mu, \sigma^2) \Rightarrow E(X) = \mu$, $Var(X) = \sigma^2$

Let $X \sim N(\mu, \sigma^2)$.
Then necessarily $E(X) = \mu$ and $Var(X) = \sigma^2$.

Proof. Note $Z = \frac{X - \mu}{\sigma}$, where $Z \sim N(0,1)$.
Thus $X = \mu + \sigma Z$, so
$$E(X) = \mu + \sigma E(Z) = \mu, \ \&$$
$$Var(X) = \sigma^2 Var(Z) = \sigma^2. \quad ☒$$

Chapter 10.2: Moments and Moment Generating Functions

MOMENTS: $E(X^k)$ (10.2.1)

Let X be a rv.
Then, the "k^{th} moment" of X is simply $E(X^k)$.

MOMENT GENERATING FUNCTIONS / MGF: $M_X(t) = E(e^{tX})$

1. Let X be a rv.
Then, the "moment generating function" of X, denoted as "$M_X(t)$", is defined to be
$$M_X(t) := E(e^{tX}),$$
provided this expectation exists for all $t \in (-h, h)$, where $h > 0$ (ie is finite).

2. In particular, if X is discrete, then
$$M_X(t) = E(e^{tX}) = \sum_x e^{tx} f(x).$$

3. If X is instead continuous, then
$$M_X(t) = E(e^{tX}) = \int_{-\infty}^{+\infty} e^{tx} f(x)\, dx.$$

4. Note that a MGF uniquely determines a probability distribution, whether discrete or continuous.

$X \sim Bin(n, p) \Rightarrow M_X(t) = (e^t p + 1 - p)^n, \quad t \in (-\infty, \infty)$ (10.2.3.1)

Let $X \sim Bin(n,p)$.
Then necessarily $M_X(t) = (e^t p + (1-p))^n$, where $t \in (-\infty, \infty)$.

Proof.
$$\begin{aligned} M_X(t) &= E(e^{tX}) \\ &= \sum_{x=0}^{n} e^{tx} f(x) \\ &= \sum_{x=0}^{n} e^{tx} \cdot \binom{n}{x} p^x (1-p)^{n-x} \\ &= \sum_{x=0}^{n} \binom{n}{x} (e^t p)^x (1-p)^{n-x} \\ &= (e^t p + (1-p))^n \quad \text{(by binomial formula)} \\ &< +\infty \quad \forall t \in (-\infty, \infty) \end{aligned}$$

$X \sim Gamma(\alpha, \beta) \Rightarrow M_X(t) = \frac{1}{(1-\beta t)^\alpha}, \quad t < \frac{1}{\beta}$ (10.2.3.2)

Let $X \sim Gamma(\alpha, \beta)$.
Then necessarily $M_X(t) = \frac{1}{(1-\beta t)^\alpha}$ $\forall t < \frac{1}{\beta}$.

Proof.
$$\begin{aligned} M_X(t) &= E(e^{tX}) \\ &= \int_0^{+\infty} e^{tx} \cdot \frac{1}{\beta^\alpha \Gamma(\alpha)} x^{\alpha-1} e^{-\frac{x}{\beta}}\, dx \\ &= \frac{1}{\beta^\alpha \Gamma(\alpha)} \int_0^{+\infty} x^{\alpha-1} \exp\left(-x\left(\tfrac{1}{\beta} - t\right)\right) dx \\ &= \frac{1}{\beta^\alpha \Gamma(\alpha)} \int_0^{+\infty} x^{\alpha-1} \exp\left(-\frac{x}{(\frac{1}{\beta} - t)^{-1}}\right) dx \\ &= \frac{1}{\beta^\alpha \Gamma(\alpha)} \cdot \left(\tfrac{1}{\beta} - t\right)^{-\alpha} \Gamma(\alpha) \quad \text{if } \tfrac{1}{\beta} - t > 0 \quad \text{(using the "gamma dist}^{\underline{n}} \text{ trick")} \\ &= \frac{1}{\beta^\alpha} \left(\frac{1-\beta t}{\beta}\right)^{-\alpha} \\ &= \frac{1}{\beta^\alpha} \cdot \frac{\beta^\alpha}{(1-\beta t)^\alpha} \\ &= \frac{1}{(1-\beta t)^\alpha}, \quad t < \tfrac{1}{\beta}. \end{aligned}$$

What if $t \geq \frac{1}{\beta}$?
Then
$$\begin{aligned} M_X(t) &= \frac{1}{\beta^\alpha \Gamma(\alpha)} \int_0^{+\infty} x^{\alpha-1} \underbrace{\exp\Big(\underbrace{-x\underbrace{\left(\tfrac{1}{\beta} - t\right)}_{-ve}}_{+ve}\Big)}_{\geq 1} dx \\ &\geq \frac{1}{\beta^\alpha \Gamma(\alpha)} \int_0^{+\infty} x^{\alpha-1}\, dx \\ &= \frac{1}{\beta^\alpha \Gamma(\alpha)} \left[\frac{x^\alpha}{\alpha}\right]_0^\infty \\ &= +\infty. \quad \boxtimes \end{aligned}$$

$X \sim Po(\lambda) \Rightarrow M_X(t) = e^{\lambda[e^t - 1]}$

Let $X \sim Po(\lambda)$.
Then necessarily $M_X(t) = e^{\lambda[e^t - 1]}$

$X \sim N(0,1) \Rightarrow M_X(t) = e^{\frac{t^2}{2}}, \quad t \in (-\infty, \infty)$ (10.2.3.3)

Let $X \sim N(0,1)$.
Then necessarily $M_X(t) = e^{\frac{t^2}{2}}$, for $t \in (-\infty, \infty)$.

$Y = aX + b \Rightarrow M_Y(t) = e^{bt} M_X(at), \quad t \in \left(\frac{-h}{|a|}, \frac{h}{|a|}\right)$ (10.2.3.4)

Let X be a rv, with MGF $M_X(t)$ $\forall t \in (-h, h)$.
Let $Y = aX + b$.
Then the MGF of Y is necessarily given by
$$M_Y(t) = e^{bt} M_X(at) \quad \forall t \in \left(\tfrac{-h}{|a|}, \tfrac{h}{|a|}\right).$$

Proof.
$$\begin{aligned} M_Y(t) &= E[e^{tY}] \\ &= E[e^{t(aX+b)}] \\ &= E[e^{taX + bt}] \\ &= E[e^{bt} e^{taX}] \\ &= e^{tb} E[e^{taX}] \\ &= e^{bt} M_X(at). \end{aligned}$$
In particular, $M_X(at)$ exists for $at \in (-h, h) \Rightarrow t \in \left(\frac{-h}{|a|}, \frac{h}{|a|}\right)$. $\boxtimes$

$X \sim N(\mu, \sigma^2) \Rightarrow M_X(t) = \exp\left(\mu t + \frac{\sigma^2 t^2}{2}\right)$

Let $X \sim N(\mu, \sigma^2)$.
Then necessarily $M_X(t) = \exp\left(\mu t + \frac{\sigma^2 t^2}{2}\right)$, $t \in (-\infty, \infty)$.

Proof. Note $X = \mu + \sigma Z$, $Z \sim N(0,1)$.
Hence
$$\begin{aligned} M_X(t) &= e^{\mu t} M_Z(\sigma t) \\ &= e^{\mu t}\left(e^{\frac{\sigma^2 t^2}{2}}\right) \\ &= e^{\mu t + \frac{\sigma^2 t^2}{2}}. \quad \boxtimes \end{aligned}$$

$X \sim Gamma(\alpha, \beta) \Rightarrow \frac{X}{\beta} \sim Gamma(\alpha, 1)$

Let $X \sim Gamma(\alpha, \beta)$. Then $\frac{X}{\beta} \sim Gamma(\alpha, 1)$.

Proof. Let $Y = \frac{X}{\beta}$. See that
$$\begin{aligned} M_Y(t) &= e^{0t} M_X\left(\tfrac{t}{\beta}\right) \\ &= \frac{1}{(1 - \beta \cdot \frac{t}{\beta})^\alpha}, \quad \tfrac{t}{\beta} < \tfrac{1}{\beta} \quad \text{(ie } t < 1\text{)} \\ &= \frac{1}{(1-t)^\alpha}. \end{aligned}$$
Note $W \sim Gamma(\alpha, 1) \iff M_W(t) = \frac{1}{(1-t)^\alpha}$.
Therefore $\frac{X}{\beta} \sim Gamma(\alpha, 1)$, as needed. $\boxtimes$

JOINT MGF: $M_{X,Y}(t_1, t_2) = E(e^{t_1 X + t_2 Y})$ (10.2.3.5)

1. Let X and Y be rv.
Then, the "joint MGF" of X & Y, denoted as "$M_{X,Y}(t_1, t_2)$", is defined to be
$$M_{X,Y}(t_1, t_2) = E(e^{t_1 X + t_2 Y}),$$
provided it exists $\forall t_1 \in (-h_1, h_1)$, $t_2 \in (-h_2, h_2)$.

2. Note that
$$M_X(t_1) = E(e^{t_1 X}) = E(e^{t_1 X + 0 \cdot Y}) = M_{X,Y}(t_1, 0),$$
and
$$M_Y(t_2) = E(e^{t_2 Y}) = E(e^{0 \cdot X + t_2 \cdot Y}) = M_{X,Y}(0, t_2).$$
* this is valid, as $0 \in (-h_1, h_1)$ & $0 \in (-h_2, h_2)$.

3. This also extends to more variables as well.

Chapter 11.1: Main Properties of MGF

MOMENTS FROM MGF:

$M_X(0) = 1, \quad M_X^{(k)}(0) = E(X^k)$ (11.1.1)

Let X be a rv with MGF $M_X(t)$, $t \in (-h, h)$.

Then necessarily $M_X(0) = 1$ and

$$M_X^{(k)}(0) = E(X^k), \quad k = 1, 2, \dots$$

where

$$M_X^{(k)}(t) = \frac{d^k}{dt^k} M_X(t).$$

Proof. Recall

$$M_X(t) = E(e^{tX}).$$

Thus

$$M_X(0) = E(e^{0 \cdot X}) = E(1) = 1.$$

Case #1: X is a drv.

Then
$$M_X^{(k)}(t) = \frac{d^k}{dt^k} M_X(t) = \frac{d^k}{dt^k} \sum_x e^{tx} f(x) = \sum_x x^k e^{tx} f(x),$$

so

$$M_X^{(k)}(0) = \sum_x x^k (1) f(x) = E(X^k) \quad \text{(by LOUS)}. \ \#$$

Case #2: X is a crv.

Then
$$M_X^{(k)}(t) = \frac{d^k}{dt^k} M_X(t) = \frac{d^k}{dt^k} \int_{-\infty}^{\infty} e^{tx} f(x)\, dx = \int_{-\infty}^{\infty} \frac{d^k}{dt^k} e^{tx} f(x)\, dx = \int_{-\infty}^{\infty} x^k e^{tx} f(x)\, dx,$$

so

$$M_X^{(k)}(0) = \int_{-\infty}^{\infty} x^k e^{(0)} f(x)\, dx = \int_{-\infty}^{\infty} x^k f(x)\, dx = E(X^k). \ \boxtimes$$

JOINT MOMENTS FROM JOINT MGF:

$$E(X^j Y^k) = \frac{\partial^{j+k}}{\partial t_1^j \partial t_2^k} M_{X,Y}(t_1, t_2)\Big|_{(t_1,t_2)=(0,0)} \quad (11.1.2)$$

Let X, Y be crv with joint MGF $M_{X,Y}(t_1, t_2)$ $\forall t_1 \in (-h_1, h_1)$, $t_2 \in (-h_2, h_2)$.

Then necessarily

$$E(X^j Y^k) = \frac{\partial^{j+k}}{\partial t_1^j \partial t_2^k} M_{X,Y}(t_1, t_2)\Big|_{(t_1,t_2)=(0,0)}.$$

Proof.
$$\frac{\partial^{j+k}}{\partial t_1^j \partial t_2^k} M_{X,Y}(t_1,t_2) = \frac{\partial^{j+k}}{\partial t_1^j \partial t_2^k} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{t_1 x} e^{t_2 y} f(x,y)\, dx\, dy$$
$$= \frac{\partial^j}{\partial t_1^j} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{\partial^k}{\partial t_2^k} e^{t_1 x} e^{t_2 y} f(x,y)\, dx\, dy$$
$$= \frac{\partial^j}{\partial t_1^j} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} y^k e^{t_1 x} e^{t_2 y} f(x,y)\, dx\, dy$$
$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{\partial^j}{\partial t_1^j} y^k e^{t_1 x} e^{t_2 y} f(x,y)\, dx\, dy$$
$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x^j y^k e^{t_1 x} e^{t_2 y} f(x,y)\, dx\, dy,$$

and so

$$\frac{\partial^{j+k}}{\partial t_1^j \partial t_2^k} M_{X,Y}(t_1,t_2)\Big|_{(t_1,t_2)=(0,0)} = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x^j y^k e^{t_1(0)} e^{t_2(0)} f(x,y)\, dx\, dy$$
$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x^j y^k f(x,y)\, dx\, dy$$
$$= E(X^j Y^k) \quad \text{(by LOUS)}. \ \boxtimes$$

Ex: $(X_1, X_2, X_3) \sim \text{Tri}(n, p_1, p_2, p_3) \Rightarrow E(X_1 X_2) = ?$

Sol$^{\underline{n}}$. See that

$$E(X_1 X_2) = \frac{\partial^2}{\partial t_1 \partial t_2} M_{X_1,X_2}(t_1,t_2)$$

where

$$M_{X_1,X_2}(t_1,t_2) = E(e^{t_1 X_1} e^{t_2 X_2}) \quad (\forall t_1, t_2 \in \mathbb{R})$$
$$= \sum_{x_1=0}^{n} \sum_{x_2=0}^{n-x_1} e^{t_1 x_1} e^{t_2 x_2} \frac{n!}{x_1! x_2! x_3!} p_1^{x_1} p_2^{x_2} p_3^{x_3}, \quad x_3 = n - x_1 - x_2 \quad \text{(by LOUS)}$$
$$= \sum_{x_1=0}^{n} \sum_{x_2=0}^{n-x_1} \frac{n!}{x_1! x_2! x_3!} (p_1 e^{t_1})^{x_1} (p_2 e^{t_2})^{x_2} p_3^{x_3}, \quad x_3 = n - x_1 - x_2$$
$$= (p_1 e^{t_1} + p_2 e^{t_2} + p_3)^n. \quad \left(\text{recall } (a+b+c)^n = \sum_{x_1=0}^{n} \sum_{x_2=0}^{n-x_1} \frac{n!}{x_1! x_2! x_3!} a^{x_1} b^{x_2} c^{x_3}, \ x_3 = n - x_1 - x_2\right)$$

Thus

$$\frac{\partial^2}{\partial t_1 \partial t_2} M_{X_1,X_2}(t_1,t_2) = \frac{\partial^2}{\partial t_1 \partial t_2} (p_1 e^{t_1} + p_2 e^{t_2} + p_3)^n$$
$$= \frac{\partial}{\partial t_1} \left(n (p_1 e^{t_1} + p_2 e^{t_2} + p_3)^{n-1} \cdot p_2 e^{t_2} \right)$$
$$= n(n-1)(p_1 e^{t_1} + p_2 e^{t_2} + p_3)^{n-2} \cdot p_2 e^{t_2} \cdot p_1 e^{t_1},$$

and so

$$E(X_1 X_2) = \frac{\partial^2}{\partial t_1 \partial t_2} M_{X_1,X_2}(t_1,t_2)\Big|_{(t_1,t_2)=(0,0)}$$
$$= n(n-1)(p_1 e^0 + p_2 e^0 + p_3)^{n-2} p_2 e^0 p_1 e^0$$
$$= n(n-1) p_1 p_2. \ \#$$

X, Y ARE INDEPENDENT $\Leftrightarrow M_{X,Y}(t_1,t_2) = M_X(t_1) M_Y(t_2)$

<< MGF INDEPENDENCE THEOREM >> (11.1.3)

Let X, Y be rv.

Then X & Y are independent iff

$$M_{X,Y}(t_1,t_2) = M_X(t_1) M_Y(t_2) \quad \forall t_1 \in (-h_1, h_1), t_2 \in (-h_2, h_2).$$

$M_X(t) \equiv M_Y(t) \Leftrightarrow X = Y$ << UNIQUENESS OF MGF >>

(11.1.4)

Let X, Y be rv such that $M_X(t) = M_Y(t)$ $\forall t \in (-h, h)$.

Then necessarily $X = Y$.

In other words, X & Y have the same distribution.

Proof. This follows from injectivity of e^{tX} & e^{tY}. $\boxtimes$

X_1, X_2 ARE INDEPENDENT $\Rightarrow M_{X_1+X_2}(t) = M_{X_1}(t) M_{X_2}(t)$

(11.1.5)

Let X_1 & X_2 be independent rv, and X_i has MGF $M_{X_i}(t)$ $\forall t \in (-h, h)$.

Then necessarily

$$M_{X_1+X_2}(t) = M_{X_1}(t) M_{X_2}(t).$$

Chapter 11.2: Convergence in Probability

CONVERGENCE IN PROBABILITY: $X_n \xrightarrow{P} b$

Let (X_n) be a sequence of rvs.
Then, we say X_n "converges in probability" to a constant b if for any $\varepsilon > 0$,

$$\lim_{n\to\infty} P[|X_n - b| \geq \varepsilon] = 0 \quad \text{or} \quad \lim_{n\to\infty} [|X_n - b| < \varepsilon] = 1,$$

and in this case we write

$$X_n \xrightarrow{P} b.$$

$|E(|X|^k)| < \infty \Rightarrow P[|X| \geq c] \leq \frac{E(|X|^k)}{c^k} \quad \forall c, k > 0$

≪MARKOV'S INEQUALITY≫ (11.2.1)

Let X be a rv, and suppose $E(|X|^k)$ is finite for each $k \in \mathbb{N}$.
Then necessarily

$$P[|X| \geq c] \leq \frac{E(|X|^k)}{c^k} \quad \forall c, k > 0.$$

Proof. We just consider the cts case; discrete case is similar.

See that

$$\begin{aligned}
\frac{E[|X|^k]}{c^k} &= \frac{1}{c^k}\left[\int_{x:|x|<c} |x|^k f(x)\,dx + \int_{x:|x|\geq c} |x|^k f(x)\,dx\right] \\
&= \int_{x:|\frac{x}{c}|<1} \left|\frac{x}{c}\right|^k f(x)\,dx + \int_{x:|\frac{x}{c}|\geq 1} \left|\frac{x}{c}\right|^k f(x)\,dx \\
&\geq 0 + \int_{x:|\frac{x}{c}|\geq 1} \left|\frac{x}{c}\right|^k f(x)\,dx \\
&\geq 0 + \int_{x:|\frac{x}{c}|\geq 1} 1^k f(x)\,dx \\
&= P\left(\left|\frac{X}{c}\right| \geq 1\right) \\
&= P(|X| \geq c),
\end{aligned}$$

as needed. ☒

INDEPENDENT & IDENTICALLY DISTRIBUTED RANDOM VARIABLES / IID

We say the rv $X_1, \dots, X_n$ are "independent & identically distributed", or just "IID", if

① each random variable has the same probability distribution (ie $X_1, \dots, X_n \sim f(x)$); and

② the variables are mutually independent.

In this case, we may also write that

$$X_i \overset{iid}{\sim} f(x).$$

X_i ARE IID, $E(X_i) = \mu$, $Var(X_i) = \sigma^2$, $\overline{X}_n = \frac{1}{n}\sum_{i=1}^n X_i$ $\Rightarrow \overline{X}_n \xrightarrow{P} \mu$ ≪ WEAK LAW OF LARGE NUMBERS ≫

Let the rvs X_i be IID, with each $E(X_i) = \mu$ & $Var(X_i) = \sigma^2$.
Denote

$$\overline{X}_n := \frac{1}{n}\sum_{i=1}^n X_i.$$

Then necessarily $\overline{X}_n \xrightarrow{P} \mu$.

Proof. First, see that

$$E(\overline{X}) = \frac{1}{n}\sum_{i=1}^n E(X_i) = \frac{n\cdot\mu}{n} = \mu; \quad \&$$

$$Var(\overline{X}) = \frac{1}{n^2}\sum_{i=1}^n Var(X_i) = \frac{1}{n^2}(n\sigma^2) = \frac{\sigma^2}{n}$$

Let $\varepsilon > 0$. We wish to show

$$\lim_{n\to\infty} P[|\overline{X} - \mu| \geq \varepsilon] = 0.$$

By Markov's Inequality,

$$P[|\overline{X} - \mu| \geq \varepsilon] \leq \frac{E[|\overline{X} - \mu|^k]}{\varepsilon^k} \quad \forall k > 0.$$

In particular, when $k = 2$,

$$\begin{aligned}
P[|\overline{X} - \mu| \geq \varepsilon] &\leq \frac{E[|\overline{X} - \mu|^2]}{\varepsilon^2} \\
&= \frac{E[(\overline{X} - \mu)^2]}{\varepsilon^2} \\
&= \frac{Var(\overline{X})}{\varepsilon^2} \\
&= \frac{\sigma^2}{n\varepsilon^2} \\
&\to 0 \quad \text{as } n \to \infty.
\end{aligned}$$

By a squeeze theorem argument, the proof follows. ☒

$X_n \xrightarrow{P} a$, $Y_n \xrightarrow{P} b \Rightarrow X_n + Y_n \xrightarrow{P} a + b$, $X_n Y_n \xrightarrow{P} ab$

Let X_n and Y_n be rv such that $X_n \xrightarrow{P} a$ & $Y_n \xrightarrow{P} b$.
Then necessarily

① $X_n + Y_n \xrightarrow{P} a + b$; and

② $X_n Y_n \xrightarrow{P} ab$.

Chapter 12.1: Convergence in Distribution

CONVERGENCE IN DISTRIBUTION : $X_n \xrightarrow{D} X$

1. We say the rvs X_n "converge in distribution" to X if

$$\lim_{n\to\infty} F_n(x) = F(x) \quad \forall x: F \text{ is cts at } x,$$

and in this case write

$$X_n \xrightarrow{D} X.$$

2. In particular,

$$P[X_n \le x] \approx P[X \le x]$$

for large n (if F is cts at x).

3. Note that $\lim_{n\to\infty} F_n(x)$ might not be a CDF!

Ex: $Y_i \overset{iid}{\sim} Unif(0,\theta)$, $X_n = \max(Y_1,\dots,Y_n)$; SHOW $X_n \xrightarrow{D} X$

Sol$^{\underline{n}}$. Note that

$$P[Y_i \le x] = \begin{cases} 0, & x \le 0 \\ \frac{x}{\theta}, & 0 < x < \theta \\ 1, & x \ge \theta \end{cases} \quad \forall i = 1,\dots,n,$$

so

$$F_n(x) = \begin{cases} 0, & x \le 0 \\ \left(\frac{x}{\theta}\right)^n, & 0 < x < \theta \\ 1, & x \ge \theta. \end{cases}$$

as

$$\begin{aligned} P[X_n \le x] &= P[\max(Y_1,\dots,Y_n) \le x] \\ &= P[Y_1 \le x, \dots, Y_n \le x] \\ &= P[Y_1 \le x] \cdots P[Y_n \le x] \quad \text{(by iid of } Y_n\text{'s)} \\ &= \left(\tfrac{x}{\theta}\right)^n. \end{aligned}$$

As $\left(\frac{x}{\theta}\right) \in (0,1)$, thus

$$\lim_{n\to\infty} F_n(x) = \begin{cases} 0, & x \le 0 \\ 0, & 0 < x < \theta \\ 1, & x \ge \theta \end{cases} = \begin{cases} 0, & x < \theta \\ 1, & x \ge \theta. \end{cases}$$

Hence

$$X_n \xrightarrow{D} X, \quad \text{where} \quad F(x) = \begin{cases} 0, & x < \theta \\ 1, & x \ge \theta \end{cases}$$

(ie X is the rv that takes value θ with probability 1). #

$\lim_{n\to\infty} \psi(n) = 0 \Rightarrow \lim_{n\to\infty} [1 + \frac{b}{n} + \frac{\psi(n)}{n}]^n = e^b$

<< e LIMIT >> (12.1.1)

If $b \in \mathbb{R}$ and $\lim_{n\to\infty} \psi(n) = 0$, then

$$\lim_{n\to\infty} \left[1 + \frac{b}{n} + \frac{\psi(n)}{n}\right]^n = e^b.$$

In particular,

$$\lim_{n\to\infty} \left[1 + \frac{b}{n}\right]^n = e^b.$$

EX: $Y_i \overset{iid}{\sim} Exp(1)$; $X_n = \max(Y_1,\dots,Y_n) - \ln(n)$; SHOW $X_n \xrightarrow{D} X$

Sol$^{\underline{n}}$. See that

$$\begin{aligned} F_n(x) = P[X_n \le x] &= P[\max(Y_1,\dots,Y_n) - \ln(n) \le x] \\ &= P[\max(Y_1,\dots,Y_n) \le x + \ln(n)] \\ &= \prod_{i=1}^{n} P[Y_i \le x + \ln(n)] \\ &= (P[Y_1 \le x + \ln(n)])^n \end{aligned} \quad \text{(by iid-ness of } Y_i\text{'s)}.$$

Then, as $Y_i \overset{iid}{\sim} Exp(1)$, thus each Y_i has pdf $f(y) = e^{-y}$, and so

$$P[Y_1 \le y] = \begin{cases} \int_0^y e^{-y}\,dy = 1 - e^{-y}, & y > 0 \\ 0, & y \le 0. \end{cases}$$

Therefore

$$F_n(x) = \begin{cases} (1 - e^{-x-\ln(n)})^n = (1 - \frac{e^{-x}}{n})^n, & x + \ln(n) > 0 \\ 0, & x + \ln(n) \le 0. \end{cases}$$

See that $x \le -\ln(n) \Rightarrow x < 0$ (cont$^{\underline{n}}$ as $x \ge 0$), so

$$\begin{aligned} \lim_{n\to\infty} F_n(x) &= \lim_{n\to\infty} \left(1 - \frac{e^{-x}}{n}\right)^n \\ &= e^{(-e^{-x})} \quad \forall x \in \mathbb{R}. \quad \text{(by e limit)} \end{aligned}$$

Therefore

$X_n \xrightarrow{D} X$, where X has CDF $e^{-e^{-x}}$ $\forall x \in \mathbb{R}$. ☒

$\lim_{n\to\infty} M_n(t) \to M(t) \quad \forall t \in (-h,h) \Rightarrow X_n \xrightarrow{D} X$

<< MGF CONVERGENCE THEOREM >> (12.1.2)

Let $X_1, X_2, \dots$ be a sequence of rvs, and let $M_1(t), M_2(t), \dots$ be their respective MGFs. Let X be a rv with MGF $M(t)$, such that there exists a $h > 0$ such that

$$\lim_{n\to\infty} M_n(t) = M(t) \quad \forall t \in (-h,h).$$

Then necessarily $X_n \xrightarrow{D} X$.

$X_n \sim Bin(n,p)$; $n \to \infty$, $p \to 0 \Rightarrow X_n \xrightarrow{D} X \sim Po(np)$

<< POISSON APPROXIMATION TO THE BINOMIAL DISTRIBUTION >>

1. Let $X_i \sim Bin(n,p)$. We can use the MGF technique to show that

$$X_i \xrightarrow{D} X \quad (\text{as } p \to 0),$$

where $X \sim Po(np)$.

Sol$^{\underline{n}}$. $X_n \sim Bin(n,p)$

$$\begin{aligned} M_n(t) &= E[e^{t \cdot X_n}], \\ &= [e^t p + (1-p)]^n, \quad t \in \mathbb{R}. \end{aligned}$$

Let $\lambda = np$. Then

$$\begin{aligned} \lim_{n\to\infty} M_n(t) &= \lim_{n\to\infty} [e^t p + (1-p)]^n \\ &= \lim_{n\to\infty} \left[1 + \frac{(e^t - 1)\lambda}{n}\right]^n \\ &= e^{\lambda(e^t - 1)} \quad \text{(by the e limit)}, \end{aligned}$$

which is exactly the MGF of Poisson dist$^{\underline{n}}$ w/ intensity parameter λ. ☒

2. Note that since X is discrete, $F(x) = P[X \le x]$ is not continuous at $x = 0, 1, 2, \dots$

So, we have to perform a continuity correction:

$$\begin{aligned} P[X_n = x] &= P[X_n \le x + 0.5] - P[X_n \le x - 0.5] \\ &\approx P[X \le x + 0.5] - P[X \le x - 0.5]. \end{aligned}$$

Chapter 12.2: Central Limit Theorem

X_i IID, $E(X_i)=\mu$, $Var(X_i)=\sigma^2$, $M_{X_i}(t)$ EXISTS

$\forall t \in (-h,h) \Rightarrow \sqrt{n}\frac{(\bar{X}-\mu)}{\sigma} \xrightarrow{D} Z \sim N(0,1)$

<< CENTRAL LIMIT THEOREM >>

Let X_i be a sequence of iid rvs, with $E(X_i)=\mu$ and $Var(X_i)=\sigma^2<\infty$.
Further suppose each MGF $M_{X_i}(t)$ exists $\forall t \in (-h,h)$, where $h>0$.
Then necessarily

$$\sqrt{n}\frac{(\bar{X}-\mu)}{\sigma} \xrightarrow{D} Z \sim N(0,1).$$

Proof. Let $Z_n = \sqrt{n}\frac{(\bar{X}-\mu)}{\sigma}$. Then

$$M_{\bar{X}}(t) = E[e^{t\cdot\frac{\sqrt{n}(\bar{X}-\mu)}{\sigma}}]$$

$$= E[e^{\frac{t}{\sqrt{n}}\sum_{i=1}^{n}\frac{(X_i-\mu)}{\sigma}}]$$

$$= E[e^{\frac{t}{\sqrt{n}}\sum_{i=1}^{n}Y_i}] \quad \text{where } Y_i = \frac{X_i-\mu}{\sigma} \quad (\text{so } E(Y_i)=0,\ Var(Y_i)=1)$$

$$= M^n_{Y_i}\left(\frac{t}{\sqrt{n}}\right), \quad (\text{by iid of } Y_i)$$

which exists for $\frac{t}{\sqrt{n}} \in (-\sigma h, \sigma h)$.

Next, recall the Taylor series of f:

$$f(x) = f(0) + f'(0)x + \frac{f^{(2)}(0)x^2}{2!} + \frac{f^{(3)}(0)x^3}{3!} + \cdots,$$

and so

$$M_{Y_i}\left(\frac{t}{\sqrt{n}}\right) = M_{Y_i}(0) + M'_{Y_i}(0)\left(\frac{t}{\sqrt{n}}\right) + \frac{M^{(2)}_{Y_i}(0)\left(\frac{t}{\sqrt{n}}\right)^2}{2!} + \frac{M^{(3)}_{Y_i}(0)\left(\frac{t}{\sqrt{n}}\right)^3}{3!} + \cdots$$

$$= 1 + E(Y_i)\frac{t}{\sqrt{n}} + \frac{1}{2}E(Y_i^2)\frac{t^2}{n} + \frac{1}{n}\left[\frac{M^{(3)}_{Y_i}(0)\,t^3}{3!\cdot\sqrt{n}} + \frac{M^{(4)}_{Y_i}(0)\,t^4}{4!\cdot n} + \cdots\right]$$

(recall $M^{(n)}_X(0) = E(X^n)$)

$$= 1 + E(Y_i)\frac{t}{\sqrt{n}} + \frac{1}{2}[Var(Y_i) - (E(Y_i))^2]\frac{t^2}{n} + \frac{\psi(n)}{n},$$

$$\psi(n) = \frac{M^{(3)}_{Y_i}(0)\,t^3}{3!\cdot\sqrt{n}} + \frac{M^{(4)}_{Y_i}(0)\,t^4}{4!\cdot n} + \cdots$$

$$= 1 + 0\cdot\frac{t}{\sqrt{n}} + \frac{1}{2}(1-0^2)\frac{t^2}{n} + \frac{\psi(n)}{n}$$

$$\therefore M_{Y_i}\left(\frac{t}{\sqrt{n}}\right) = 1 + \frac{1}{2}\cdot\frac{t^2}{n} + \frac{\psi(n)}{n}.$$

Then, notice

$\psi(n) \to 0$ as $n \to +\infty$,

and so as $M_{\bar{X}}(t) = \left(M_{Y_i}\left(\frac{t}{\sqrt{n}}\right)\right)^n$, it follows that

$$\lim_{n\to\infty} M_{\bar{X}}(t) = \lim_{n\to\infty}\left(1 + \frac{t^2}{2n} + \frac{\psi(n)}{n}\right)^n$$

$$= e^{\frac{t^2}{2}}, \quad (\text{by the } e \text{ limit})$$

which is exactly the MGF of $Z \sim N(0,1)$.

Hence, by the MGF convergence theorem,

$$\sqrt{n}\frac{(\bar{X}-\mu)}{\sigma} = Z_n \xrightarrow{D} Z \sim N(0,1). \quad \square$$